The Distribution Of Prime Numbers And Continued Fractions

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Introduction

- Gandhi's formula:

$$p_n = \left\lfloor 1 - log_2\left(-\frac{1}{2} + \sum_{d\left|\prod_{k=1}^{n-1} p_k\right.} \frac{\mu(d)}{2^d - 1}\right)\right\rfloor$$

- Mill's formula:

$$\left\lfloor \theta^{3^n} \right\rfloor, where\ \theta = 1.3064\ ...$$

- Willans' formula

$$p_n = 1 + \sum_{i=1}^{2^n} \left\lfloor \left(\frac{n}{\sum_{j=1}^{i} \left\lfloor \left(\cos\frac{(j-1)!+1}{x}\pi\right)^2 \right\rfloor} \right)^{\frac{1}{n}} \right\rfloor$$

- Rowland's sequence

$$r(n) = r(n-1) + gcd\big(n, r(n-1)\big)$$

Where the initial condition $r(1) = 7$.

The sequence $\gcd(n, r(n-1))$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 47, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 233, 3, 1, 467, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, … (See A132199)

$\gcd(n, r(n-1))$ takes only 1's and primes.

The continued fraction

For all integers $n \geq 3$.

$$\cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\,(n-1)-\cfrac{n}{m}}}}} = \frac{mb(n-3)-nb(n-4)}{n(m-n+2)-m}$$

- Where $b(n) = (n+2)\big(b(n-1) - b(n-2)\big)$
  With the initial conditions $b(-1) = 0$ and $b(0) = 1$.

- And m is a polynomial in term n.

Our sequence

$$a(n) = \frac{n^2 - n - 1}{gcd\left(n^2 - \mathrm{n} - 1,\ b(n-3) + nb(n-4)\right)}$$

1, 5, 11, 19, 29, 41, 11, 71, 89, 109, 131, 31, 181, 19, 239, 271, 61, 31, 379, 419, 461, 101, 29, 599, 59, 701, 151, 811, 79, 929, 991, 211, 59, 41, 1259, 1, 281, 1481, 1559, 149, 1721, 1, 61, 1979, 2069, 2161, 1, 2351, 79, 2549, 241, 1, 2861, 2969, 3079, 3191, 661, 311, 3539, 3659, 199, 71, 139, 4159, 4289, 4421, 911, 4691, 439, 4969, 269, 1051, 491, 179, 139, 5851, 1201, 101, 89, 1, 229, 1361, 6971, 1, 7309, 7481, 1531, 191, 8009, 431, 761, 1, 8741, 8929, 829, 9311, 1901, 109, 521, 10099, 10301, 191, 10711, 179, 359, 1031, 2311, 149, 631, 421, 401, 2531, 1171, 13109, 13339, 331, 251, 739, 131, 14519, 509, 3001, 151, 1409, 15749, 16001, 3251, 1, 409, 17029, 17291, 3511, 251, 18089, 1669, 601, 199, 19181, 1, 19739, 20021, 1, 349, 20879, 21169, 1951, 229, 22051, 22349, 1, 389, 4651, 23561, 23869, 24179, 1289, 1, 25121, 25439, 25759, 2371, 5281, 26731, 27059, 449, 1459, 181, 1, 28729, 709, 29411, 541, 971, 30449, 1621, 31151, 6301, 211, 1, 32579, 32941, 6661, 3061, 34039, 1811, 34781, 1, 35531, 241, 3299, 36671, 7411, 37441, 1, 38219, 38611, 269, 1, 39799, 659, 3691, 1, 41411, 1, 349,.. (See A356247)

$a(n)$ takes only 1's and primes.

Conjecture 1.

For all integers k > 0 and $n \geq 3$, The sequence

$$a_k(n) = \frac{n^2 + (k-2)n - k}{gcd\left(n^2 + (k-2)n - k,\ kb(n-3) + nb(n-4)\right)}$$

$a_k(n)$ takes only 1's and primes.

Remark 1.

$a_k(n)$ is the denominator of the continued fraction for

$m = -k.$

Table 1: The sequence $a_k(n)$ for $k = 1, 2, 3, 4, 5$.

k	$a_k(n)$	Oeis
1	5, 11, 19, 29, 41, 11, 71, 89, 109, 131, 31, 181, 19, 239, 271, 61, 31, 379, 419, 461, 101, 29, 599, 59, 701, 151, 811, 79, 929, 991, 211, 59, 41, 1259, 1, 281, 1481,…	A356247
2	7, 7, 23, 17, 47, 31, 79, 7, 17, 71, 167, 97, 223, 127, 41, 23, 359, 199, 439, 241, 31, 41, 89, 337, 727, 1, 839, 449, 137, 73, 1087, 577, 1223, 647, 1367, 103, 1, 47, 73,…	A363102
3	3, 17, 9, 13, 53, 23, 29, 107, 43, 17, 179, 23, 79, 269, 101, 113, 29, 139, 1, 503, 61, 199, 647, 233, 251, 809, 17, 103, 43, 1, 373, 1187, 419, 443, 61, 1, 173, 1637, 191, 601,…	A362086
4	11, 5, 31, 11, 59, 19, 19, 29, 139, 41, 191, 1, 251, 71, 29, 89, 79, 109, 479, 131, 571, 31, 61, 181, 41, 1, 179, 239, 1019, 271, 1151, 61, 1291, 1, 1439, 379, 1, 419, 1759,…	A363347
5	13, 23, 7, 49, 13, 83, 103, 5, 149, 1, 29, 233, 53, 23, 67, 373, 59, 1, 499, 109, 593, 643, 139, 107, 1, 863, 71, 197, 1049, 223, 1, 179, 53, 1399, 59, 1553, 71, 1, 257, 1,…	A363482

- Except for 9, the sequence $a_3(n)$ contains only 1's and the primes.
- Except for 49, the sequence $a_5(n)$ contains only 1's and the primes.

Conjecture 2.

For all integers k > 0 and $n \geq 3$, The sequence

$$a_k(n) = \frac{(k+1)n - k}{gcd\big((k+1)n - k, b(n-2) + kb(n-3)\big)}$$

$a_k(n)$ takes only 1's and primes.

Remark 2.

$a_k(n)$ is the denominator of the continued fraction for $m = n + k$.

Table 2: The sequence $a_k(n)$ for $k = 1, 2, 3, 4, 5$.

k	$a_k(n)$	Oeis
1	5, 7, 3, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1,…	A356360
2	7, 5, 13, 2, 19, 11, 5, 1, 31, 17, 37, 1, 43, 23, 1, 1, 1, 29, 61, 1, 67, 1, 73, 1, 79, 41, 1, 1, 1, 47, 97, 1, 103, 53, 109, 1, 1, 59, 1, 1, 127, 1, 1, 1, 139, 71, 1, 1, 151, 1, 157, 1, 163,…	A369797
3	3, 13, 17, 7, 5, 29, 11, 37, 41, 1, 7, 53, 19, 61, 1, 23, 73, 1, 1, 1, 89, 31, 97, 101, 1, 109, 113, 1, 1, 1, 43, 1, 137, 47, 1, 149, 1, 157, 1, 1, 1, 173, 59, 181, 1, 1, 193, 197, 67,…	A370726
4	11, 4, 7, 13, 31, 1, 41, 23, 17, 1, 61, 1, 71, 19, 1, 43, 1, 1, 101, 53, 37, 29, 1, 1, 131, 1, 47, 73, 151, 1, 1, 83, 1, 1, 181, 1, 191, 1, 67, 103, 211, 1, 1, 113, 1, 59, 241, 1, 251, 1, 1,…	
5	13, 19, 5, 31, 37, 43, 7, 11, 61, 67, 73, 79, 17, 1, 97, 103, 109, 23, 11, 127, 1, 139, 29, 151, 157, 163, 1, 1, 181, 1, 193, 199, 41, 211, 1, 223, 229, 47, 241, 1, 1, 1, 53, 271,...	

- Except for 4, the sequence $a_4(n)$ contains only 1's and the primes.