



THEOREM OF THE DAY



The Bungers–Lehmer Theorem on Cyclotomic Coefficients *The cyclotomic polynomials, taken over all products of three distinct primes, contain arbitrarily large coefficients.*

The image on the right depicts the 105-th roots of unity, $\sqrt[105]{1}$, whose values are given by $e^{\tau ik/105} = \cos \frac{\tau k}{105} + i \sin \frac{\tau k}{105}$ ($k \geq 0, \tau = 2\pi$). Those marked with a star are *primitive roots*: when k is coprime to 105, i.e. $\text{GCD}(k, 105) = 1$, the successive powers of $e^{\tau ik/105}$ will generate every other 105th root. The n -th cyclotomic polynomial $\Phi_n(x)$ is the unique monic polynomial whose roots are precisely the primitive n -th roots of unity. Now $105 = 3 \times 5 \times 7$ and there are 48 integers less than 105 and coprime to 105 (the sequence 1, 2, 4, 8, 11, ..., 104, anticlockwise from far right on the horizontal axis). We denote this by $\varphi(105) = 48$, φ being the Euler totient function. Now we have $\Phi_{105}(x) = (x - e^{\tau i/105})(x - e^{2\tau i/105})(x - e^{4\tau i/105}) \dots (x - e^{104\tau i/105})$; remarkably the cyclotomic polynomials, defined in this way, always expand to give a polynomial of degree $\varphi(n)$ all of whose coefficients are integers! Indeed, for our example we get:

$$\begin{aligned} \Phi_{105}(x) = & 1 + x + x^2 - x^5 - x^6 - 2x^7 - x^8 - x^9 + x^{12} + x^{13} + x^{14} \\ & + x^{15} + x^{16} + x^{17} - x^{20} - x^{22} - x^{24} - x^{26} - x^{28} \\ & + x^{31} + x^{32} + x^{33} + x^{34} + x^{35} + x^{36} - x^{39} - x^{40} \\ & - 2x^{41} - x^{42} - x^{43} + x^{46} + x^{47} + x^{48}. \end{aligned}$$

An equivalent definition of $\Phi_n(x)$ sheds light on this phenomenon:

$$\Phi_n(x) = \prod_{d|n} (1 - x^{n/d})^{\mu(d)},$$

the product running over all divisors d of n , with μ being the Möbius function:

$$\mu(k) = \begin{cases} -1 & k \text{ is a product of an odd number of distinct primes} \\ 1 & k \text{ is 1 or a product of an even number of distinct primes} \\ 0 & k \text{ is not square-free} \end{cases}$$

(giving factors in the product of negative index, which must somehow conspire to cancel). The coefficient of -2 appearing in the expansion of $\Phi_{105}(x)$ is notable: it marks the first occurrence of a cyclotomic polynomial coefficient which lies outside the set $\{-1, 1, 0\}$.

1883 Adolph Migotti, a student of Lazarus Fuchs, demonstrates that no integer n with fewer than three distinct prime factors may have coefficients in $\Phi_n(x)$ outside the set $\{-1, 1, 0\}$. Moreover, three primes do not necessarily guarantee the converse, e.g. all coefficients of $\Phi_{3 \times 7 \times 11}(x)$ are ≤ 1 in absolute value.

1934 Rolf Bungers proves that, *provided there is an infinitude of twin primes*, taking products of just three distinct primes is sufficient to produce cyclotomic polynomials containing coefficients of arbitrarily large absolute value.

1931 Issai Schur, in a letter to Edmund Landau, shows that, for n a product of sufficiently many distinct primes, the coefficients of $\Phi_n(x)$ may become arbitrarily large in absolute value. (In 1987, his proof is shown by Jiro Susuki to yield *any* integer as a coefficient of some cyclotomic polynomial).

1936 Emma Lehmer gives a proof of Bungers' result which removes the reliance on there being an infinitude of twin primes. Paul Erdős subsequently observes that her coefficients grow as $\sqrt[3]{n}$, subsequently proving super-polynomial growth.

Moral: cyclotomic coefficients get big fast!

Web link: www.maths.lancs.ac.uk/~jameson/cyp.pdf.

Further reading: *Polynomials* by Victor Prasolov, Springer, 2nd printing, 2009, chapter 3.

