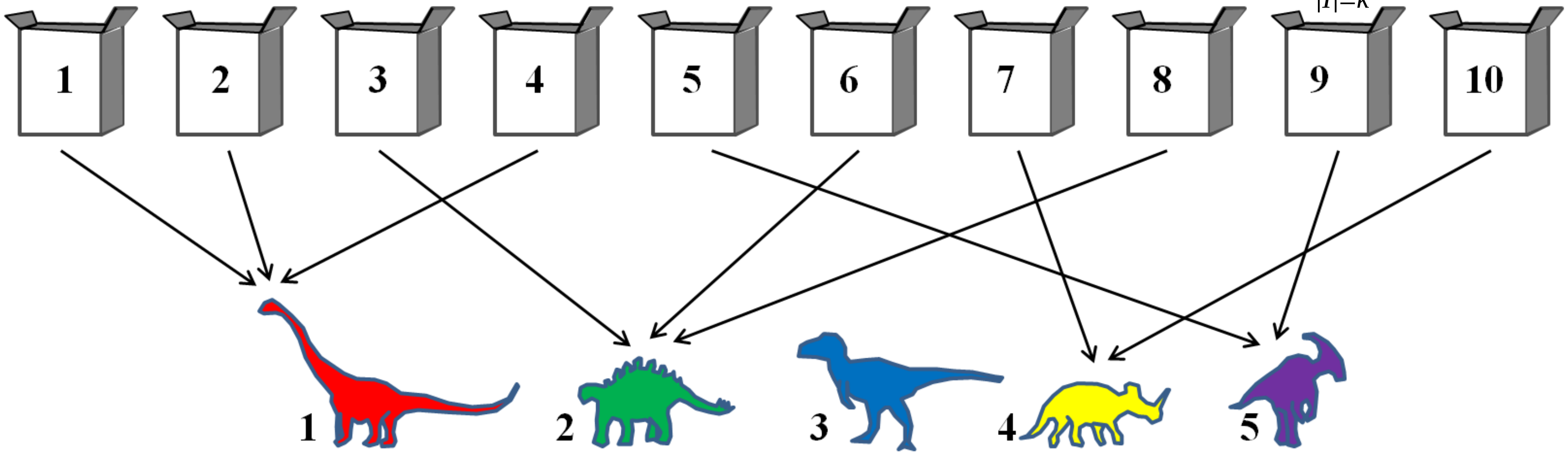
# THEOREM OF THE DAY

**The Inclusion-Exclusion Principle** *If $A_1, A_2, \ldots, A_n$ are subsets of a set then*

$$
\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| \;=\;& |A_1| + |A_2| + \ldots + |A_n| \\
& -(|A_1 \cap A_2| + |A_1 \cap A_3| + \ldots + |A_{n-1} \cap A_n|) \\
& +(|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\
\ldots\; & +(-1)^{n-1}|A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n| = \sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{I\subseteq[n]\\|I|=k}} |\cap_{i\in I}A_i| \quad ([n]=\{1,2,\ldots,n\}).
\end{aligned}
$$




We illustrate with the classic **Coupon Collector's Problem**: Suppose that we have $n$ different dinosaur models to collect from cereal packets each containing one dinosaur selected uniformly at random from $n$. What is the probability that we will have all $n$ dinosaurs after munching our way through $m$ packets of cereals?

Putting dinosaurs into packets is modelled as a function from packets to dinosaurs. We have all dinosaurs if the function is **surjective** (aka **onto**: every dinosaur gets at least one packet mapped to it). The illustration above shows a non-surjective sequence of breakfasts: we are still waiting for a packet to map to T. rex.

There are $n^m$ functions from $[m]$ to $[n]$, so our probability is (number of surjections)$/n^m$. Let $A_i$ be the subset of functions which *fail* to map to $i$. Then $|A_1 \cup A_2 \cup \ldots \cup A_n|$ counts the number of functions which fail to be surjections. This means that number of surjections $= n^m - |A_1 \cup A_2 \cup \ldots \cup A_n|$. Now the number of functions which do not map to $I \subset [n]$ is $|\cap_{i\in I}A_i|$. This is the same as the number of functions $[m] \longrightarrow [n-|I|]$, so $|\cap_{i\in I}A_i| = (n-|I|)^m$. We are ready to apply Inclusion-Exclusion:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{I\subseteq[n]\\|I|=k}}|\cap_{i\in I}A_i| = \sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{I\subseteq[n]\\|I|=k}}(n-|I|)^m = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m = \sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}(n-k)^m \text{ (since summand is zero for } k=n).$$

So **number of surjections** $= n^m - |A_1 \cup A_2 \cup \ldots \cup A_n| = n^m - \sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}(n-k)^m = \binom{n}{0}(n-0)^m + \sum_{k=1}^{n-1}(-1)^{k}\binom{n}{k}(n-k)^m = \boldsymbol{\sum_{k=0}^{n-1}(-1)^{k}\binom{n}{k}(n-k)^m}$.

... and that's just 10 packs for a BETTER THAN 50% chance of getting ALL FIVE, folks!

This principle, almost self-evident but immensely versatile, traces back to Abraham De Moivre in 1718.

**Weblink:** www.gresham.ac.uk/lectures-and-events/the-mathematics-that-counts.

**Further reading:** The Doctrine of Chances: Probabilistic Aspects of Gambling by Stewart N. Ethier, Springer-Verlag, 2010.