On an extended divisor product summatory function (accumulation of products of *all* divisors, positive and negative)

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1. Background

In mathematical number theory, a *divisor function* is an arithmetic function related to the divisors of integers.

The *sum of positive divisors function* $\sigma_x(n)$, for a real (or complex) number x, is defined as the sum of the xth powers of the positive divisors of n. It can be expressed in sigma notation as

$$\sigma_x(n) = \sum_{d|n} d^x$$

where $n > 0$, $d > 0$, and $d|n$ is shorthand for "d divides n" (which means that $n = m \cdot d$ for some $m \in \mathbb{N}$).

When x is 0, the function $\sigma_x(n)$ is referred to as *the number-of-divisors function* or simply *the divisor function*. The notations $d(n)$, $\nu(n)$, and $\tau(n)$, are often used instead of $\sigma_0(n)$, but I will use $\sigma_0(n)$ here:[1]

$$\sigma_0(n) = \sum_{d|n} 1$$

$\sigma_0(n)$ counts the number of (positive) divisors d of n. For $n = 1, 2, 3, \ldots$, the first few values of $\sigma_0(n)$ are 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, … (sequence A000005 in OEIS).

Lemma 1. If n is non-square positive integer, then $\sigma_0(n)$ is even. If n is a square number, then $\sigma_0(n)$ is odd.

Proof. The proof is well-known. Let n be a positive integer. According to the fundamental theorem of arithmetic, n has a unique prime factorization, so $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ for some primes p_i with exponents a_i $(i = 1, 2, \ldots, r)$. All positive divisors of n must then be of the form $p_1^{b_1} p_2^{b_2} \ldots p_r^{b_r}$, where $0 \le b_i \le a_i$, otherwise they would not be divisors of n. Since $0 \le b_i \le a_i$, we have $a_i + 1$ possible values for each exponent b_i. Thus, the total number of divisors is $\sigma_0(n) = (a_1 + 1)(a_2 + 1) \ldots (a_r + 1)$. For $\sigma_0(n)$ to be odd, all its factors $a_i + 1$ must be odd, so all a_i must be even (let's say that $a_i = 2m_i$, where each m_i is a positive integer), and thus, $n = p_1^{2m_1} p_2^{2m_2} \ldots p_r^{2m_r} = \left(p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}\right)^2$ is a perfect square. Consequently, the number of all positive divisors of an integer is always even, except when the integer is a perfect square. ■

When x is 1, the function $\sigma_x(n)$ above is called the *sigma function* or *sum-of-divisors function*, and the subscript is often omitted, so $\sigma(n)$ is equivalent to $\sigma_1(n)$:

$$\sigma(n) = \sum_{d|n} d$$

By analogy with this *sum-of-divisors function*, let

$$\pi(n) = \prod_{d|n} d$$

denote the *product* of the positive divisors d of n (including n itself).[2] For $n = 1, 2, 3, \ldots$, the first values are 1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, … (OEIS sequence A007955).

Lemma 2. The *divisor product* $\pi(n)$, as defined above, satisfies the identity $\pi(n) = \sqrt{n^{\sigma_0(n)}}$.

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[1] The denotation $\tau(n)$ comes from the German word *Teiler*, meaning *divisor*. It can be confused with the Ramanujan tau function. Thus, $\tau(n)$ might be a poorly choice. I use d as denotation for the divisors of a number, so $d(n)$ could also be confusing here.

[2] I have chosen the denotation $\pi(n)$ for the divisor product to highlight its relation to the *pi notation* (Π) used in its definition, the same way the denotation $\sigma(n)$ relates to the *sigma notation* (Σ). The denotation $\pi(n)$ is somewhat unfortunate, since it is also commonly used for the prime counting function, and it has absolutely no relation to the constant π. But it is commonly accepted (I borrowed it from Wolfram MathWorld). I have seen several other denotations for divisor products, such as $T(n)$ (Sándor, 2009), and $f(n)$ (Šalát & Tomanová, 2002).

Proof. The proof is well-known. Let n be a positive integer and the positive divisors of n be $d_1 < d_2 < \cdots < d_t$, where $t = \sigma_0(n)$. The trick is to pair these *sorted* divisors, in a way so that each pair consists of two divisors whose product is n. Obviously, $d_1 d_t = n$, since d_1 and d_t are the trivial divisors 1 and n, respectively. The next pair consists of d_2 and d_{t-1}, with the product $d_2 d_{t-1} = n$, and so on (the symmetry arises from the fact that the divisors are sorted, and, strictly speaking, from the commutativity of multiplication). If n is not a square integer, t is even, according to Lemma 1.
In this case, the last pair of divisors will be $d_{\frac{t}{2}}$ and $d_{1+\frac{t}{2}}$, with the product $d_{\frac{t}{2}} d_{1+\frac{t}{2}} = n$, and we get $\frac{t}{2}$ products, all equal to n. When multiplying these, we get the desired result, $d_1 d_2 \ldots d_t = n^{\frac{t}{2}}$. When n is a square integer, t is odd, so the last pair will consist of $d_{\frac{t-1}{2}}$ and $d_{2+\frac{t-1}{2}}$. Multiplication of these $\frac{t-1}{2}$ pairs (all equal to n), together with the middle divisor $d_{1+\frac{t-1}{2}}$, which is $\sqrt{n}$ (due to the symmetry), yields $d_1 d_2 \ldots d_t = n^{\frac{t-1}{2}} \cdot \sqrt{n} = n^{\frac{1}{2}+\frac{t-1}{2}} = n^{\frac{t}{2}}$. Since $t = \sigma_0(n)$, we have shown that $\pi(n) = \sqrt{n^{\sigma_0(n)}}$ for both even and odd integers. ■

2. The extended divisor product

In many situations, only the *positive* divisors of a positive integer, n, are of relevance (and sometimes only the *proper* divisors). There are several reasons for this, not least that the symmetric nature of positive versus negative divisors makes the *sum-of-divisors function* $\sigma_1(n)$ yield 0 for all n when its domain is extended to include negative divisors. When it comes to divisor *products*, it is more interesting to include negative divisors.

Let us first have a look at the number of all divisors (negative *and* positive). For $n = 1, 2, 3, \ldots$, the first values are 2, 4, 4, 6, 4, 8, 4, 8, 6, 8, 4, 12, 4, 8, … (OEIS sequence A062011). This sequence is generated simply by doubling the function $\sigma_0(n)$:

$$2\sigma_0(n) = 2\sum_{d|n} d^0 = 2\sum_{d|n} 1 = \sum_{d^*|n} 1$$

where $d^* \in \mathbb{Z}^*$, and $d^*|n$ is shorthand for "d^* divides n".[3]

Let us now look at the *product of all divisors*, both positive and negative, which I have denoted $\pi_*(n)$:[4]

$$\pi_*(n) = \prod_{d^*|n} d^*$$

This *extended divisor product* (i.e., the product of *all* divisors d^* of n) satisfies the identity

$$\pi_*(n) = (-n)^{\sigma_0(n)}$$

where $\sigma_0(n)$, as usual, is the number of *positive* divisors d of n. The proof is trivial. For $n = 1, 2, 3, \ldots$, the first values of $\pi_*(n)$ are $-1, 4, 9, -64, 25, 1296, 49, 4096, -729, 10000, 121, \ldots$ (OEIS sequence A217854).

Definition 1. Let n be a positive integer such that $\pi_*(n) < 0$. Then n is called a *divisorial-negative number*. Let n be a positive integer such that $\pi_*(n) > 0$. Then n is called a *divisorial-positive number*.[5]

We see that $\pi_*(n)$ is negative if and only if n is a square number (it follows directly from Lemma 1).
So, square numbers are divisorial-negative. All other natural numbers are divisorial-positive.

Definition 2. Let n be a positive integer such that $\pi_*(n) < \pi_*(k)$ for all positive $k < n$. Then n is called a *highly divisorial-negative number*.[6] Let n be a positive integer such that $\pi_*(n) > \pi_*(k)$ for all positive $k < n$. Then n is called a *highly divisorial-positive number*.

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[3] $\mathbb{Z}^*$ is the set $\{x \in \mathbb{Z} \mid x \neq 0\} = \mathbb{Z} \setminus \{0\}$. Since we look at both positive and negative divisors $d^* \in \mathbb{Z}^*$, I use a superscript asterisk to distinguish d^* from d.

[4] Since $d^* \in \mathbb{Z}^*$, using a superscript asterisk (i.e., π^* instead of π_*) would have been a better way to distinguish $\pi_*(n)$ from $\pi(n)$. But the denotation $\sigma^*(n)$ is standard for the *sum-of-unitary-divisors* function, so $\pi^*(n)$ would instead be an appropriate denotation for a *product-of-unitary-divisors* function. Thus, $\pi_*(n)$ will have to do here.

[5] The terms *divisorial-negative* and *divisorial-positive* are chosen because the (positive) divisor product $\pi(n)$ is sometimes, but not often, called *the divisorial of* n (see https://oeis.org/wiki/Divisorial). The *extended* divisor product $\pi_*(n)$ can be both positive and negative, so while *divisor product* is an established term for the product of *positive* divisors, I suggest that *divisorial* is used for $\pi_*(n)$. Then, *divisorial-negative* and *divisorial-positive* simply refers to numbers with negative and positive divisorials, respectively.

[6] I use the adjective *highly* in the same way it is used in several other divisor-related definitions (for instance that of *highly composite numbers*).

It is easy to see, that all highly divisorial-negative numbers are also divisorial-negative, and all highly divisorial-positive numbers are also divisorial-positive. The integer 1 is a square number, so it is divisorial-negative and included in the set of all highly divisorial-negative numbers. It is *not* regarded as highly divisorial-positive.

The first highly divisorial-negative numbers are 1, 4, 9, 16, 36, 100, 144, 324, 400, 576, 900, 1764, 2304, 3600, 7056, 8100, 14400, 28224, 32400, 44100, 57600, 108900, … (OEIS sequence A363657).

The first highly divisorial-positive numbers are 2, 3, 5, 6, 8, 10, 12, 18, 20, 24, 30, 40, 42, 48, 60, 72, 84, 90, 96, 108, 120, 168, 180, 240, 336, 360, 420, 480, 504, 540, 600, … (OEIS sequence A363658).

I suggest further studies of highly divisorial-negative and highly divisorial-positive numbers. How are they related to *abundant numbers*, *highly abundant numbers*, *perfect numbers*, *highly totient numbers*, *smooth numbers*, *rough numbers*, and the *primes* themselves (just to mention a few categories)? These are interesting questions. This paper does not go into detail about all this. What follows are some observations regarding the relation to *highly composite numbers*.

Definition 3. A *highly composite number* is a natural number which has more positive divisors than any lower natural number, i.e., a positive integer n such that $\sigma_0(n) > \sigma_0(k)$ for all positive $k < n$.
A *largely composite number* is a positive integer n such that $\sigma_0(n) \geq \sigma_0(k)$ for all positive $k < n$.

The first highly composite numbers are 1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, … (OEIS sequence A002182). The first largely composite numbers are 1, 2, 3, 4, 6, 8, 10, 12, 18, 20, 24, 30, 36, 48, 60, 72, 84, 90, 96, 108, … (OEIS sequence A067128). It is obvious that all highly composite numbers are also largely composite.

Lemma 3. All highly composite numbers (except 1, 4 and 36) are highly divisorial-positive.

Proof. Let n be a highly composite number (not 1, 4 or 36) with the unique prime factorization $p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ for some primes p_i with positive exponents a_i $(i = 1, 2, \dots, r)$, where $a_1 \geq a_2 \geq \dots \geq a_r$ and $p_1 < p_2 < \dots < p_r$. Using the argument from the proof of Lemma 1, the number of divisors is $\sigma_0(n) = (a_1 + 1)(a_2 + 1) \dots (a_r + 1)$. It has been proved that a_r must equal 1, except when $n \in \{1,4,36\}$ (Ramanujan 1915). Consequently, $\sigma_0(n)$ is even, and thus $\pi_*(n) = (-n)^{\sigma_0(n)} = n^{\sigma_0(n)}$ is clearly positive. According to the definition of highly composite numbers, $\sigma_0(k) < \sigma_0(n)$ for all positive $k < n$. For all k with odd $\sigma_0(k)$, we see that $\pi_*(k) = (-k)^{\sigma_0(k)} < n^{\sigma_0(n)} = \pi_*(n)$ because $(-k)^{\sigma_0(k)}$ is negative, and $n^{\sigma_0(n)}$ is positive. For all k with even $\sigma_0(k)$, we see that $(-k)^{\sigma_0(k)} = k^{\sigma_0(k)}$, and $k^{\sigma_0(k)} < n^{\sigma_0(n)}$, since $k < n$, and $\sigma_0(k) < \sigma_0(n)$.
The only *square* highly composite numbers are 1, 4 and 36 (Ramanujan 1915), and thus they are divisorial-negative.
All other highly composite numbers are highly divisorial-positive. ■

Theorem 1. All largely composite numbers (except 1, 4 and 36) are highly divisorial-positive.

Proof. Let n be a largely composite number. If n is highly composite, n is highly divisorial-positive (Lemma 3). Suppose n is not highly composite. If n is non-square, $\sigma_0(n)$ is even, and the line of reasoning used in the previous proof can be applied again (since for all a,b, if $a < b$ then $a \leq b$), which shows that n is highly divisorial-positive. It is obvious that n cannot be a square number, because there must exist a highly composite number $m < n$, such that $\sigma_0(m) = \sigma_0(n)$, otherwise n would be highly composite, and m cannot be square (unless it is 1, 4 or 36), because, as in the former proof, its largest prime factor has an exponent that equals 1 (Ramanujan 1915). ■

The highly divisorial-negative numbers are all square numbers, so only three of them (1, 4 and 36) are largely composite (a consequence of Theorem 1).

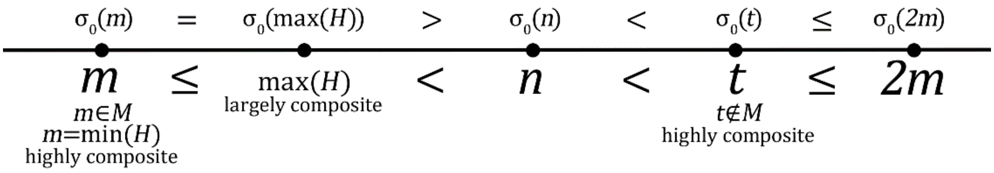
When it comes to the highly divisorial-positive numbers, all of them are largely composite, except 5, 40 and 42 (see Theorem 2 below).

Lemma 4. Let n be a highly composite number. Then there exist a highly composite number t such that $n < t \leq 2n$.

Proof. The proof is well-known (Alaoglu & Erdős 1944), but I include it for completeness. Let the unique prime factorization of n be $p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ for some primes p_i with positive exponents a_i $(i = 1, 2, \dots, r)$, where $a_1 \geq a_2 \geq \dots \geq a_r$ and $p_1 < p_2 < \dots < p_r$. Then, $\sigma_0(n) = (a_1 + 1)(a_2 + 1) \dots (a_r + 1)$, according to the argument from the proof of Lemma 1. Since n is highly composite, p_1 equals 2, and it is clearly sufficient to increase a_1 by 1 (i.e., multiplying n by 2) to get a number with more divisors than n. ■

Theorem 2. All highly divisorial-positive numbers (except 5, 40 and 42) are largely composite.

Proof outline. Let n be a highly divisorial-positive number (not 5, 40 or 42). Assume that n is *not* largely composite. This implies that $\sigma_0(k) > \sigma_0(n)$ for one or more positive $k < n$, since $\sigma_0(k) \leq \sigma_0(n)$ for all $k < n$ would contradict n not being largely composite. In other words, the set $M = \{k \in \mathbb{N}: k < n \wedge \sigma_0(k) > \sigma_0(n)\}$ cannot be empty. Now, from M we create a subset $H = \{h \in M: \sigma_0(h) \geq \sigma_0(i) \text{ for all } i \in M\}$. The set H contains all the elements in M with the highest number of divisors. H cannot be empty, since M is not empty (and obviously $\sigma_0(h) \geq \sigma_0(i)$, when $i = h$), and all elements in H have the same number of divisors, i.e., $\sigma_0(\min(H)) = \sigma_0(j) = \sigma_0(\max(H))$ for all $j \in H$.[7] Furthermore, all elements in H are largely composite, because if there were an element in H that were not largely composite, that would imply the existence of a number $w \notin M$, such that $w < n$ and $\sigma_0(w) > \sigma_0(n)$, which contradicts of definition of M. The smallest element in H, $\min(H)$, is highly composite, according to Definition 3. Let $m = \min(H)$ be the minimal element of H. Lemma 4 tells us that there exist a highly composite number, say t, such that $m < t \leq 2m$. Since $\min(H)$ is the only highly composite number in H, we see that $m < t < n$ is impossible, so $t \geq n$. Thus, assuming $2m < n$ leads to a contradiction. Then, assuming $2m = n$ contradicts n not being largely composite, since $2m$ would have to be largely composite with $t = 2m = n$. Thus, $n < 2m$.



Since m is highly composite, Lemma 3 tells us that m is divisorial-positive. Since both m and n are divisorial-positive (thus, non-square), they both have an even number of divisors, i.e., $\sigma_0(m)$ and $\sigma_0(n)$ are both even, so $\sigma_0(m) > \sigma_0(n) + 1$, and $\pi_*(m) = (-m)^{\sigma_0(m)} = m^{\sigma_0(m)}$, and $\pi_*(n) = (-n)^{\sigma_0(n)} = n^{\sigma_0(n)}$. Because $\sigma_0(n) + 1 < \sigma_0(m)$, clearly both $m^{\sigma_0(n)+1} < m^{\sigma_0(m)}$ and $n^{\sigma_0(n)} < n^{\sigma_0(m)-1}$. Now, since n is highly divisorial-positive, $\pi_*(k) < \pi_*(n)$ for all positive $k < n$. When $k = m$, we get the inequalities $m^{\sigma_0(n)+1} < m^{\sigma_0(m)} = \pi_*(m) < \pi_*(n) = n^{\sigma_0(n)} < n^{\sigma_0(m)-1}$.
To summarize, we have the following system of inequalities:

$$\begin{cases} m^{\sigma_0(n)+1} < n^{\sigma_0(m)-1} \\ \sigma_0(n) + 1 < \sigma_0(m) \\ 0 < m < n < 2m \end{cases}$$

Solving this system gives us two integer solutions: $n = 7$ (with $m = 6$) and $n = 692$ (with $m = 686$).[8] We have a contradiction, since neither 7 nor 692 is a highly divisorial-positive number, but n is highly divisorial-positive. So, the assumption that that n is *not* largely composite leads to a contradiction. Conclusively, n is largely composite. The number 4 is largely composite with $\sigma_0(4) = 3$, but since $\pi_*(4) = -64$ is negative, 4 is not included in the highly divisorial-positive numbers. Instead, 5 takes its place with $\sigma_0(5) = 2$. The same goes for 36 (replaced by both 40 and 42). (■)

Corollary 1. The largest highly divisorial-positive number that is not largely composite is 42.

Proof. It follows directly from Theorem 2. I couldn't resist writing this equivalent version as a separate statement.[9] ■

Theorem 1 together with Theorem 2 implies that the sequence of all largely composite numbers from the 14th term (the number 48) is identical to the sequence of all highly divisorial-positive numbers from the 14th term (also 48) and forth. So, the highly divisorial-positive numbers are essentially the same as the largely composite numbers. This is an important relationship between the number of divisors and the divisor product.

It has been proved that there exist infinitely many highly composite numbers (Ramanujan 1915). Thus, according to Lemma 3, there exist infinitely many divisorial-positive numbers.[10] Interestingly, it is a proven fact, that only a finite number of *highly abundant numbers* can be highly composite (Alaoglu & Erdős 1944).[11] So, only a finite number of highly abundant numbers can be highly divisorial-positive.

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[7] I use $min(H)$ and $max(H)$ a bit intuitively here, but since H is a totally ordered set, the *minimal element* of H and the *maximal element* of H are the same as the *greatest element* of H and the *least element* of H, respectively. So, no confusion should arise.

[8] I took a shortcut (hence the *Proof outline* part), and found the solutions with Wolfram Mathematica using the Solve function:
```
Solve[m^(DivisorSigma[0,n]+1)<n^(DivisorSigma[0,m]-1) && DivisorSigma[0,n]<DivisorSigma[0,m]-1 && 0<m<n<2*m, n]
```
I also ran various tests to ensure that Solve had not missed any larger solutions.

[9] All readers of Douglas Adams' novel *The Hitchhiker's Guide to the Galaxy* (1979) know why.

[10] This can also be shown by means of the unboundedness of $\sigma_0(n)$, which might be a more profound way to take.

[11] A *highly abundant number* is a positive integer n such that $\sigma_1(n) > \sigma_1(k)$ for all positive $k < n$.

Table 1 below shows the first 132 highly divisorial-positive numbers (n) and their values of $\sigma_0(n)$. An **x** in the HCN-column means that n is highly composite. The table contains all highly divisorial-positive numbers with less than 500 positive divisors. It was produced with the Python script given in Appendix A.

n	$\sigma_0(n)$	HCN	n	$\sigma_0(n)$	HCN	n	$\sigma_0(n)$	HCN	n	$\sigma_0(n)$	HCN
2	2	X	672	24		42840	96		831600	240	
3	2		720	30	X	43680	96		942480	240	
5	2		840	32	X	45360	100	X	982800	240	
6	4	X	1080	32		50400	108	X	997920	240	
8	4		1260	36	X	55440	120	X	1053360	240	
10	4		1440	36		65520	120		1081080	256	X
12	6	X	1680	40	X	75600	120		1330560	256	
18	6		2160	40		83160	128	X	1413720	256	
20	6		2520	48	X	98280	128		1441440	288	X
24	8	X	3360	48		110880	144	X	1663200	288	
30	8		3780	48		131040	144		1801800	288	
40	8		3960	48		138600	144		1884960	288	
42	8		4200	48		151200	144		1965600	288	
48	10	X	4320	48		163800	144		2106720	288	
60	12	X	4620	48		166320	160	X	2162160	320	X
72	12		4680	48		196560	160		2827440	320	
84	12		5040	60	X	221760	168	X	2882880	336	X
90	12		7560	64	X	262080	168		3326400	336	
96	12		9240	64		277200	180	X	3603600	360	X
108	12		10080	72	X	327600	180		4324320	384	X
120	16	X	12600	72		332640	192	X	5405400	384	
168	16		13860	72		360360	192		5654880	384	
180	18	X	15120	80	X	393120	192		5765760	384	
240	20	X	18480	80		415800	192		6126120	384	
336	20		20160	84	X	443520	192		6320160	384	
360	24	X	25200	90	X	471240	192		6486480	400	X
420	24		27720	96	X	480480	192		7207200	432	X
480	24		30240	96		491400	192		8648640	448	X
504	24		32760	96		498960	200	X	1081080	480	X
540	24		36960	96		554400	216	X	1225224	480	
600	24		37800	96		655200	216		1297296	480	
630	24		40320	96		665280	224	X	1369368	480	
660	24		41580	96		720720	240	X	1413720	480	

Table 1. The first 132 highly divisorial-positive numbers.

We see that $\sigma_0(n)$ is non-decreasing in Table 1, which correlate with the fact that highly divisorial-positive numbers are largely composite. Different values of $\sigma_0(n)$ are separated by horizontal lines in the table. For $n > 42$, the number of rows between two such horizontal lines is given by OEIS sequence A308530, starting at the 8th element. This sequence is defined as (a_k), where a_k is the number of largely composite numbers having the same number of divisors as the kth highly composite number. In the table, the visible part of the sequence is 1, 6, 2, 1, 2, 9, 1, 2, 2, 2, 8, 1, 2, 3, 2, 1, 1, 9, 1, 1, 3, 2, 5, 2, 2, 2, 8, 1, 2, 1, 6, 3, 6, 2, 2, 1, 6, 1, 1, 1, 5. Each of the 1's in this sequence correspond to a single **x**-marked row in the table surrounded by two horizontal lines, the first being $n = 48$, the next being $n = 180$, and so on. The numbers in those rows are highly composite and their number of divisors is smaller than the number of divisors for any following largely composite number. They can be found in OEIS sequence A308531.

Three highly composite numbers (the square numbers 1, 4, and 36) are missing in Table 1. It has been proved that these three numbers are the only highly composite numbers that are also square numbers (Ramanujan 1915).

Now, let's investigate the relationship between the functions σ_0, π, and π_* a bit further. It has been conjectured that the sequence of numbers whose product of (positive) divisors is larger than that of any smaller number (OEIS sequence A034287) is identical to the sequence of largely composite numbers (OEIS sequence A067128).[12] So, let us start with that.

Theorem 3. Let n be a positive integer. Then n is largely composite if and only if $\pi(n) > \pi(k)$ for all positive integers $k < n$.[13]

[12] I do not know whether this conjecture has been formally stated earlier, but according to the comments section on OEIS sequences A034287 and A067128, it *is* an open question. Furthermore, according to the same website, the identity has been verified for the first 105834 terms (all terms less than 10^{150}).

[13] The denotation $\pi(n)$ means the product of all *positive* divisors of n (see the Background section of this paper).

Proof outline. It is easy to see that $\pi(n)^2 = |\pi_*(n)| = n^{\sigma_0(n)}$ for all $n \in \mathbb{N}$. When n is divisorial-positive, we get $\pi(n) = \sqrt{\pi_*(n)}$, and when n is divisorial-negative, we get $\pi(n) = \sqrt{-\pi_*(n)}$.

Let us first show that if n is largely composite, then $\pi(n) > \pi(k)$ for all positive integers $k < n$. Assume that n is largely composite. The only divisorial-negative numbers (thus, square numbers) that are also largely composite are 1, 4 and 36 (Ramanujan 1915), and it is easily checked that $0 < k < n \Leftrightarrow \pi(n) > \pi(k)$ when n is 1, 4 or 36. For all $n \notin \{1, 4, 36\}$, Theorem 1 tells us that n is highly divisorial-positive. Thus, for all $k < n$, we have $\pi_*(n) > \pi_*(k)$. When k is *divisorial-positive*, this means that $\sqrt{\pi_*(n)} > \sqrt{\pi_*(k)} \Rightarrow \pi(n) > \pi(k)$. When it comes to all *divisorial-negative* $k < n$, we know that $\sigma_0(k)$ is odd. Since n is largely composite, we know that $\sigma_0(n) \geq \sigma_0(k)$. Since $\sigma_0(n)$ is even, $\sigma_0(n) \neq \sigma_0(k)$, so $\sigma_0(n) > \sigma_0(k)$, and since $n > k > 0$, we get $n^{\sigma_0(n)} > k^{\sigma_0(k)}$, so $\pi(n) > \pi(k)$ for all divisorial-negative $k < n$.

Let us show that if $\pi(n) > \pi(k)$ for all positive integers $k < n$, then n is largely composite. Assume $\pi(n) > \pi(k)$ for all positive integers $k < n$. Since $\pi(n) > \pi(k) > 0$, we get $\pi(n)^2 > \pi(k)^2 \Leftrightarrow |\pi_*(n)| > |\pi_*(k)|$. Assume that n is divisorial-positive. Then $\pi_*(n) = |\pi_*(n)| > |\pi_*(k)| \geq \pi_*(k)$, which means that n is highly divisorial-positive, and then Theorem 2 tells us that n is largely composite (if $\pi(n) > \pi(k)$ for all positive integers $k < n$, then n cannot be 5, 40 or 42, since $\pi(4) > \pi(5)$, $\pi(36) > \pi(40)$, and $\pi(36) > \pi(42)$, so those three exceptions are irrelevant here).

Now, assume that n is divisorial-negative, i.e., n is a square number. If n is *not* highly divisorial-negative, then it is obvious that $\pi(n) > \pi(k)$ for all positive integers $k < n$ cannot be true, since there exist a divisorial-negative number $m < n$ such that $\pi_*(m) \leq \pi_*(n) \Rightarrow \sqrt{-\pi_*(n)} \leq \sqrt{-\pi_*(m)} \Rightarrow \pi(n) \leq \pi(m)$. Thus, n must be highly divisorial-negative, so $\pi_*(n) < \pi_*(k)$ for all $k < n$. We need to prove that $\pi(n) > \pi(k)$ for all positive integers $k < n$ cannot be true when n is highly divisorial-negative. Proving this last part is equivalent to proving Conjecture 1 below, and I leave that task to the reader, while referring to the reasoning presented in the proof of Theorem 2, and to Lemma 4. (■)

The concepts of *divisorial-negative* and *divisorial-positive* numbers might seem superfluous, since they are nothing more than *square* numbers and *non-square* numbers, respectively, and we don't need more synonyms for those. But the *highly* divisorial-negative numbers (and perhaps the *highly* divisorial-positive numbers) might play an important role in future studies or come in handy as a tool in certain situations. Due to time constraints, I am unable to fully explore these possibilities. Consequently, I conclude this section with a conjecture.

Conjecture 1. If n is a highly divisorial-negative number (not 1, 4 or 36), then $|\pi_*(n)| \leq |\pi_*(m)|$ for some $m < n$.

The following lemma, together with the discussion above, might be useful in the proof, so it's included here (even though it is rather trivial).

Lemma 5. Let n be a positive integer. Then $\sigma_0(n) \leq 2\sqrt{n}$.

Proof. The positive divisors of n occur in pairs $\{d_i, \frac{n}{d_i}\}$, where $d_i | n$, and $1 \leq i \leq \frac{\sigma_0(n)}{2}$. The largest possible value of i would generate the pair $\{\sqrt{n}, \frac{n}{\sqrt{n}}\}$. Therefore, $\sigma_0(n) \leq 2\sqrt{n}$. ■

3. An extended divisor product summatory function

Let us now have a look at a new summatory function. I will denote it with the capital Greek letter P (rho):

$$P(n) = \sum_{k=1}^{n} \pi_*(k)$$

which can, of course, also be written as

$$P(n) = \sum_{k=1}^{n} (-k)^{\sigma_0(k)}$$

The first values of this sequence is $-1, 3, 12, -52, -27, 1269, 1318, 5414, 4685, 14685, 14806, 3000790, \ldots$ (I got this approved as OEIS sequence A224914 some years ago). $P(1)$, $P(4)$, and $P(5)$ are negative, and from there it continues with positive values until $P(36) = -100792120241072$. After $P(36)$, the sequence is negative until we reach $P(48) = 64840521809262990$, and after this, $P(n)$ is, most likely, always positive.

Conjecture 2. Define $P(n)$ as above. Then $P(n) > 0$ for all $n > 47$.

When $k > 1$ is a *square number*, $P(k) = P(k-1) - k^{\sigma_0(k)}$. When k is *non-square*, $P(k) = P(k-1) + k^{\sigma_0(k)}$. When $k > 1$ is a *prime*, we have $P(k) = P(k-1) + k^2$. These are trivial identities.

Divisorial-positive numbers make the summatory function P *grow*, and highly divisorial-positive numbers make it *grow fast*. This strongly indicates that Conjecture 2 is true, but it does not prove it, because on the other hand, the primes make P grow much slower, and the divisorial-negative numbers make it *shrink*. The highly divisorial-negative numbers make it *shrink quite a lot*, such as $\pi_*(14400) < -10^{260}$, but still, the negative values of $\pi_*(n)$ will (probably) never outweigh the positive, because as n increases, the distance between the negative terms in the sum $P(n)$ also increases. This gives rise to some interesting questions. Does ℕ have prime-dense regions with no or few largely composite numbers around square numbers with a very large number of divisors? And would this be enough to make $P(n)$ negative for some n in such regions? I am convinced that this is not the case, but my knowledge about the distribution of both primes and largely composite numbers are, so far, too limited to tackle the problem analytically.

Conjecture 3. $P(n) > |\pi_*(n)|$ for all $n > 48$.

Conjecture 4. There exist a positive integer s, such that $P(n) \geq |\pi_*(n)|^2$ for all $n > s$.

If Conjecture 3 is true, then obviously Conjecture 2 is true. I have checked it for $n \leq 10^9$ with the Python script given in Appendix A. This part will be updated in the next edition of this paper (see the next section).

I find Conjecture 4 interesting. The first terms in the sequence of positive integers n, such that $P(n) < |\pi_*(n)|^2$ are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26, … (not yet submitted to OEIS).
The sequence has 827 terms for $n < 10000$, and it has 6710 terms for $n < 1000000$.

The first terms in the sequence of positive integers n, such that $P(n) \geq |\pi_*(n)|^2$ are 11, 13, 17, 19, 23, 25, 29, 31, 49, 51, 53, 55, 57, 58, 59, 61, 62, 65, 67, 69, … (not yet submitted to OEIS).

Calculating the values in the $P(n)$ sequence requires a little bit of programming. As an example, $P(20000)$ is
2168489866725755264761185623571575807224769134715706592111461785349784984124287026560761083449646871799070410688695275479836591631035800056532030798251475385579948765593267522654867541838735544979889617361898232220700243571313828111062277359888411262590613784776667067027154835485598377017612539310282549037387739134583575161922565503766788 78.

I refer to Appendix A for the Python code used here. Appendix B contains a table of the first hundred values of $P(n)$ with corresponding values of n, $\sigma_0(n)$, and $\pi_*(n)$.

4. About this paper

This text is a result of an exploration of various number-theoretical concepts purely for recreational purposes. The content has not undergone peer review (nor has it been proofread), as I am currently not affiliated with any mathematical community. Consequently, it is reasonable to assume that I may have made a few mistakes. I hope that these mistakes are of minimal significance and do not overshadow the ideas presented here.
I encourage my readers to reach out to me if they come across any irregularities in the text, and particularly if they can construct the missing proofs. My e-mail address is information@simonjensen.com.

This version was published the 17th of June (with minor typos corrected 31st of August) 2023 at my website www.simonjensen.com. I intend to complete the proofs for Theorem 2, Theorem 3, and the aforementioned conjectures, unless someone else manages to do so before me. The latest version of this paper will always be available at https://www.simonjensen.com/pdf/On_an_extended_divisor_product_summatory_function.pdf.

5. References

Ramanujan, S., Highly composite numbers, Proceedings of the London Mathematical Society, 2, XIV (1915), pp. 347–409.

Sándor, J., The product of divisors minimum and maximum functions, Scientia Magna, vol. 5, no. 3 (2009), pp. 13+. Gale Academic OneFile.

Šalát, T., Tomanová, J., On the product of divisors of a positive integer, Mathematica Slovaca, Vol. 52, Issue 3 (2002), pp. 271–287, ISSN: 0139-9918.

Alaoglu, L., Erdős, P., On Highly Composite and Similar Numbers, Transactions of the American Mathematical Society, Vol. 56, No. 3 (1944), pp. 448–469.

Appendix A

Python code used to generate the tables and the various sequences mentioned in the paper:

```
from math import isqrt
from math import sqrt

def create_positive_divisors(n):
    global n_divisors
    n_divisors = set()
    for i in range(1, isqrt(n)+1):
        if n % i == 0:
            n_divisors.add(i)
            n_divisors.add(n//i)
    n_divisors = sorted(n_divisors)

def get_number_of_positive_divisors():
    return len(n_divisors)

def get_extended_divisor_product(n):
    return (-n)**(get_number_of_positive_divisors())

def is_highly_divisorial_negative():
    global previous_minimum_extended_divisor_product
    if n_extended_divisor_product < previous_minimum_extended_divisor_product:
        previous_minimum_extended_divisor_product = n_extended_divisor_product
        return True
    return False

def is_highly_divisorial_positive():
    global previous_maximum_extended_divisor_product
    if n_extended_divisor_product > previous_maximum_extended_divisor_product:
        previous_maximum_extended_divisor_product = n_extended_divisor_product
        return True
    return False

# Input (maximum value of n in loop)
limit=int(input("Limit: "))

# Initial values
n_divisors = set()
n_accumulated_divisor_product = 0 # P(n)
previous_HCN_number_of_positive_divisors = 0 # σ0(m) for largest highly composite number m ≤ n in loop
previous_minimum_extended_divisor_product = 0
previous_maximum_extended_divisor_product = 0

# Loop generating values for each n and accumulated values
for n in range(1, limit + 1):
    create_positive_divisors(n)
    n_extended_divisor_product = get_extended_divisor_product(n)
    n_number_of_positive_divisors = get_number_of_positive_divisors()
    n_prime = True if n_number_of_positive_divisors == 2 else False
    n_square = isqrt(n) == sqrt(n)
    n_HCN = True if n_number_of_positive_divisors > previous_HCN_number_of_positive_divisors else False
    n_HDP = is_highly_divisorial_positive()
    n_HDN = is_highly_divisorial_negative()
    if n_number_of_positive_divisors > previous_HCN_number_of_positive_divisors:
        previous_HCN_number_of_positive_divisors = n_number_of_positive_divisors
    n_accumulated_divisor_product += n_extended_divisor_product

    # USEFUL VARIABLES AVAILABLE HERE (all these can be printed with the function print):
    # limit                         = the maximum value of n (the loop runs from 1 to limit)
    # n                             = 1, 2, ..., limit
    # n_divisors                    = the set of all positive divisors of n
    # n_number_of_positive_divisors = σ0(n)
    # n_extended_divisor_product    = π*(n)
    # n_accumulated_divisor_product = P(n)
    # n_HCN                         = True when n is highly composite, else False
    # n_HDP                         = True when n is highly divisorial-positive, else False
    # n_HDN                         = True when n is highly divisorial-negative, else False
    # n_square                      = True when n is a square number, else False
    # n_prime                       = True when n a prime, else False

    # Output (modify according to preferences)
    if n_HCN:
        print(f"n = {n}, σ0(n) = {n_number_of_positive_divisors}, n is highly composite")
    else:
        print(f"n = {n}, σ0(n) = {n_number_of_positive_divisors}, n is not highly composite ")
    if n_accumulated_divisor_product < abs(n_extended_divisor_product):
        print(f"n = {n}, P={n_accumulated_divisor_product}, P(n) < |π*(n)|")

print(f"\nDone.")
```

Appendix B

Below are the first 100 values of n, $\sigma_0(n)$, $\pi_*(n)$, and $P(n)$. Negative values are **green**.

n	$\sigma_0(n)$	$\pi_*(n)$	$P(n)$
1	1	-1	-1
2	2	4	3
3	2	9	12
4	3	-64	-52
5	2	25	-27
6	4	1296	1269
7	2	49	1318
8	4	4096	5414
9	3	-729	4685
10	4	10000	14685
11	2	121	14806
12	6	2985984	3000790
13	2	169	3000959
14	4	38416	3039375
15	4	50625	3090000
16	5	-1048576	2041424
17	2	289	2041713
18	6	34012224	36053937
19	2	361	36054298
20	6	64000000	100054298
21	4	194481	100248779
22	4	234256	100483035
23	2	529	100483564
24	8	110075314176	110175797740
25	3	-15625	110175782115
26	4	456976	110176239091
27	4	531441	110176770532
28	6	481890304	110658660836
29	2	841	110658661677
30	8	656100000000	766758661677
31	2	961	766758662638
32	6	1073741824	767832404462
33	4	1185921	767833590383
34	4	1336336	767834926719
35	4	1500625	767836427344
36	9	-101559956668416	-100792120241072
37	2	1369	-100792120239703
38	4	2085136	-100792118154567
39	4	2313441	-100792115841126
40	8	6553600000000	-94238515841126
41	2	1681	-94238515839445
42	8	9682651996416	-84555863843029
43	2	1849	-84555863841180
44	6	7256313856	-84548607527324
45	6	8303765625	-84540303761699
46	4	4477456	-84540299284243
47	2	2209	-84540299282034
48	10	64925062108545024	64840521809262990
49	3	-117649	64840521809145341
50	6	15625000000	64840537434145341
51	4	6765201	64840537440910542
52	6	19770609664	64840557211520206
53	2	2809	64840557211523015
54	8	72301961339136	64912859172862151
55	4	9150625	64912859182012776
56	8	96717311574016	65009576493586792
57	4	10556001	65009576504142793
58	4	11316496	65009576515459289
59	2	3481	65009576515462770
60	12	2176782336000000000000	2176847345576515462770
61	2	3721	2176847345576515466491
62	4	14776336	2176847345576530242827
63	6	62523502209	2176847345639053745036
64	7	-4398046511104	2176847341241007233932
65	4	17850625	2176847341241025084557
66	8	360040606269696	2176847701281631354253
67	2	4489	2176847701281631358742
68	6	98867482624	2176847701380498841366
69	4	22667121	2176847701380521508487
70	8	576480100000000	2176848277860621508487
71	2	5041	2176848277860621513528
72	12	19408409961765342806016	21585258239625964319544

73	2	5329	2158525823962596432487З
74	4	29986576	21585258239625994311449
75	6	177978515625	21585258239803972827074
76	6	192699928576	21585258239996672755650
77	4	35153041	21585258239996707908691
78	8	1370114370683136	21585259610111078591827
79	2	6241	21585259610111078598068
80	10	10737418240000000000	21595997028351078598068
81	5	-3486784401	21595997028347591813667
82	4	45212176	21595997028347637025843
83	2	6889	21595997028347637032732
84	12	123410307017276135571456	145006304045623772604188
85	4	52200625	145006304045623824804813
86	4	54700816	145006304045623879505629
87	4	57289761	145006304045623936795390
88	8	3596345248055296	145006307641969184850686
89	2	7921	145006307641969184858607
90	12	282429536481000000000000	427435844122969184858607
91	4	68574961	427435844122969253433568
92	6	606355001344	427435844123575608434912
93	4	74805201	427435844123575683240113
94	4	78074896	427435844123575761315009
95	4	81450625	427435844123575842765634
96	12	612709757329767363772416	1040145601453343206538050
97	2	9409	1040145601453343206547459
98	6	885842380864	1040145601454229048928323
99	6	941480149401	1040145601455170529077724
100	9	-1000000000000000000	1040144601455170529077724