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Frederick E. Daum was born in Orange, NJ, on March 20, 1947. He received the B.S. degree in electrical engineering from the Newark College of Engineering, Newark, NJ, in 1969, and the M.S. degree in decision and control theory from Harvard University, Cambridge, MA, in 1972.

From 1972 to 1974 he did research on nonlinear continuous-time estimation theory at Harvard University. He is currently a Principal Engineer at Raytheon Company, Wayland, MA, on the staff of the Advanced System Engineering De-



Robert J. Fitzgerald was born in Hamilton, Ont., Canada. He received the B.A.Sc. degree from the University of Toronto, Toronto, Ont., Canada, in 1956, and the S.M., Mech. E., and Ph.D. degrees from the Massachusetts Institute of Technology, Cambridge, in 1957, 1958, and 1964, respectively. He also studied at the Ecole Nationale Supérieure de l'Aéronautique, Paris, France.

Since 1964 he has been employed in various divisions of the Raytheon Company, Bedford, MA, where his principal activities have been concerned with the application of estimation and control theory to radar tracking, inertial navigation, and missile intercept problems.

Utilization of Modified Polar Coordinates for Bearings-Only Tracking

VINCENT J. AIDALA, MEMBER, IEEE, AND SHERRY E. HAMMEL, MEMBER, IEEE

Abstract—Previous studies have shown that the Cartesian coordinate extended Kalman filter exhibits unstable behavior characteristics when utilized for bearings-only target motion analysis (TMA). In contrast, formulating the TMA estimation problem in modified polar (MP) coordinates leads to an extended Kalman filter which is both stable and asymptot-

ically unbiased. Exact state equations for the MP filter are derived without imposing *any* restrictions on own-ship motion; thus, prediction accuracy inherent in the traditional Cartesian formulation is completely preserved. In addition, these equations reveal that MP coordinates are well-suited for bearings-only TMA because they automatically decouple observable and unobservable components of the estimated state vector. Such decoupling is shown to prevent covariance matrix ill-conditioning, which is the primary cause of filter instability. Further investigation also confirms that the MP state estimates are asymptotically unbiased. Realistic simulation data are presented to support these findings and to compare algorithm performance with respect to the Cramer-Rao lower bound (ideal) as well as the Cartesian and pseudolinear filters.

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The authors are with the U.S. Naval Underwater Systems Center, Newport, RI 02840.

INTRODUCTION

PASSIVE localization and tracking problems arise in a variety of important practical applications [1]–[4]. In the ocean environment, two-dimensional bearings-only target motion analysis (TMA) is perhaps familiar [4]–[8]. Here, a single moving observer (own-ship) monitors noisy sonar bearings from a radiating acoustic source (target) assumed to be traveling with constant velocity, and subsequently processes these measurements to obtain estimates of source position and velocity. The geometric configuration is depicted in Fig. 1, where own-ship and target are presumed to lie in the same horizontal plane.

Unfortunately, this particular estimation problem is not amenable to simple solution. Intrinsic system nonlinearities preclude the rigorous application of conventional linear analysis. When pseudolinear formulations [9], [10] are employed, the resulting algorithms exhibit biased estimation properties [11], [12]. Moreover, since bearing measurements are extracted from only one sensor, the process remains unobservable until own-ship executes a maneuver [5], [13]. It is this prerequisite maneuver which distinguishes bearings-only TMA from more conventional localization and tracking procedures (e.g., classical triangulation ranging, etc.) and introduces added complexity to the problem.

Despite the aforementioned difficulties, numerous techniques have been devised for bearings-only TMA [6]. One method of solution which has received considerable attention in recent years is the extended Kalman filter [14]. As is well known, utilization of this filter requires explicit mathematical models for both the measurement process and the state dynamics. When addressing these modeling requirements, it is important to recognize that the pertinent analytical equations often acquire entirely dissimilar properties when expressed in different coordinate systems (e.g., linear equations are transformed into nonlinear equations and vice versa). Accordingly, the final filter configuration for any specific problem will ultimately depend upon which reference frame is employed during problem formulation. It is not surprising then, to find that Cartesian coordinates are used extensively, if not exclusively, to formulate TMA estimation problems in the context of an extended Kalman filter. Indeed, this reference frame permits a simple linear representation of the state dynamics; all system nonlinearities are embedded in a single scalar measurement equation [10]. Such a modeling structure is especially appealing for practical applications because it minimizes filter computational requirements.

While the question of choosing "optimal" tracking coordinates has been previously addressed in the literature [3], [6], [15], [16], until recently, sufficient evidence was not available to legitimately infer that non-Cartesian filters may possess significantly different, and perhaps better, performance characteristics than their Cartesian counterparts. Furthermore, the underlying cause of these differences was never clearly identified or well understood. Now, however, new theoretical and experimental findings have been published [10], [17]–[19] which conclusively

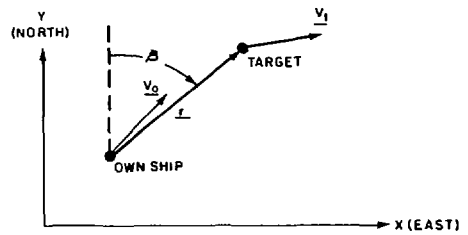


Fig. 1.

demonstrate that the Cartesian filter is unstable for single sensor bearings-only TMA. Specifically, these results show that the unique interaction and feedback of estimation errors within this filter render it highly susceptible to premature covariance collapse and solution divergence.

One method which has been successfully utilized to eliminate Cartesian filter instability involves replacing the measured bearings with pseudolinear measurement residuals [9], [10]. While this procedure is particularly simple to implement, it has not gained widespread acceptance within the TMA community because the resulting algorithm (pseudolinear filter) generates biased estimates whenever noisy measurements are processed [12]. Instead, research efforts have focused upon the analysis and development of alternative estimation schemes which are both stable and asymptotically unbiased.

In this paper, a candidate TMA algorithm with the desired attributes is rigorously analyzed and subsequently evaluated under realistic operating conditions. The pertinent equations of state and measurement are formulated in modified polar (MP) coordinates [20], [21], while the algorithm itself is configured as an extended Kalman filter [14]. This coordinate system is shown to be well-suited for bearings-only TMA because it automatically decouples observable and unobservable components of the estimated state vector. Such decoupling prevents covariance matrix ill-conditioning, which is the primary cause of filter instability. Further investigation also confirms that the resulting state estimates are asymptotically unbiased, as required.

The MP state vector is comprised of the following four components: bearing, bearing rate, range rate divided by range, and the reciprocal of range. In theory, the first three can be determined from single-sensor bearing data without an own-ship maneuver; the fourth component, however, should remain unobservable until this maneuver requirement is satisfied. These theoretical observability properties are implicitly preserved in the MP filter formulation. In essence, the state estimates are constrained to behave as predicted by theory, even in the presence of measurement errors. Under similar conditions, standard Cartesian filters often experience covariance matrix ill-conditioning which precipitates false observability.

Exact state equations for the MP filter are rigorously derived without imposing *any* restrictions on own-ship motion; thus, prediction accuracy inherent in the traditional Cartesian formulation is completely preserved. These equations also reveal that the choice of reciprocal range as the fourth state is optimal, at least from the viewpoint of

minimizing system nonlinearities. In addition, the aforementioned expressions can be readily generalized to satisfy the nonlinear differential equations of *arbitrary* particle motion. To the authors' knowledge, these exact solutions have not been previously documented in the literature.

Realistic tactical scenarios are simulated to illustrate and evaluate MP filter performance. Employing an "idealized" filter, optimal performance (i.e., the Cramer-Rao lower bound [22]) is first determined, and subsequently used as an evaluation basis. Simulation data depicting relative performance are then presented. Finally, the MP filter is compared to the standard Cartesian and pseudolinear filters, both of which are utilized in operational systems. These results clearly demonstrate the stability and overall efficacy of the MP filter.

II. DEVELOPMENT OF THE FILTER EQUATIONS

As noted earlier, formulating the TMA estimation problem in Cartesian coordinates leads to a linear representation of the state dynamics and a nonlinear scalar measurement relation,¹ viz.

$$\mathbf{x}(t) = A_x(t, t_o)\mathbf{x}(t_o) - \mathbf{w}_o(t, t_o) \quad (1a)$$

$$\tilde{\beta}(t) = h_x[\mathbf{x}(t)] + n(t) \quad (1b)$$

where $\mathbf{x}(t)$ defines the Cartesian state vector of unknown target motion parameters. To facilitate the ensuing analysis, it is implicitly assumed that $\mathbf{w}_o(t, t_o)$ is deterministic and $n(t)$ is zero-mean Gaussian white noise with variance $\sigma^2(t)$. For completeness, initial estimates of the state vector and its associated error covariance matrix are also presumed to be specified.

Ostensibly, it is difficult to obtain an equivalent representation of equation set (1) in MP coordinates. This is especially true if conventional modeling techniques are employed. Significant analytical complications arise because the transformed equations of motion are highly nonlinear and not readily amenable to direct integration. These encumbrances can be avoided, however, by recognizing that the desired representation is also derivable entirely by algebraic manipulation (see Appendix B for details). Briefly, if $\mathbf{y}(t)$ denotes the MP state vector, then $\mathbf{x}(t)$ and $\mathbf{y}(t)$ will be related at all instants of time by the nonlinear one-to-one transformations

$$\mathbf{x}(t) = f_x[\mathbf{y}(t)] \quad (2a)$$

$$\mathbf{y}(t) = f_y[\mathbf{x}(t)]. \quad (2b)$$

Letting $t = t_o$ in (2a), and substituting the result into (1a) to eliminate $\mathbf{x}(t_o)$, yields

$$\mathbf{x}(t) = A_x(t, t_o) f_x[\mathbf{y}(t_o)] - \mathbf{w}_o(t, t_o). \quad (3)$$

Substitution of (3) into (2b) subsequently leads to the relation

$$\mathbf{y}(t) = f[\mathbf{y}(t_o); t, t_o] \quad (4)$$

where

$$f[\mathbf{y}(t_o); t, t_o] \triangleq f_y[A_x(t, t_o) f_x[\mathbf{y}(t_o)] - \mathbf{w}_o(t, t_o)]. \quad (5)$$

In a similar manner, the measurement relation can be formally expressed in MP coordinates by substituting (2a) into (1b), viz.

$$\tilde{\beta}(t) = h_y[\mathbf{y}(t)] + n(t) \quad (6)$$

where

$$h_y[\mathbf{y}(t)] \triangleq h_x[f_x[\mathbf{y}(t)]] = [0, 0, 1, 0] \mathbf{y}(t). \quad (7)$$

Equations (4) and (6) are the exact MP analogs of (1a) and (1b), respectively. As might be expected, the state equations now exhibit nonlinearities and are considerably more complicated than their Cartesian counterparts. The derivation outlined here, however, remains comparatively simple; all difficulties associated with integrating the transformed equations of motion have been expeditiously circumvented without sacrificing mathematical rigor. Moreover, since (4) is valid for *arbitrary* own-ship motion, prediction accuracy inherent in the traditional Cartesian formulation is completely preserved.

Although the preceding results are expressed in continuous form, discrete time equations of state and measurement may be readily deduced by assigning appropriate values to t and t_o . In particular, if $t = kT$ and $t_o = (k-1)T$ where $k = 1, 2, 3, \dots$ and $T =$ constant sampling period, straightforward application of the extended Kalman filter [14] to (4) and (6) will yield the MP estimation algorithm given below:

$\mathbf{y}(0/0)$ = initial estimate of the MP state vector

$P(0/0)$ = initial estimate of the MP state vector error covariance matrix

$$\mathbf{y}(k/k-1) = f[\mathbf{y}(k-1/k-1); kT, (k-1)T] \quad (8a)$$

$$A_y(k, k-1) = \frac{\partial f[\mathbf{y}(k-1/k-1); kT, (k-1)T]}{\partial \mathbf{y}(k-1/k-1)} \quad (8b)$$

$$P(k/k-1) = A_y(k, k-1)P(k-1/k-1)A_y'(k, k-1) \quad (8c)$$

$$H = [0, 0, 1, 0] \quad (8d)$$

$$G(k) = P(k/k-1)H'[HP(k/k-1)H' + \sigma^2(k)]^{-1} \quad (8e)$$

$$\mathbf{y}(k/k) = \mathbf{y}(k/k-1) + G(k)[\tilde{\beta}(k) - H\mathbf{y}(k/k-1)] \quad (8f)$$

$$P(k/k) = [I - G(k)H]P(k/k-1) \quad k = 1, 2, 3, \dots \quad (8g)$$

¹Mathematical details of the Cartesian and modified polar coordinate modeling processes are presented in Appendix A and B, respectively, along with a precise description of the various quantities.

where the "prime" symbol ($'$) denotes matrix transposition. Here, $y(i/j)$ and $P(i/j)$ for $i, j = 1, 2, 3, \dots$ denote estimates of the true state vector $y(i)$ and its associated error covariance matrix $P(i)$, respectively, based upon j data measurements.

While equation set (8) is well-suited for numerical work, it is not readily amenable to analysis. In contrast, nonrecursive representations of this filter are often more tractable, despite their computational inefficiency. One such representation is obtained by reformulating the original dynamic estimation problem for $y(t)$ as a static estimation problem for $y(t_0)$. Note that since $w_0(t, t_0)$ is deterministic, (4) allows the unknown state vector to be uniquely specified at any arbitrary time t in terms of its value at some fixed time t_0 . The required measurement relation then follows by substituting (4) into (6). It is unnecessary to explicitly specify state equations because $y(t_0)$ is time invariant. Subsequent application of the extended Kalman filter to the discretized expressions obtained by letting $t = kT$ and $t_0 = 0$ yields a recursive algorithm which may be algebraically recast into the nonrecursive form shown below:

$y(0/0)$ = initial estimate of the MP state vector

$P(0/0)$ = initial estimate of the MP state vector
error covariance matrix

$$\begin{aligned} y(0/k) &= P(0/k)P^{-1}(0/0)y(0/0) \\ &+ P(0/k) \sum_{j=1}^k M'(j)\sigma^{-2}(j)M(j)y(0/j-1) \\ &+ P(0/k) \sum_{j=1}^k M'(j)\sigma^{-2}(j) \\ &\cdot [\hat{\beta}(j) - Hy(j/j-1)] \end{aligned} \quad (9a)$$

$$P(0/k) = [P^{-1}(0/0) + \sum_{j=1}^k M'(j)\sigma^{-2}(j)M(j)]^{-1} \quad (9b)$$

where H is given by (8d), and

$$M(j) = HA_y(j, 0) \quad (10a)$$

$$A_y(j, 0) = \frac{\partial f[y(0/j-1); jT, 0]}{\partial y(0/j-1)} \quad (10b)$$

$$y(j/j-1) = f[y(0/j-1); jT, 0]. \quad (10c)$$

Finally, discrete estimates of the current state vector and its associated error covariance matrix can be obtained via the relations

$$y(k/k) = f[y(0/k); kT, 0] \quad (11a)$$

$$P(k/k) = A_y(k, 0)P(0/k)A_y'(k, 0). \quad (11b)$$

We remark that the static and dynamic estimation algorithms described here possess similar mathematical properties and exhibit virtually identical behavior characteristics.

III. BEHAVIOR CHARACTERISTICS OF THE FILTER

In practical applications, $y(k/k)$ and $P(k/k)$ are of special interest since they statistically characterize the *current* state vector. For analysis purposes, however, it is more convenient to work directly with $y(0/k)$ and $P(0/k)$ which do not vary explicitly with time. By utilizing this approach, extraneous time variations are automatically suppressed. There is no loss of generality either because all four quantities are related by equation set (11). Pertinent behavior characteristics of $y(k/k)$ and $P(k/k)$ can always be determined *a posteriori* from knowledge of the behavior of $y(0/k)$ and $P(0/k)$. Consequently, an examination of equation set (9) will reveal the underlying mechanism governing MP filter stability.

To begin the discussion, consider the closed form expression for $P(0/k)$ given by (9b). Utilizing conventional matrix multiplication rules, it can be shown that

$$\sum_{j=1}^k M'(j)\sigma^{-2}(j)M(j) = C'(k)D^{-1}(k)C(k) \quad (12)$$

where $D(k)$ is a $k \times k$ diagonal matrix given by

$$D(k) = \text{diag}[\sigma^2(1), \sigma^2(2), \dots, \sigma^2(k)] \quad (13)$$

and $C(k)$ is a $k \times 4$ matrix which may be written in the partitioned form

$$C(k) = [M'(1)|M'(2)|\dots|M'(k)]'. \quad (14)$$

Important structural characteristics of $P(0/k)$ can now be discerned by analyzing the ancillary matrix $C(k)$. Combining equation set (10) with (B10) of Appendix B yields

$$M(j) = \begin{bmatrix} M_{11}(j) & \left| \begin{array}{l} \frac{\partial y_3(j/j-1)}{\partial y_4(0/j-1)} \end{array} \right. \end{bmatrix} \quad (15)$$

where

$$M_{11}(j) = \begin{bmatrix} \frac{\partial y_3(j/j-1)}{\partial y_1(0/j-1)}, \frac{\partial y_3(j/j-1)}{\partial y_2(0/j-1)}, \frac{\partial y_3(j/j-1)}{\partial y_3(0/j-1)} \end{bmatrix} \quad (16)$$

$$\begin{aligned} y_3(j/j-1) &= f_3[y(0/j-1); jT, 0] \\ &= y_3(0/j-1) \\ &+ \tan^{-1}[S_3(jT, 0)/S_4(jT, 0)] \end{aligned} \quad (17)$$

and

$$\begin{aligned} S_3(jT, 0) &= jTy_1(0/j-1) - y_4(0/j-1) \\ &\cdot [w_{o3}(jT, 0) \cos y_3(0/j-1) \\ &- w_{o4}(jT, 0) \sin y_3(0/j-1)] \end{aligned} \quad (18a)$$

$$\begin{aligned} S_4(jT, 0) &= 1 + jTy_2(0/j-1) - y_4(0/j-1) \\ &\cdot [w_{o3}(jT, 0) \sin y_3(0/j-1) \\ &+ w_{o4}(jT, 0) \cos y_3(0/j-1)]. \end{aligned} \quad (18b)$$

The quantities $w_{o3}(jT,0)$ and $w_{o4}(jT,0)$ that appear in equation set (18) depend upon own-ship acceleration and can be deduced from (A9) of Appendix A by letting $t = jT$ and $t_o = 0$. It is especially important to recognize that $w_{o3}(jT,0) = 0$ and $w_{o4}(jT,0) = 0$ prior to the first own-ship maneuver. In addition, the fourth element of $M(j)$ will also vanish under these circumstances, as can be seen by performing the indicated differentiation, viz.

$$\frac{\partial y_3(j/j-1)}{\partial y_4(0/j-1)} = \frac{w_{o4}(jT,0) \sin y_3(j/j-1) - w_{o3}(jT,0) \cos y_3(j/j-1)}{\sqrt{S_3^2(jT,0) + S_4^2(jT,0)}} \quad (19)$$

These important null characteristics may be exploited further by repartitioning the matrix $C(k)$ as follows:

$$C(k) = [C_{11}(k)|C_{12}(k)] \quad (20)$$

where $C_{11}(k)$ is a $k \times 3$ matrix given by

$$C_{11}(k) = [M'_{11}(1)|M'_{11}(2)|\dots|M'_{11}(k)]' \quad (21a)$$

and $C_{12}(k)$ is a $k \times 1$ column matrix of the form

$$C_{12}(k) = \left[\frac{\partial y_3(1/0)}{\partial y_4(0/0)}, \frac{\partial y_3(2/1)}{\partial y_4(0/1)}, \dots, \frac{\partial y_3(k/k-1)}{\partial y_4(0/k-1)} \right]' \quad (21b)$$

Next, assume that $P(0/0)$ is chosen so as to satisfy the relation

$$P(0/0) = \begin{bmatrix} \Gamma_o^{-1} & 0 \\ 0 & \sigma_o^2 \end{bmatrix} \quad (22)$$

where Γ_o is a 3×3 positive definite symmetric matrix and σ_o is a nonzero scalar. Note that this covariance matrix structure is physically realistic and implies that errors associated with observable and unobservable components of the initial state vector are uncorrelated. Combining (9b) with (12), (20), and (22) subsequently yields

$$P(0/k) = \begin{bmatrix} \Gamma_o^{-1} + C'_{11}(k)D^{-1}(k)C_{11}(k) & C'_{11}(k)D^{-1}(k)C_{12}(k) \\ C'_{12}(k)D^{-1}(k)C_{11}(k) & \sigma_o^{-2} + C'_{12}(k)D^{-1}(k)C_{12}(k) \end{bmatrix}^{-1} \quad (23)$$

Recall that, in theory, the first three components of $y(0/k)$ can be determined without an own-ship maneuver, whereas the fourth component remains unobservable until this maneuver requirement is satisfied. Formulating the TMA estimation problem in MP coordinates leads to a natural decoupling of observable and unobservable states. The resulting structure of $P(0/k)$ depicted by (23) illustrates this decoupling property and provides the key to MP filter stability. From (19) and (21b) it is evident that $C_{12}(k)$ reduces to a null matrix for unaccelerated own-ship

motion. In this case, the off-diagonal terms in (23) vanish and the lower diagonal term reduces to σ_o^{-2} . Straightforward inversion [23] of the resulting simplified matrix leads to

$$P(0/k) = \begin{bmatrix} [\Gamma_o^{-1} + C'_{11}(k)D^{-1}(k)C_{11}(k)]^{-1} & 0 \\ 0 & \sigma_o^2 \end{bmatrix} \quad (24)$$

Examination of (24) reveals that the variance associated with $y_4(0/k)$ remains unchanged (i.e., equal to σ_o^2) prior to the first own-ship maneuver. The automatic decoupling that accrues from utilizing MP coordinates thus prevents covariance matrix ill-conditioning and premature collapse. As a result, $P(0/k)$ will behave exactly as predicted by theory, even in the presence of measurement errors.

The implications of component decoupling on $y(0/k)$ are also readily exposed by substituting (15), (19), (22), and (24) into (9a). Assuming no own-ship acceleration, and partitioning the state vector into observable and unobservable components, leads to the expression

$$y(0/k) = \begin{bmatrix} y_o(0/k) \\ y_4(0/k) \end{bmatrix} = \begin{bmatrix} y_1(0/k) \\ y_2(0/k) \\ y_3(0/k) \\ y_4(0/k) \end{bmatrix} \quad (25)$$

where

$$y_o(0/k) = \left[\Gamma_o^{-1} + C'_{11}(k)D^{-1}(k)C_{11}(k) \right]^{-1} \cdot \left[\Gamma_o^{-1}y_o(0/0) + \sum_{j=1}^k M'_{11}(j)\sigma^{-2}(j)M_{11}(j)y_o(0/j-1) + \sum_{j=1}^k M'_{11}(j)\sigma^{-2}(j)[\tilde{\beta}(j) - \beta(j/j-1)] \right] \quad (26a)$$

$$y_4(0/k) = y_4(0/0) \quad (26b)$$

and $\beta(j/j-1)$ is given by the formula

$$\beta(j/j-1) = y_3(0/j-1) + \tan^{-1} \left[\frac{jTy_1(0/j-1)}{1 + jTy_2(0/j-1)} \right] \quad (27)$$

which follows from (17) for $w_{o3}(jT,0) \equiv 0$ and $w_{o4}(jT,0) \equiv 0$.

Note that any measurements processed before the first own-ship maneuver will affect only $y_o(0/k)$, while $y_4(0/k)$

remains unaltered. This result is not surprising since data acquired under such conditions cannot realistically yield information about the unobservable component. In fact, for unaccelerated own-ship motion, (27) clearly shows that bearings extracted from a single sensor are independent of the fourth state. Hence, $y(0/k)$ exhibits theoretically consistent behavior characteristics analogous to $P(0/k)$.

Unlike its MP counterpart, the Cartesian filter does not provide a natural decoupling of observable and unobservable states. Consequently, there is no internal mechanism to effectively prevent covariance matrix ill-conditioning and premature collapse. The unimpeded feedback and interaction of estimation errors resulting from this deficiency often precipitate false observability. Under such conditions, the filter erroneously attempts to estimate all four state components, even though only three are theoretically observable. Since fundamental uniqueness requirements are violated, the corresponding state estimates will necessarily exhibit unstable behavior characteristics. In essence, Cartesian filter instability is intrinsically related to its development.

IV. SIMULATION RESULTS

To illustrate MP filter performance characteristics, the estimation algorithm described by equation set (8) was programmed on a digital computer and subsequently tested with simulated data. Representative results for two typical TMA scenarios are summarized in Figs. 2–4 (additional data depicting MP, Cartesian, and pseudolinear filter performance may also be found in [7], [10]–[12], [17], [20], [21]). Fig. 2 describes the target and own-ship trajectories for both scenarios, which differ only in their respective values of initial range. The solution errors plotted in Fig. 3 correspond to an initial range of 2700 yds, while those in Fig. 4 correspond to an initial range of 27 000 yds. The initial bearing is 0° , and the target maintains a steady course of 0° with a constant speed of 20 knots. Own-ship also maintains a constant speed of 28.28 knots, but periodically executes 90° course changes as follows:

$$\begin{aligned} \text{from } 45^\circ \text{ to } 315^\circ \text{ at } t = (4 + 17k) \text{ min} \\ [k = 0, 1, 2, 3, 4] \end{aligned}$$

$$\begin{aligned} \text{from } 315^\circ \text{ to } 45^\circ \text{ at } t = (12.5 + 17k) \text{ min} \\ [k = 0, 1, 2, 3, 4]. \end{aligned}$$

Own-ship turning rate is constrained to $3^\circ/\text{s}$; thus, each maneuver requires 0.5 min to complete.

To realistically simulate measurement errors and account for the effects of data preprocessing, all raw bearings were corrupted by additive zero-mean Gaussian noise and then sequentially time-averaged in blocks of twenty 1 s samples. Three different noise levels were used.² The graphs labeled (A–a)–(A–d) depict TMA solution errors obtained with

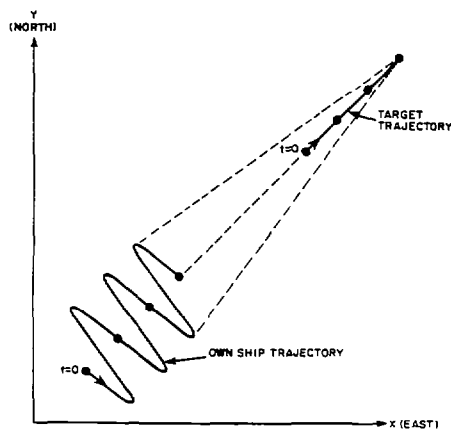


Fig. 2.

a rms noise level of 2° ; those labeled (B–a)–(B–d) and (C–a)–(C–d) correspond to rms noise levels of 4° and 6° , respectively. Estimation errors associated with four different TMA filters are plotted simultaneously on each graph labeled as follows:

A—the “idealized” MP filter, viz. an extended Kalman filter formulated in MP coordinates and linearized about the *true* state vector. This filter provides a measure of optimal performance since the error covariance matrix coincides with the Cramer–Rao lower bound [22],

B—the MP filter defined by equation set (8),

C—the pseudolinear filter described in [10], [12],

D—the Cartesian filter described in [10].

Finally, all results have been ensemble averaged over 25 Monte Carlo runs.

Some comments on filter initialization are perhaps in order here. Under actual operating conditions, it is extremely difficult, and indeed rare, to obtain reliable initial estimates of the state vector and associated covariance matrix. Consequently, existing procedures for accomplishing this task are necessarily ad hoc and somewhat arbitrary. While no claim of optimality is implied, the MP filter and its “idealized” counterpart were found to perform quite satisfactorily when initialized according to the scheme

$$\begin{aligned} y(0/0) &= [0, 0, \tilde{\beta}(0), 10^{-4}]' \\ P(0/0) &= \text{diag}[10^{-4}, 10^{-4}, 10^{-4}, 10^{-8}]. \end{aligned}$$

The pseudolinear and Cartesian filters were subsequently initialized in accordance with commonly prescribed procedures [10], viz.

$$\begin{aligned} x(0/0) &= [0, 0, 0, 0]' \\ P(0/k) &= \text{diag}[1, 1, 1, 1] \end{aligned}$$

for the pseudolinear filter, and

$$\begin{aligned} x(0/0) &= [0, 0, 10^4 \cdot \sin \tilde{\beta}(0), 10^4 \cdot \cos \tilde{\beta}(0)]' \\ P(0/0) &= \text{diag}[15^2, 15^2, 10^8, 10^8] \end{aligned}$$

for the Cartesian filter.

Examination of Figs. 3 and 4 reveals that the MP filter performs exactly as predicted by theory. Indeed, for both

²Note that data preprocessing reduces the rms random noise level by a factor of $1/\sqrt{20}$.

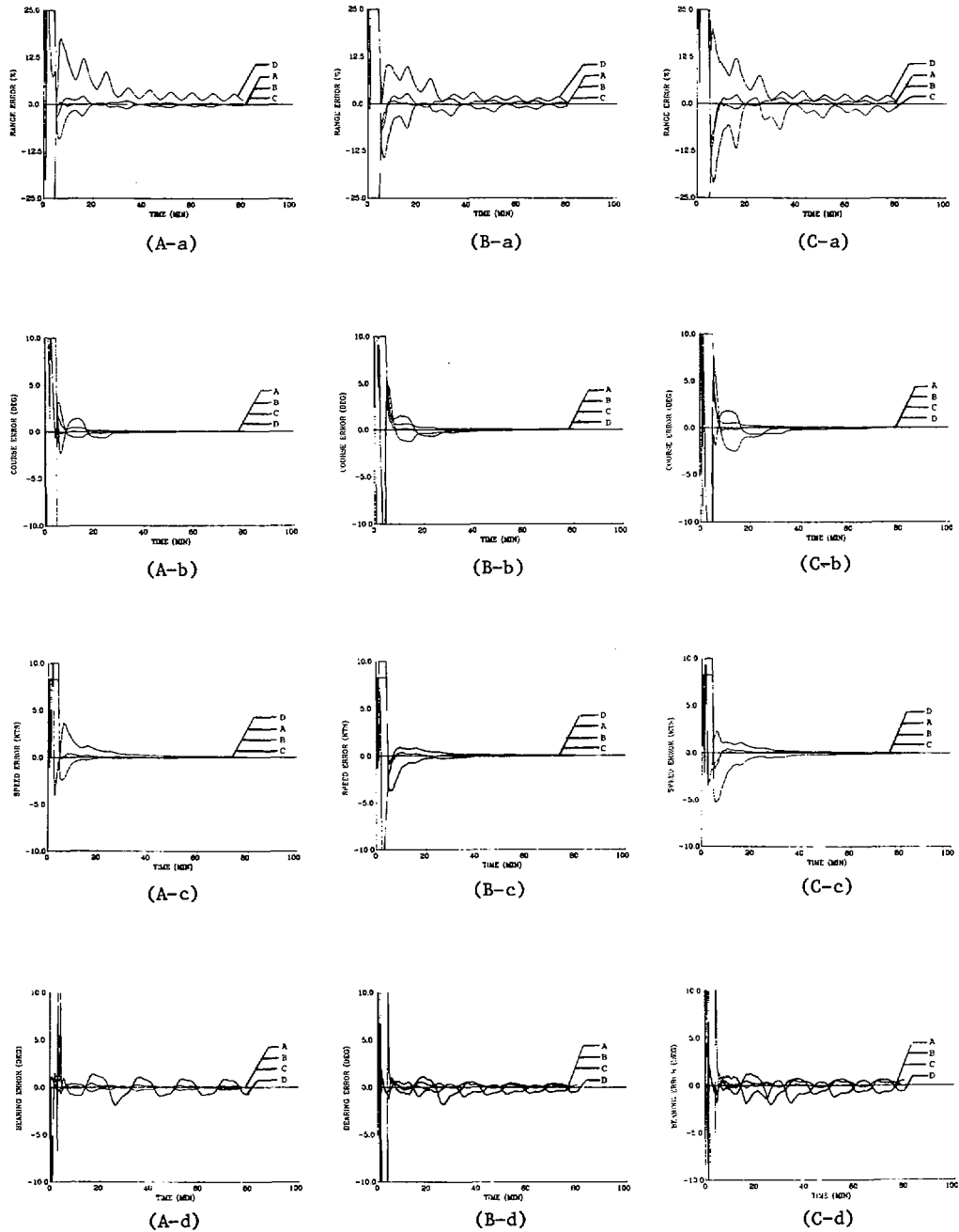


Fig. 3.

short and long range scenarios, the state estimates begin converging to their true values immediately after own-ship executes a maneuver. There is also no evidence of instability; subsequent own-ship maneuvers simply enhance convergence so that the final estimates become asymptotically unbiased. Overall efficacy of the MP filter can also be discerned by noting how rapidly it approaches "idealized" filter performance.

Behavior characteristics exhibited by the pseudolinear filter were also in agreement with previously documented theoretical and experimental findings [12]. Specifically, this filter generated biased range estimates which are readily apparent in the long range scenario data (see Fig. 4). However, the pertinent bias errors are known to be geo-

metrically dependent and become negligibly small for TMA scenarios characterized by high bearing rates and/or low measurement noise levels. This fact is substantiated by the simulation results presented in Fig. 3. Under such conditions, the pseudolinear filter possesses excellent tracking capabilities, comparable even to the MP filter.

In contrast with the other filters tested, Cartesian filter performance was generally poor, and erratic at best. For the short range scenario, this filter converged very slowly, despite relatively favorable tracking conditions. Indeed, five own-ship maneuvers were required to obtain steady-state range estimation errors of less than 5 percent. Abnormal behavior was also manifest in the long range scenario. Here, the Cartesian filter converges to the wrong

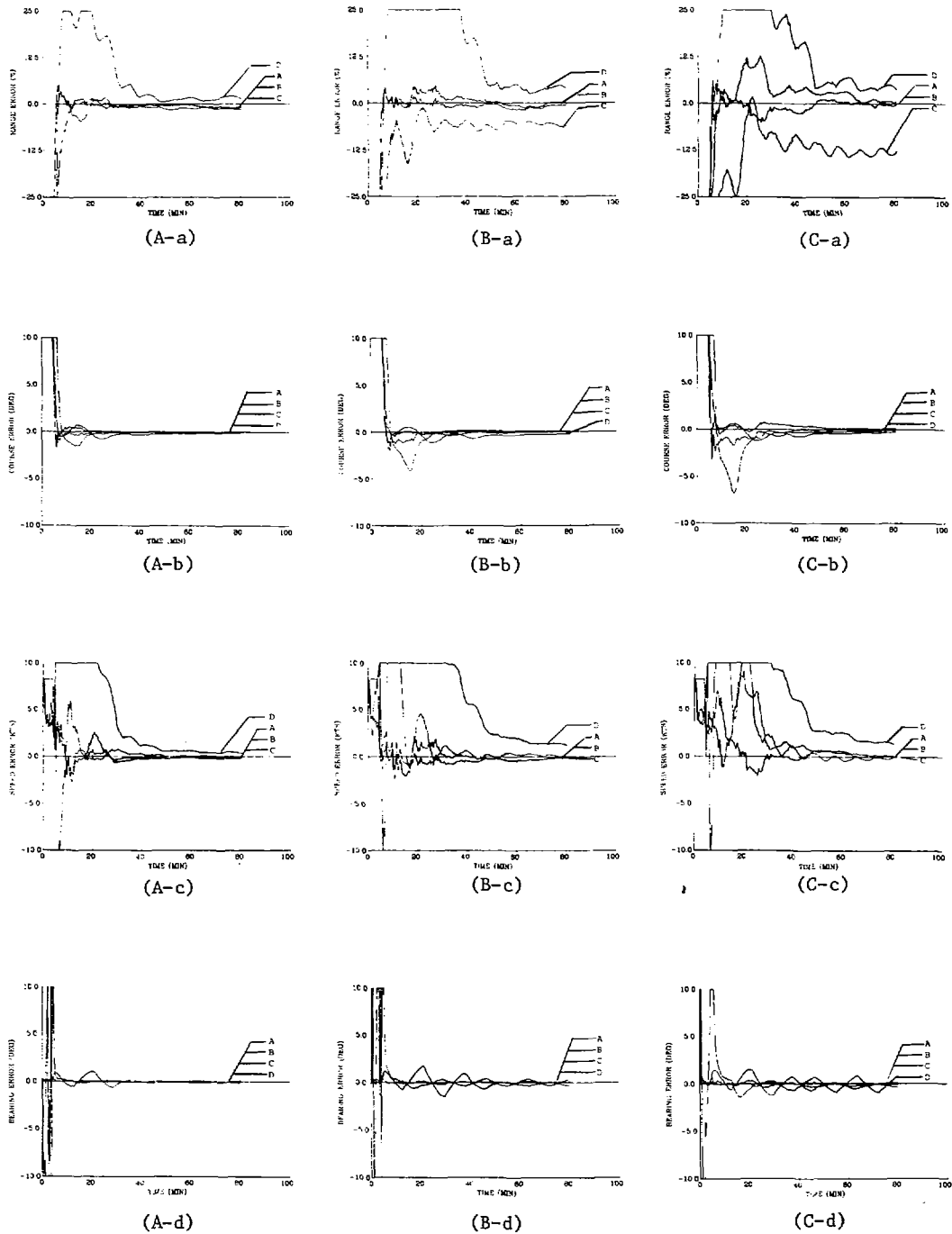


Fig. 4.

solution after erratic transient response. The unstable estimation characteristics of this filter are clearly evident.

V. SUMMARY AND CONCLUSIONS

We have attempted to elucidate the advantages of utilizing MP coordinates for bearings-only tracking via an extended Kalman filter. Exact state and measurement relations were rigorously derived for the general case of arbitrary vehicle motion. Subsequent analysis revealed that observable and unobservable components of the MP state

vector are automatically decoupled prior to the first own-ship maneuver. It was further shown that the covariance matrix structure accurately reflects this decoupling property. As a result, estimates generated by the MP filter are constrained to behave exactly as predicted by theory, even in the presence of measurement errors.

A realistic description of MP filter performance is provided by the data shown in Figs. 3 and 4. While only a small representative sampling of results is presented here, additional experiments have also been conducted elsewhere [24], [25] using both real and simulated measurements. A wide variety of geometries and noise levels were examined.

In all these tests, the MP filter behaved in a completely predictable manner. No evidence of instability was observed, and the state estimates were always asymptotically unbiased. Under actual operating conditions, it is important to recognize that filter performance will still be affected by such factors as own-ship tactics, environmental disturbances, measurement errors, etc.. However, estimation difficulties associated with bearings-only TMA unobservability have been eliminated.

The foregoing theoretical and experimental findings demonstrate that filter performance is intrinsically related to its development. Since other reference frames that permit a natural decoupling of observable and unobservable components also lead to more complicated equations of state and measurement, it is concluded that MP coordinates are ideally suited for bearings-only TMA.

APPENDIX A
 CARTESIAN COORDINATE FORMULATION OF THE
 BEARINGS-ONLY TMA PROBLEM

Derivation of the State Equations

Consider the geometry depicted in Fig. 1, with target and own-ship confined to the same horizontal plane.³ The Cartesian state vector for this two-dimensional configuration is defined by

$$\mathbf{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} v_x(t) \\ v_y(t) \\ r_x(t) \\ r_y(t) \end{bmatrix} \quad (\text{A1})$$

where

$$\begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} = \begin{bmatrix} v_{ix}(t) - v_{ox}(t) \\ v_{iy}(t) - v_{oy}(t) \end{bmatrix} \quad (\text{A2a})$$

and

$$\begin{bmatrix} r_x(t) \\ r_y(t) \end{bmatrix} = \begin{bmatrix} r_{ix}(t) - r_{ox}(t) \\ r_{iy}(t) - r_{oy}(t) \end{bmatrix} \quad (\text{A2b})$$

denote *relative* target velocity and position, respectively. A mathematical model of the system dynamics may now be specified in general form via the linear differential equations of motion

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} a_x(t) \\ a_y(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} \quad (\text{A3})$$

where

$$\begin{bmatrix} a_x(t) \\ a_y(t) \end{bmatrix} = \begin{bmatrix} a_{ix}(t) - a_{ox}(t) \\ a_{iy}(t) - a_{oy}(t) \end{bmatrix} \quad (\text{A4})$$

depicts *relative* acceleration. Integrating (A3) directly and expressing the result in matrix notation subsequently yields

$$\mathbf{x}(t) = A_x(t, t_o)\mathbf{x}(t_o) + \mathbf{w}(t, t_o) \quad (\text{A5})$$

where

$$A_x(t, t_o) \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (t - t_o) & 0 & 1 & 0 \\ 0 & (t - t_o) & 0 & 1 \end{bmatrix} \quad (\text{A6})$$

$$\mathbf{w}(t, t_o) \triangleq \begin{bmatrix} w_1(t, t_o) \\ w_2(t, t_o) \\ w_3(t, t_o) \\ w_4(t, t_o) \end{bmatrix} = \begin{bmatrix} \int_{t_o}^t a_x(\lambda) d\lambda \\ \int_{t_o}^t a_y(\lambda) d\lambda \\ \int_{t_o}^t (t - \lambda) a_x(\lambda) d\lambda \\ \int_{t_o}^t (t - \lambda) a_y(\lambda) d\lambda \end{bmatrix} \quad (\text{A7})$$

and t_o denotes any arbitrary fixed value of time.

Although (A5) remains valid for unconstrained vehicle motion, solution uniqueness requirements necessitate that the bearings-only TMA estimation problem be formulated under more restrictive assumptions [5], e.g., constant target velocity. In this case $a_{ix}(t) \equiv a_{iy}(t) \equiv 0$ and $\mathbf{w}(t, t_o)$ reduces to a deterministic input vector which depends only upon the characteristics of own-ship acceleration. Specifically,

$$\mathbf{w}(t, t_o) = -\mathbf{w}_o(t, t_o) \quad \text{iff} \quad \begin{bmatrix} a_{ix}(t) \\ a_{iy}(t) \end{bmatrix} \equiv \mathbf{0} \quad (\text{A8})$$

$$\mathbf{w}_o(t, t_o) \triangleq \begin{bmatrix} w_{o1}(t, t_o) \\ w_{o2}(t, t_o) \\ w_{o3}(t, t_o) \\ w_{o4}(t, t_o) \end{bmatrix} = \begin{bmatrix} \int_{t_o}^t a_{ox}(\lambda) d\lambda \\ \int_{t_o}^t a_{oy}(\lambda) d\lambda \\ \int_{t_o}^t (t - \lambda) a_{ox}(\lambda) d\lambda \\ \int_{t_o}^t (t - \lambda) a_{oy}(\lambda) d\lambda \end{bmatrix} \quad (\text{A9})$$

The Cartesian state equations then take the familiar form

$$\dot{\mathbf{x}}(t) = A_x(t, t_o)\mathbf{x}(t_o) - \mathbf{w}_o(t, t_o). \quad (\text{A10})$$

Derivation of the Measurement Equation

As the name implies, bearings-only TMA is distinguished by the fact that measured data consist entirely of passive sonar bearings extracted from a single sensor. Consequently, the measurement process is described by a

³The bearings-only TMA problem is typically formulated under this assumption.

time-varying scalar equation of the form (see Fig. 1)

$$\tilde{\beta}(t) = h_x[\mathbf{x}(t)] + \eta(t) \quad (\text{A11})$$

where

$$h_x[\mathbf{x}(t)] = \tan^{-1}[x_3(t)/x_4(t)] \quad (\text{A12})$$

and $\tilde{\beta}(t)$ represents the measured target bearing corrupted by additive measurement noise $\eta(t)$. If the pertinent acoustic sensor is properly aligned and calibrated, it may be realistically assumed that $\eta(t)$ is zero-mean Gaussian white noise with variance $\sigma^2(t)$, i.e.,

$$E[\eta(t)] = 0 \quad (\text{A13a})$$

$$E[\eta(t)\eta(t+\lambda)] = \begin{cases} \sigma^2(t) & \lambda = 0 \\ 0 & \lambda \neq 0 \end{cases} \quad (\text{A13b})$$

where $E[\cdot]$ denotes the statistical expectation operator.

APPENDIX B MODIFIED POLAR COORDINATE FORMULATION OF THE BEARINGS-ONLY TMA PROBLEM

Derivation of the State Equations

Again referring to the geometric configuration depicted in Fig. 1, the MP state vector is defined by

$$\mathbf{y}(t) \triangleq \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} \dot{\beta}(t) \\ \dot{r}(t)/r(t) \\ \beta(t) \\ 1/r(t) \end{bmatrix} \quad (\text{B1})$$

where

$$r(t) = \|\mathbf{r}(t)\| = \sqrt{r_x^2(t) + r_y^2(t)} \quad (\text{B2a})$$

and

$$\beta(t) = \tan^{-1}[r_x(t)/r_y(t)] \quad (\text{B2b})$$

represent the *relative* target range and bearing angle, respectively. In this coordinate system, the differential equations for arbitrary vehicle motion take the form

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \\ \dot{y}_4(t) \end{bmatrix} = \begin{bmatrix} -2y_1(t)y_2(t) + y_4(t)[a_x(t)\cos y_3(t) - a_y(t)\sin y_3(t)] \\ y_1^2(t) - y_2^2(t) + y_4(t)[a_x(t)\sin y_3(t) + a_y(t)\cos y_3(t)] \\ y_1(t) \\ -y_2(t)y_4(t) \end{bmatrix} \quad (\text{B3})$$

where $a_x(t)$ and $a_y(t)$ are the Cartesian components of *relative* acceleration defined in Appendix A by (A4).

A comparison of (A3) to (B3) reveals that the MP equations of motion are considerably more complicated than their Cartesian counterparts and not readily amenable

to direct integration. Despite this apparent difficulty, the *exact* general solutions of these differential equations can be obtained by straightforward algebraic manipulation. To this end, observe from Fig. 1 that

$$r_x(t) = r(t)\sin\beta(t) \quad (\text{B4a})$$

$$r_y(t) = r(t)\cos\beta(t). \quad (\text{B4b})$$

Differentiating these expressions with respect to time yields the familiar relations

$$v_x(t) = \dot{r}(t)\sin\beta(t) + r(t)\dot{\beta}(t)\cos\beta(t) \quad (\text{B5a})$$

$$v_y(t) = \dot{r}(t)\cos\beta(t) - r(t)\dot{\beta}(t)\sin\beta(t). \quad (\text{B5b})$$

A one-to-one transformation which maps the MP state vector into its Cartesian counterpart can now be deduced by combining (A1) with (B1), (B4), and (B5). The result is

$$\mathbf{x}(t) = \mathbf{f}_x[\mathbf{y}(t)] = \frac{1}{y_4(t)} \begin{bmatrix} y_2(t)\sin y_3(t) + y_1(t)\cos y_3(t) \\ y_2(t)\cos y_3(t) - y_1(t)\sin y_3(t) \\ \sin y_3(t) \\ \cos y_3(t) \end{bmatrix}. \quad (\text{B6})$$

Observe that (B6) is valid for all values of t such that $y_4(t) \neq 0$. Accordingly, letting $t = t_o$, this transformation may be applied to the right-hand side of (A5) to eliminate $\mathbf{x}(t_o)$. Performing the required algebraic operations eventually yields

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \frac{1}{y_4(t_o)} \begin{bmatrix} S_1(t, t_o)\cos y_3(t_o) + S_2(t, t_o)\sin y_3(t_o) \\ S_2(t, t_o)\cos y_3(t_o) - S_1(t, t_o)\sin y_3(t_o) \\ S_3(t, t_o)\cos y_3(t_o) + S_4(t, t_o)\sin y_3(t_o) \\ S_4(t, t_o)\cos y_3(t_o) - S_3(t, t_o)\sin y_3(t_o) \end{bmatrix} \quad (\text{B7})$$

where

$$S_1(t, t_o) = y_1(t_o) + y_4(t_o)[w_1(t, t_o)\cos y_3(t_o) - w_2(t, t_o)\sin y_3(t_o)] \quad (\text{B8a})$$

$$S_2(t, t_o) = y_2(t_o) + y_4(t_o)[w_1(t, t_o) \sin y_3(t_o) + w_2(t, t_o) \cos y_3(t_o)] \quad (\text{B8b})$$

$$S_3(t, t_o) = (t - t_o)y_1(t_o) + y_4(t_o)[w_3(t, t_o) \cos y_3(t_o) - w_4(t, t_o) \sin y_3(t_o)] \quad (\text{B8c})$$

$$S_4(t, t_o) = 1 + (t - t_o)y_2(t_o) + y_4(t_o)[w_3(t, t_o) \sin y_3(t_o) + w_4(t, t_o) \cos y_3(t_o)] \quad (\text{B8d})$$

and $[w_i(t, t_o); i = 1, 2, 3, 4]$ are defined in Appendix A by (A7).

The inverse transformation which maps Cartesian states into MP states may also be derived in a straightforward manner by differentiating (B2) with respect to time and then combining the resulting expressions with (A1), (A3), (B1), and (B2). These manipulations lead to

$$y(t) = f_y[x(t)] = \begin{bmatrix} [x_1(t)x_4(t) - x_2(t)x_3(t)] / [x_3^2(t) + x_4^2(t)] \\ [x_1(t)x_3(t) + x_2(t)x_4(t)] / [x_3^2(t) + x_4^2(t)] \\ \tan^{-1}[x_3(t)/x_4(t)] \\ 1/\sqrt{x_3^2(t) + x_4^2(t)} \end{bmatrix} \quad (\text{B9})$$

Substituting (B7) into (B9) finally yields, after some elementary algebra, the exact general solution to (B3), viz.

$$y(t) = f[y(t_o); t, t_o] = \begin{bmatrix} f_1[y(t_o); t, t_o] \\ f_2[y(t_o); t, t_o] \\ f_3[y(t_o); t, t_o] \\ f_4[y(t_o); t, t_o] \end{bmatrix} = \begin{bmatrix} [S_1(t, t_o)S_4(t, t_o) - S_2(t, t_o)S_3(t, t_o)] / [S_3^2(t, t_o) + S_4^2(t, t_o)] \\ [S_1(t, t_o)S_3(t, t_o) + S_2(t, t_o)S_4(t, t_o)] / [S_3^2(t, t_o) + S_4^2(t, t_o)] \\ y_3(t_o) + \tan^{-1}[S_3(t, t_o)/S_4(t, t_o)] \\ y_4(t_o) / \sqrt{S_3^2(t, t_o) + S_4^2(t, t_o)} \end{bmatrix} \quad (\text{B10})$$

Observe that (B10) is valid for *arbitrary* vehicle motion. Consequently, to obtain MP state equations for bearings-only TMA, components of the vector $w(t, t_o)$ appearing in (B8) must be replaced with those of $-w(t, t_o)$ before $[S_i(t, t_o); i = 1, 2, 3, 4]$ are substituted into the solution. This simple modification will incorporate the required motion constraint (i.e., constant target velocity) into the generalized expressions.

Derivation of the Measurement Equation

Since target bearing is a component of the MP state vector, the measurement equation for bearing-only TMA may be expressed in the simple linear form

$$\tilde{\beta}(t) = h_y[y(t)] + \eta(t) \quad (\text{B11})$$

where

$$h_y[y(t)] = [0, 0, 1, 0] y(t) \quad (\text{B12})$$

and $\tilde{\beta}(t), \eta(t)$ are defined in Appendix A.

Although (B11) follows directly from geometric considerations (see Fig. 1), it can also be rigorously derived by combining (A1), (A11), and (A12) with (B6). Utilization of this latter procedure leads to the important functional relation

$$h_x[f_x[y(t)]] = h_y[y(t)]. \quad (\text{B13})$$

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Vincent J. Aidala (M'75) was born in Providence, RI, on February 26, 1942. He received the B.S. degree in applied mathematics and the M.S. degree in electrical engineering from Brown University, Providence, RI, in 1964 and 1966, respectively.

Since 1967 he has been with the Naval Underwater Systems Center, Newport, RI. His current research interests are in the areas of adaptive estimation and systems identification.

Mr. Aidala is a member of Sigma Xi.



Sherry E. Hammel (M'82) was born in Bainbridge, MD, on May 23, 1957. She received the B.S. and M.S. degrees in applied mathematics from the University of Rhode Island, Kingston, in 1979 and 1981, respectively.

From 1979 to 1981 she was a Mathematics Instructor at the University of Rhode Island. Since July 1981, she has been with the Naval Underwater Systems Center, Newport, RI, where she is currently pursuing the Ph.D. degree in applied mathematics at the University of Rhode Island.

Her research interests include control systems analysis, estimation theory, and its applications.

Ms. Hammel is a member of Phi Beta Kappa and Phi Kappa Phi.

Estimation and Prediction for Maneuvering Target Trajectories

RUSSELL F. BERG, MEMBER, IEEE

Abstract—The Kalman filter is well suited for application to the problem of anti-aircraft gun fire control. In this paper we make use of the Kalman filter theory to develop an accurate, numerically efficient scheme for estimating and predicting the present and future position of maneuvering fixed-wing aircraft. This scheme was implemented in a radar tracker gun fire control system and tested against a variety of fixed-wing aircraft targets. Actual field test results are presented to demonstrate the high accuracy pointing which can be achieved by this approach.

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The author is with the General Dynamics Corporation, Pomona, CA 91766.

I. INTRODUCTION

THE classical problem of anti-aircraft gun fire control is the accurate prediction of the future position of a given target at the time of projectile intercept. Having obtained this information, the correct gun-pointing angles can be ascertained. Current approaches to the solution of this problem typically employ the use of modern estimation techniques (Kalman filtering) to estimate target velocity and acceleration on the basis of target position