School of Physics and Astronomy

Fourier Analysis

Prof. John A. Peacock

jap@roe.ac.uk

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1 Introduction

Describing continuous signals as a superposition of waves is one of the most useful concepts in physics, and features in many branches - acoustics, optics, quantum mechanics for example. The most common and useful technique is the *Fourier* technique, which were invented by Joseph Fourier in the early 19th century. In many cases of relevance in physics, the waves evolve independently, and this allows solutions to be found for many ordinary differential equations (ODEs) and partial differential equations (PDEs). We will also explore some other techniques for solving common equations in physics, such as Green's functions, and separation of variables, and investigate some aspects of digital sampling of signals.

As a reminder of notation, a single wave mode might have the form

$$\psi(x) = a\cos(kx + \phi). \tag{1.1}$$

Here, a is the wave *amplitude*; ϕ is the *phase*; and k is the *wavenumber*, where the wavelength is $\lambda = 2\pi/k$. Equally, we might have a function that varies in time: then we would deal with $\cos\omega t$, where ω is *angular frequency* and the *period* is $T = 2\pi/\omega$. In what follows, we will tend to assume that the waves are functions of x, but this is an arbitrary choice: the mathematics will apply equally well to functions of time.

2 Fourier Series

Learning outcomes

In this section we will learn how Fourier series (real and complex) can be used to represent functions and sum series. We will also see what happens when we use truncated Fourier series as an approximation to the original function, including the Gibbs phenomenon for discontinuous functions.

2.1 Overview

Fourier series are a way of expressing a function as a sum, or *linear superposition*, of waves of different frequencies:

$$f(x) = \sum_i a_i \cos(k_i x + \phi_i). \tag{2.1}$$

This becomes more well specified if we consider the special case where the function is *periodic* with a period $2L$. This requirement means that we can only consider waves where a whole number of wavelengths fit into $2L$: $2L = n\lambda \Rightarrow k = n\pi/L$. Unfortunately, this means we will spend a lot of time writing $n\pi/L$, making the formulae look more complicated than they really are.

A further simplification is to realize that the phase of the waves need not be dealt with explicitly. This is because of the trigonometric identity (which you should know)

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B). \tag{2.2}$$

Thus a single wave mode of given phase can be considered to be the combination of a sin and a cos mode, both of zero phase.

- Fourier Series deal with functions that are *periodic* over a *finite interval.* e.g. $-1 < x < 1$. The function is assumed to repeat outside this interval.
- Fourier Series are useful if (a) the function really is periodic, or (b) we only care about the function in a finite range (e.g. $-\pi < x < \pi$). We'll discuss this more in Sec. 2.7.
- If the range is infinite, we can use a Fourier Transform (see section 3).
- We can decompose *any* function we like in this way (well, any that satisfy some very mild mathematical restrictions).
- The sines and cosines are said to form a *complete set.* This means the same as the last bullet point. We won't prove this.
- One can decompose functions in other complete sets of functions (e.g. powers: the Taylor series is an example of this), but the Fourier Series is perhaps the most common and useful. Most of this course will be concerned with Fourier Series and Fourier Transforms (see later).

2.2 Periodic Functions

Periodic functions satisfy

$$f(t+T) = f(t) \tag{2.3}$$

for all t. T is then the period. Similarly, a function can be periodic in space: $f(x+X) = f(x)$.

Exercise: Show that if $f(t)$ and $g(t)$ are periodic with period T, then so are $af(t) + bg(t)$ and $cf(t)g(t)$, where a, b, c are constants.

Note that a function which is periodic with a period X is also periodic with period $2X$, and indeed periodic with period nX, for any integer n. The *smallest* period is called the *fundamental period.*

Note also that the function does not have to be continuous.

Examples:

- $\sin x$ and $\cos x$ both have a fundamental period of 2π.
- $\sin\left(\frac{n\pi x}{L}\right)$ has a period of $2L/n$, where n is an integer.
- So $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$ all have periods $2L$ as well as $2L/n$ (for all integer n).
- Note that the boundary of the period can be anything convenient: $0 \le x \le 2L$ for example, or $a \le x \le a + 2L$ for any a. Since it is periodic, it doesn't matter.

2.3 The Fourier expansion

Within the interval $-L \le x \le L$, we can write a general (real-valued) function as a *linear superposition* of these Fourier modes:

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \end{aligned} \tag{2.4}$$

where a_n and b_n are (real-valued) *expansion coefficients*, also known as *Fourier components*. The reason for the unexpected factor 1/2 multiplying a_0 will be explained below.

2.3.1 What about $n < 0$?

We don't need to include negative n because the Fourier modes have a well defined symmetry (even or odd) under $n \to -n$: let's imagine we included negative n and that the expansion coefficients are A_n and B_n:

$$f(x) = \frac{A_0}{2} + \sum_{\pm n} \left[A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{2.5}$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi x}{L}\right) + A_{-n} \cos\left(\frac{-n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) + B_{-n} \sin\left(\frac{-n\pi x}{L}\right) \right] . \tag{2.6}$$

Now, $\cos\left(-\frac{n\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(-\frac{n\pi x}{L}\right) = -\sin\left(\frac{n\pi x}{L}\right)$, so we can rewrite this as

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[(A_n + A_{-n}) \cos\left(\frac{n\pi x}{L}\right) + (B_n - B_{-n}) \sin\left(\frac{n\pi x}{L}\right) \right] . \tag{2.7}$$

At this point A_n and A_{-n} are unknown constants. As they only appear summed together (rather than separately) we may as well just rename them as a single, unknown constant $a_0 = A_0$, $a_n \equiv A_n + A_{-n}$, $(n \geq 1)$. We do the same for $b_n \equiv B_n - B_{-n}$. So, overall it is sufficient to consider just positive values of n in the sum.

2.4 Orthogonality

Having written a function as a sum of Fourier modes, we would like to be able to calculate the components. This is made easy because the Fourier mode functions are *orthogonal* i.e. for non-zero integers m and n,

$$\int_{-L}^{L} dx \, \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} \tag{2.8}$$

$$\int_{-L}^{L} dx \, \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} \tag{2.9}$$

$$\int_{-L}^{L} dx \, \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = 0 \, . \tag{2.10}$$

You can do the integrals using the trigonometry identities in Eqn. (2.14) below. Note that one of the Fourier modes is a constant (the $a_0/2$ term), so we will also need

$$\int_{-L}^{L} dx \, \cos\left(\frac{n\pi x}{L}\right) = \begin{cases} 0 & n \neq 0 \\ 2L & n = 0 \end{cases} \tag{2.11}$$

$$\int_{-L}^{L} dx \, \sin\left(\frac{n\pi x}{L}\right) = 0 \tag{2.12}$$

Note the appearance of $2L$ here, rather than L in the $n > 0$ cases above.

FOURIER ANALYSIS: LECTURE 2

The orthogonality is the fact that we get zero in each case if $m \neq n$. We refer to the collected Fourier modes as an *orthogonal set* of functions.

Let us show one of these results. If $m \neq n$,

$$\begin{aligned}
\int_{-L}^{L} dx\ \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) &= \frac{1}{2}\int_{-L}^{L} dx\ \left[\cos\left\{\frac{(m+n)\pi x}{L}\right\} + \cos\left\{\frac{(m-n)\pi x}{L}\right\}\right] \\
&= \frac{1}{2}\left[\frac{L\sin\left\{\frac{(m+n)\pi x}{L}\right\}}{(m+n)\pi} + \frac{L\sin\left\{\frac{(m-n)\pi x}{L}\right\}}{(m-n)\pi}\right]_{-L}^{L} \\
&= 0 \qquad \text{if m} \neq \text{n}. \qquad (2.13)
\end{aligned}$$

If $m = n$, the second cosine term is $\cos 0 = 1$, which integrates to L.

ASIDE: useful trigonometric relations To prove the orthogonality, the following formulæ are useful:

$$\begin{aligned}
2\cos A\cos B &= \cos(A+B) + \cos(A-B) \\
2\sin A\cos B &= \sin(A+B) + \sin(A-B) \\
2\sin A\sin B &= -\cos(A+B) + \cos(A-B) \\
2\cos A\sin B &= \sin(A+B) - \sin(A-B) \qquad (2.14)
\end{aligned}$$

To derive these, we write $e^{i(A\pm B)} = e^{iA}e^{\pm iB}$, and rewrite each exponential using $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$. Add or subtract the two $\pm$ expressions and take real or imaginary parts as appropriate to get each of the four results. Alternatively, the orthogonality can be proved using the complex representation directly: $\cos(kx) = [\exp(ikx) + \exp(-ikx)]/2$, so a product of cosines always generates oscillating terms like $\exp(-ix\Delta k)$; these always integrate to zero, unless $\Delta k = 0$.

2.5 Calculating the Fourier components

The Fourier basis functions are always the same. When we expand different functions as Fourier series, the difference lies in the values of the expansion coefficients. To calculate these Fourier components we exploit the orthogonality proved above. The approach will be the same as we follow when we extract components of vectors, which are expressed as a sum of components times basis functions: $v = \sum_i a_i \boldsymbol{e}_i$. The basis vectors are orthonormal, so we extract the j^{th} component just by taking the dot product with $\boldsymbol{e}_j$ to project along that direction:

$$\boldsymbol{e}_j \cdot \boldsymbol{v} = \boldsymbol{e}_j \cdot \sum_i a_i \boldsymbol{e}_i = a_j. \qquad (2.15)$$

This works because all the terms in the series give zero, except the one we want. The procedure with Fourier series is exactly analogous:

1. Choose which constant we wish to calculate (i.e. a_m or b_m for some fixed, chosen value of m)
2. Multiply both sides by the corresponding Fourier mode (e.g. $\cos\left(\frac{m\pi x}{L}\right)$ if we are interested in a_m or $\sin\left(\frac{m\pi x}{L}\right)$ if we are trying to find b_m)

3. Integrate over the full range ($-L \leq x \leq L$ in this case)
4. Rearrange to get the answer.

So, to get a_m:

$$\int_{-L}^{L} dx \ \cos\left(\frac{m\pi x}{L}\right) f(x) \tag{2.16}$$

$$= \frac{1}{2}a_0 \int_{-L}^{L} dx \ \cos\left(\frac{m\pi x}{L}\right) \tag{2.17}$$

$$+ \sum_{n=1}^{\infty} \left[a_n \int_{-L}^{L} dx \ \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) + b_n \int_{-L}^{L} dx \ \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \right] \tag{2.18}$$

$$= a_0.L \ \delta_{m0} + \sum_{n=1}^{\infty} L \, a_n \ \delta_{mn} \tag{2.19}$$

$$= L \, a_m. \tag{2.20}$$

$$\tag{2.21}$$

δ_{mn} is the *Kronecker delta function*:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m <> n \end{cases} \tag{2.22}$$

Rearranging:

$$a_m = \frac{1}{L} \int_{-L}^{L} dx \ \cos\left(\frac{m\pi x}{L}\right) f(x) \tag{2.23}$$

$$\text{Similarly,} \quad b_m = \frac{1}{L} \int_{-L}^{L} dx \ \sin\left(\frac{m\pi x}{L}\right) f(x) \ . \tag{2.24}$$

So this is why the constant term is defined as $a_0/2$: it lets us use the above expression for a_m for all values of m, including zero.

2.6 Even and odd expansions

What if the function we wish to expand is even: $f(-x) = f(x)$, or odd: $f(-x) = -f(x)$? Because the Fourier modes are also even ($\cos\left(\frac{n\pi x}{L}\right)$) or odd ($\sin\left(\frac{n\pi x}{L}\right)$), we can simplify the Fourier expansions.

2.6.1 Expanding an even function

Consider first the case that $f(x)$ is even:

$$b_m = \frac{1}{L} \int_{-L}^{L} dx \ \sin\left(\frac{m\pi x}{L}\right) f(x) = \frac{1}{L} \int_{0}^{L} dx \ \sin\left(\frac{m\pi x}{L}\right) f(x) + \frac{1}{L} \int_{-L}^{0} dx \ \sin\left(\frac{m\pi x}{L}\right) f(x) \tag{2.25}$$

In the second integral, make a change of variables $y = -x \Rightarrow dy = -dx$. The limits on y are $L \to 0$, and use this minus sign to switch them round to $0 \to L$. $f(x) = f(-y) = +f(y)$ because it is an even function, whereas $\sin\left(-\frac{m\pi y}{L}\right) = -\sin\left(\frac{m\pi y}{L}\right)$ as it is odd. Overall, then:

$$b_m = \frac{1}{L} \int_{0}^{L} dx \ \sin\left(\frac{m\pi x}{L}\right) f(x) - \frac{1}{L} \int_{0}^{L} dy \ \sin\left(\frac{m\pi y}{L}\right) f(y) = 0 \tag{2.26}$$

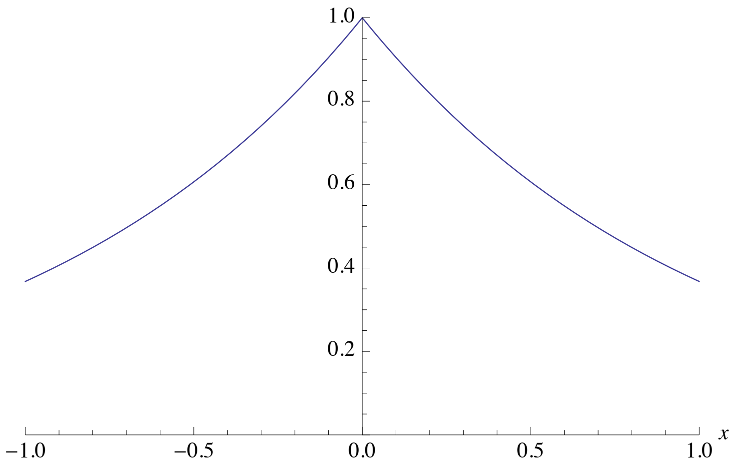

Figure 2.1: $e^{-|x|}$ in $-1 < x < 1$.

i.e. the Fourier decomposition of an even function contains only even Fourier modes. Similarly, we can show that

$$a_m = \frac{1}{L}\int_0^L dx\ \cos\left(\frac{m\pi x}{L}\right) f(x) + \frac{1}{L}\int_0^L dy\ \cos\left(\frac{m\pi y}{L}\right) f(y) = \frac{2}{L}\int_0^L dx\ \cos\left(\frac{m\pi x}{L}\right) f(x). \quad (2.27)$$

2.6.2 Expanding an odd function

For an odd function we get a similar result: all the a_m vanish, so we only get odd Fourier modes, and we can calculate the b_m by doubling the result from integrating from $0 \to L$:

$$a_m = 0 \quad (2.28)$$

$$b_m = \frac{2}{L}\int_0^L dx\ \sin\left(\frac{m\pi x}{L}\right) f(x) \quad (2.29)$$

We derive these results as before: split the integral into regions of positive and negative x; make a transformation $y = -x$ for the latter; exploit the symmetries of $f(x)$ and the Fourier modes $\cos\left(\frac{m\pi x}{L}\right)$, $\sin\left(\frac{m\pi x}{L}\right)$.

Example: $f(x) = e^{-|x|}$ for $-1 < x < 1$. The fundamental period is 2.

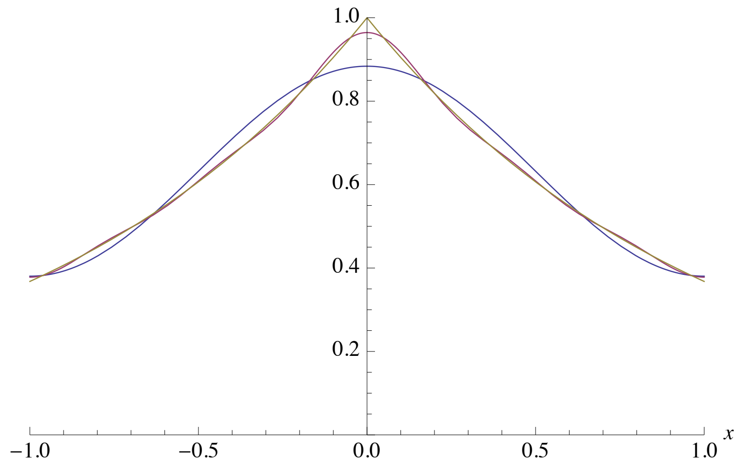

Figure 2.2: Fourier Series for $e^{-|x|}$ in $-1 < x < 1$ summed up to $m = 1$ and to $m = 5$.

The function is symmetric, so we seek a cosine series, with $L = 1$:

$$\begin{aligned}
a_m &= \frac{2}{L}\int_0^L dx \cos\left(\frac{m\pi x}{L}\right) f(x) \\
&= 2\int_0^1 dx \cos(m\pi x) e^{-x} \\
&= 2\int_0^1 dx \frac{1}{2}\left(e^{im\pi x} + e^{-im\pi x}\right) e^{-x} \\
&= \int_0^1 dx \left(e^{im\pi x - x} + e^{-im\pi x - x}\right) \\
&= \left[\frac{e^{(im\pi-1)x}}{im\pi - 1} + \frac{e^{-(im\pi+1)x}}{-(im\pi+1)}\right]_0^1 && (2.30) \\
& && (2.31)
\end{aligned}$$

Now $e^{im\pi} = (e^{i\pi})^m = (-1)^m$, and similarly $e^{-im\pi} = (-1)^m$, so (noting that there is a contribution from $x = 0$)

$$\begin{aligned}
a_m &= \frac{(-1)^m e^{-1} - 1}{im\pi - 1} - \frac{(-1)^m e^{-1} - 1}{im\pi + 1} \\
&= [(-1)^m e^{-1} - 1]\left[\frac{1}{im\pi - 1} - \frac{1}{im\pi + 1}\right] \\
&= [(-1)^m e^{-1} - 1]\frac{2}{(im\pi - 1)(im\pi + 1)} \\
&= \frac{2[(-1)^m e^{-1} - 1]}{-m^2\pi^2 - 1} \\
&= \frac{2[1 - (-1)^m e^{-1}]}{1 + m^2\pi^2}. && (2.32)
\end{aligned}$$

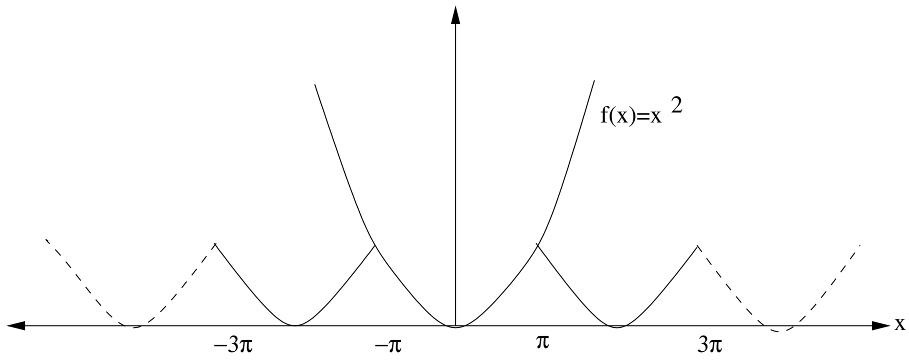

Figure 2.3: $f(x) = x^2$ as a periodic function.

FOURIER ANALYSIS: LECTURE 3

2.7 Periodic extension, or what happens outside the range?

To discuss this, we need to be careful to distinguish between the original function that we expanded $f(x)$ (which is defined for all x) and the Fourier series expansion $f_{\rm FS}(x)$ that we calculated (which is valid only for $-L \le x \le L$.

Inside the expansion range $f_{\rm FS}(x)$ is guaranteed to agree exactly with $f(x)$. Outside this range, the Fourier expansion $f_{\rm FS}(x)$ will not, in general, agree with $f(x)$.

As an example, let's expand the function $f(x) = x^2$ between $-L$ and L (L is some number, which we might decide to set equal to π). This is an *even* function so we know $b_n = 0$. The other coefficients are:

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} dx\; x^2 = \frac{2}{L}\int_0^L dx\; x^2 = \frac{2}{L}\frac{L^3}{3} = \frac{2L^2}{3} \\
a_m &= \frac{1}{L}\int_{-L}^{L} dx\; x^2 \cos\left(\frac{m\pi x}{L}\right) = \frac{2}{L}\int_0^L dx\; x^2 \cos\left(\frac{m\pi x}{L}\right) = \frac{2L^2}{m^3\pi^3}\left[y^2\sin y + 2y\cos y - 2\sin y\right]_0^{m\pi} \\
&= \frac{2L^2}{m^3\pi^3} \times 2m\pi(-1)^m = \frac{4L^2(-1)^m}{m^2\pi^2}
\end{aligned} \tag{2.33}$$

For details, see below.

So, overall our Fourier series is

$$f_{\rm FS}(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right) . \tag{2.34}$$

Inside the expansion range $f_{\rm FS}(x)$ agrees exactly with the original function $f(x)$. *Outside*, however, it does not: $f(x)$ keeps rising quadratically, whereas $f_{\rm FS}(x)$ repeats with period $2L$. We say the Fourier series has *periodically extended* the function $f(x)$ outside the expansion range. This is shown in Fig. 2.3.

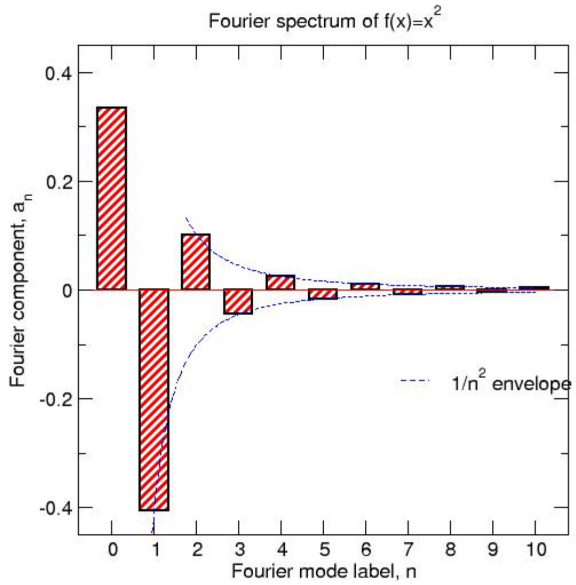

Figure 2.4: The Fourier spectrum a_n (with y-axis in units of L^2) for function $f(x) = x^2$.

There are some special cases where $f_{\rm FS}(x)$ does agree with $f(x)$ outside the range. If $f(x)$ is itself periodic with period $2L/p$ (i.e. the size of the range divided by some integer p s.t. $f(x + 2L/p) = f(x)$), then $f_{\rm FS}(x)$ will agree with $f(x)$ for all x.

Another special case is where $f(x)$ is only defined in the finite range of expansion e.g. because we are only considering a string extending from 0 to L. Physically, then it does not matter if $f_{\rm FS}(x)$ deviates from $f(x)$ outside the range.

A plot of the coefficients, $\{c_n\}$ versus n, is known as the *spectrum* of the function: it tells us how much of each frequency is present in the function. The process of obtaining the coefficients is often known as *spectral analysis*. We show the spectrum for $f(x) = x^2$ in Fig. 2.4.

Choice of periodic extension There is no unique way of casting $f(x)$ as a periodic function, and there may be good and bad ways of doing this. For example, suppose we were interested in representing $f(x) = x^2$ for $0 < x < L$: we have already solved this by considering the even function x^2 over $-L < x < L$, so the periodicity can be over a range that is larger than the range of interest. Therefore, we could equally well make an *odd* periodic function by adopting $+x^2$ for $0 < x < L$ and $-x^2$ for $-L < x < 0$. This is then suitable for a sin series. The coefficients for this are

$$\begin{aligned}
b_m &= \frac{1}{L}\int_0^L dx\; x^2 \sin\left(\frac{m\pi x}{L}\right) + \frac{1}{L}\int_{-L}^0 dx\; (-x^2)\sin\left(\frac{m\pi x}{L}\right) \\
&= \frac{2}{L}\int_0^L dx\; x^2 \sin\left(\frac{m\pi x}{L}\right) = \frac{2L^2}{m^3\pi^3}\left[-y^2\cos y + 2y\sin y + 2\cos y\right]_0^{m\pi} \\
&= \frac{2L^2}{m^3\pi^3} \times \left[(-1)^{m+1} m^2\pi^2 + 2(-1)^m - 2\right] \qquad (2.35)
\end{aligned}$$

So now we have two alternative expansions, both of which represent $f(x) = x^2$ over $0 < x < L$. To lowest order, these are

$$\cos: \quad f(x) = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\cos\left(\frac{\pi x}{L}\right) + \cdots \tag{2.36}$$

$$\sin: \quad f(x) = \frac{2L^2}{\pi}\sin\left(\frac{\pi x}{L}\right) + \cdots . \tag{2.37}$$

It should be clear that the first of these works better, since the function does behave quadratically near $x = 0$, whereas the single sin term is nothing like the target function. In order to get comparable accuracy, we need many more terms for the sin series than the cos series: the coefficients for the former decline as $1/m^2$, as against only $1/m$ for the latter at large m, showing very poor convergence.

Doing the integrals for the x^2 expansion We need to do the integral

$$a_m = \frac{2}{L}\int_0^L dx\ x^2 \cos\left(\frac{m\pi x}{L}\right) \tag{2.38}$$

The first stage is to make a substitution that simplifies the argument of the cosine function:

$$y = \frac{m\pi x}{L} \quad \Rightarrow \quad dy = \frac{m\pi}{L}dx \tag{2.39}$$

which also changes the upper integration limit to $m\pi$. So

$$a_m = \frac{2}{L}\int_0^{m\pi} \frac{L}{n\pi}dy\ \frac{L^2y^2}{m^2\pi^2}\cos y = \frac{2L^2}{m^3\pi^3}\int_0^{m\pi} dy\ y^2\cos y\ . \tag{2.40}$$

We now solve this simplified integral by integrating by parts. An easy way of remembering integration by parts is

$$\int u\frac{dv}{dx}dx = [uv] - \int v\frac{du}{dx}dx \tag{2.41}$$

and in this case we will make $u = y^2$ and $dv/dy = \cos y$. Why? Because we want to differentiate y^2 to make it a simpler function:

$$\int dy\ y^2\cos y = y^2\sin y - \int dy\ 2y\sin y\ . \tag{2.42}$$

We now repeat the process for the integral on the RHS, setting $u = 2y$ for the same reason:

$$\int dy\ y^2\cos y = y^2\sin y - \left[-2y\cos y + \int dy\ 2\cos y\right] = y^2\sin y + 2y\cos y - 2\sin y\ . \tag{2.43}$$

So, using $\sin m\pi = 0$ and $\cos m\pi = (-1)^m$:

$$a_m = \frac{2L^2}{m^3\pi^3}\left[y^2\sin y + 2y\cos y - 2\sin y\right]_0^{m\pi} = \frac{2L^2}{m^3\pi^3}\times 2m\pi(-1)^m = \frac{4L^2(-1)^m}{m^2\pi^2}. \tag{2.44}$$

2.8 Complex Fourier Series

Sines and cosines are one Fourier basis i.e. they provide one way to expand a function in the interval $[-L, L]$. Another, very similar basis is that of complex exponentials.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\phi_n(x) \quad \text{where} \quad \phi_n(x) = e^{+ik_nx} = e^{in\pi x/L} \tag{2.45}$$

$k_n = n\pi/L$ is the *wavenumber*.

This is a *complex Fourier series*, because the expansion coefficients, c_n, are in general complex numbers even for a real-valued function $f(x)$ (rather than being purely real as before). Note that the sum over n runs from $-\infty$ in this case. (The plus sign in phase of the exponentials is a convention chosen to match the convention used for Fourier Transforms in Sec. 3.)

Again, these basis functions are orthogonal and the orthogonality relation in this case is

$$\int_{-L}^{L} dx\ \phi_m(x)\phi_n^*(x) = \int_{-L}^{L} dx\ e^{i(k_m-k_n)x} = \left\{ \begin{array}{ccc} [x]_{-L}^{L} & = 2L & \text{(if } n = m) \\ \left[\frac{\exp(i(k_m-k_n)x)}{i(k_m-k_n)}\right]_{-L}^{L} & = 0 & \text{(if } n \neq m) \end{array} \right\} = 2L\ \delta_{mn}. \tag{2.46}$$

Note the complex conjugation of $\phi_m(x)$: this is needed in order to make the terms involving k cancel; without it, we wouldn't have an orthogonality relation.

For the case $n \neq m$, we note that $m - n$ is a non-zero integer (call it p) and

$$\begin{aligned} \exp[i(k_m - k_n)L] - \exp[i(k_m - k_n)(-L)] &= \exp[ip\pi] - \exp[-ip\pi] \qquad (2.47) \\ &= (\exp[i\pi])^p - (\exp[-i\pi])^p = (-1)^p - (-1)^p = 0\ . \end{aligned}$$

For $p = m - n = 0$ the denominator is also zero, hence the different result.

We can use (and need) this orthogonality to find the coefficients. If we decide we want to calculate c_m for some chosen value of m, we multiply both sides by *the complex conjugate of* $\phi_m(x)$ and integrate over the full range:

$$\int_{-L}^{L} dx\ \phi_m^*(x)f(x) = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} dx\ \phi_m^*(x)\phi_n(x) = \sum_{n=-\infty}^{\infty} c_n.2L\delta_{mn} = c_m.2L \tag{2.48}$$

$$\Rightarrow \quad c_m = \frac{1}{2L}\int_{-L}^{L} dx\ \phi_m^*(x)f(x) = \frac{1}{2L}\int_{-L}^{L} dx\ e^{-ik_m x} f(x) \tag{2.49}$$

As before, we exploited that the integral of a sum is the same as a sum of integrals, and that c_n are constants.

2.8.1 Relation to real Fourier series

The complex approach may seem an unnecessary complication. Obviously it is needed if we have to represent a complex function, but for real functions we need to go to some trouble in order to make sure that the result is real:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \Rightarrow f(x) = f^*(x) = \sum_{n=-\infty}^{\infty} c_n^* e^{-in\pi x/L} \tag{2.50}$$

Equating the coefficients of the $e^{im\pi x/L}$ mode, we see that the Fourier coefficients have to be *Hermitian*:

$$c_{-m}^* = c_m. \tag{2.51}$$

This shows why it was necessary to consider both positive and negative wavenumbers, unlike in the sin and cos case.

But the advantage of the complex approach is that it is often much easier to deal with integrals involving exponentials. We have already seen this when discussing how to prove the orthogonality relations for sin and cos. Also, doing things this way saves having to do twice the work in obtaining coefficients for sin and cos series separately.

The fact that complex exponentials contain both the sin and cos series in a single term also makes sense of the extra factor of $\frac{1}{2}$ everywhere in the orthogonality relations, relative to the real Fourier series: this arises from the definition of sine and cosine in terms of complex exponentials.

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2.8.2 Example

To show the complex Fourier approach in action, we revisit our example of expanding $f(x) = x^2$ for $x \in [-L, L]$ (we could choose $L = \pi$ if we wished). The general expression for the Fourier coefficients, c_m, takes one of the following forms, depending on whether or not m is zero:

$$c_{m=0} = \frac{1}{2L} \int_{-L}^{L} dx\ x^2 = \frac{L^2}{3} \tag{2.52}$$

$$c_{m\neq 0} = \frac{1}{2L} \int_{-L}^{L} dx\ x^2 e^{-im\pi x/L} = \frac{2L^2(-1)^m}{m^2\pi^2} \tag{2.53}$$

See below for details of how to do the second integral. We notice that in this case all the c_m are real, but this is not the case in general.

ASIDE: doing the integral We want to calculate

$$c_m \equiv \frac{1}{2L} \int_{-L}^{L} dx\ x^2 e^{-im\pi x/L} \tag{2.54}$$

To make life easy, we should change variables to make the exponent more simple (whilst keeping y real) i.e. set $y = m\pi x/L$, for which $dy = (m\pi/L)\, dx$. The integration limits become $\pm m\pi$:

$$c_m = \frac{1}{2L} \int_{-m\pi}^{m\pi} dy\ \frac{L}{m\pi} \times \frac{L^2 y^2}{m^2\pi^2} e^{-iy} = \frac{L^2}{2m^3\pi^3} \int_{-m\pi}^{m\pi} dy\ y^2 e^{-iy}. \tag{2.55}$$

Now we want to integrate by parts. We want the RHS integral to be simpler than the first, so we set $u = y^2 \ \Rightarrow\ du = 2y\, dy$ and $dv/dy = e^{-iy} \ \Rightarrow\ v = e^{-iy}/(-i) = ie^{-iy}$ (multiplying top and bottom by i and recognising $-i \times i = 1$). So

$$c_m = \frac{L^2}{2m^3\pi^3} \left\{ \left[iy^2 e^{-iy} \right]_{-m\pi}^{m\pi} - \int_{-m\pi}^{m\pi} dy\ 2y.ie^{-iy} \right\} = \frac{L^2}{2m^3\pi^3} \left\{ \left[iy^2 e^{-iy} \right]_{-m\pi}^{m\pi} - 2i \int_{-m\pi}^{m\pi} dy\ ye^{-iy} \right\} \tag{2.56}$$

The integral is now simpler, so we play the same game again, this time with $u = y \;\Rightarrow\; du/dy = 1$ to get:

$$
\begin{aligned}
c_m &= \frac{L^2}{2m^3\pi^3}\left\{\left[iy^2 e^{-iy}\right]_{-m\pi}^{m\pi} - 2i\left(\left[iye^{-iy}\right]_{-m\pi}^{m\pi} - \int_{-m\pi}^{m\pi} dy\, ie^{-iy}\right)\right\} && (2.57)\\
&= \frac{L^2}{2m^3\pi^3}\left\{\left[iy^2 e^{-iy}\right]_{-m\pi}^{m\pi} - 2i\left(\left[iye^{-iy}\right]_{-m\pi}^{m\pi} - i\left[ie^{-iy}\right]_{-m\pi}^{m\pi}\right)\right\} && (2.58)\\
&= \frac{L^2}{2m^3\pi^3}\left[iy^2 e^{-iy} - 2i.i.ye^{-iy} + 2i.i.i\, e^{-iy}\right]_{-m\pi}^{m\pi} && (2.59)\\
&= \frac{L^2}{2m^3\pi^3}\left[e^{-iy}\left(iy^2 + 2y - 2i\right)\right]_{-m\pi}^{m\pi} && (2.60)
\end{aligned}
$$

We can now just substitute the limits in, using $e^{im\pi} = e^{-im\pi} = (-1)^m$ (so e^{-iy} has the same value at both limits). Alternately, we can note that the first and third terms in the round brackets are even under $y \to -y$ and therefore we will get zero when we evaluate between *symmetric* limits $y = \pm m\pi$ (N.B. this argument only works for symmetric limits). Only the second term, which is odd, contributes:

$$
\begin{aligned}
c_m &= \frac{L^2}{2m^3\pi^3}\left[2ye^{-iy}\right]_{-m\pi}^{m\pi} = \frac{L^2}{2m^3\pi^3}\left[2m\pi e^{-im\pi} - (-2m\pi)e^{im\pi}\right]\\
&= \frac{L^2}{2m^3\pi^3} \times 4m\pi(-1)^m = \frac{2L^2(-1)^m}{m^2\pi^2}. && (2.61)
\end{aligned}
$$

2.9 Comparing real and complex Fourier expansions

There is a strong link between the real and complex Fourier basis functions, because cosine and sine can be written as the sum and difference of two complex exponentials:

$$
\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right) \quad \Rightarrow \quad \begin{cases} \cos\left(\frac{n\pi x}{L}\right) = \dfrac{1}{2}[\phi_n(x) + \phi_{-n}(x)] \\[2ex] \sin\left(\frac{n\pi x}{L}\right) = \dfrac{1}{2i}[\phi_n(x) - \phi_{-n}(x)] \end{cases} \tag{2.62}
$$

so we expect there to be a close relationship between the real and complex Fourier coefficients. Staying with the example of $f(x) = x^2$, we can rearrange the complex Fourier series by splitting the sum as follows:

$$
f_{\mathrm{FS}}(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=-1}^{-\infty} c_n e^{in\pi x/L} \tag{2.63}
$$

we can now relabel the second sum in terms of $n' = -n$:

$$
f_{\mathrm{FS}}(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n'=1}^{\infty} c_{-n'} e^{-in'\pi x/L} \tag{2.64}
$$

Now n and n' are just dummy summation indices with no external meaning, so we can now relabel $n' \to n$ and the second sum now looks a lot like the first. Noting from Eqn. (2.61) that in this case $c_{-m} = c_m$, we see that the two sums combine to give:

$$
f_{\mathrm{FS}}(x) = c_0 + \sum_{n=1}^{\infty} c_n\left[\phi_n(x) + \phi_{-n}(x)\right] = c_0 + \sum_{n=1}^{\infty} 2c_n \cos\left(\frac{n\pi x}{L}\right) \tag{2.65}
$$

So, this suggests that our real and complex Fourier expansions are identical with $a_n = 2c_n$ (and $b_n = 0$). Comparing our two sets of coefficients in Eqns. (2.33) and (2.61), we see this is true.

2.10 Some other features of Fourier series

In this subsection, we're going to look at some other properties and uses of Fourier expansions.

2.10.1 Differentiation and integration

If the Fourier series of $f(x)$ is differentiated or integrated, *term by term*, the new series (if it converges, and in general it does) is the Fourier series for $f'(x)$ or $\int dx\ f(x)$, respectively (in the latter case, only if $a_0 = 0$; if not, there is a term $a_0 x/2$, which would need to be expanded in sin and cos terms).

This means that we do not have to do a second expansion to, for instance, get a Fourier series for the derivative of a function.

It can also be a way to do a difficult integral. Integrals of sines and cosines are relatively easy, so if we need to integrate a function it may be more straightforward to do a Fourier expansion first.

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2.10.2 Solving ODEs

Fourier Series can be very useful if we have a differential equation with coefficients that are *constant*, and where the equation is *periodic*[1]. For example

$$\frac{d^2z}{dt^2} + p\frac{dz}{dt} + rz = f(t) \tag{2.66}$$

where the driving term $f(t)$ is periodic with period T. i.e. $f(t+T) = f(t)$ for all t. We solve this by expanding both f and z as Fourier Series, and relating the coefficients. Note that $f(t)$ is a given function, so we can calculate its Fourier coefficients.

Writing

$$\begin{aligned} z(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \qquad (2.67) \\ f(t) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t) \end{aligned}$$

where the fundamental frequency is $\omega = 2\pi/T$.

$$\begin{aligned} \frac{dz}{dt} &= -\sum_{n=1}^{\infty} n\omega a_n \sin(n\omega t) + \sum_{n=1}^{\infty} n\omega b_n \cos(n\omega t) \qquad (2.68) \\ \frac{d^2z(t)}{dt^2} &= -\sum_{n=1}^{\infty} n^2\omega^2 a_n \cos(n\omega t) - \sum_{n=1}^{\infty} n^2\omega^2 b_n \sin(n\omega t) \end{aligned}$$

[1]For non-periodic functions, we can use a *Fourier Transform*, which we will cover later

Then the l.h.s. of the differential equation becomes

$$\frac{d^2z}{dt^2}+p\frac{dz}{dt}+rz = \frac{r}{2}a_0+\sum_{n=1}^{\infty}\left[\left(-n^2\omega^2 a_n + pn\omega b_n + ra_n\right)\cos(n\omega t) + \left(-n^2\omega^2 b_n - pn\omega b_n + rb_n\right)\sin(n\omega t)\right]. \tag{2.69}$$

Now we compare the coefficients with the coefficients of expansion of the r.h.s. driving term $f(t)$. The constant term gives

$$ra_0 = A_0 \Rightarrow a_0 = \frac{A_0}{r}. \tag{2.70}$$

Equating the coefficients of $\cos(n\omega t)$ and $\sin(n\omega t)$ gives:

$$\begin{aligned} -n^2\omega^2 a_n + pn\omega b_n + ra_n &= A_n \\ -n^2\omega^2 b_n - pn\omega a_n + rb_n &= B_n \end{aligned} \tag{2.71}$$

This is a pair of simultaneous equations for a_n, b_n, which we can solve, e.g. with matrices:

$$\begin{pmatrix} -n^2\omega^2 + r & pn\omega \\ -pn\omega & -n^2\omega^2 + r \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} A_n \\ B_n \end{pmatrix} \tag{2.72}$$

Calculating the inverse of the matrix gives

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{D}\begin{pmatrix} -n^2\omega^2 + r & -pn\omega \\ pn\omega & -n^2\omega^2 + r \end{pmatrix}\begin{pmatrix} A_n \\ B_n \end{pmatrix} \tag{2.73}$$

where

$$D \equiv (r - n^2\omega^2)^2 + p^2n^2\omega^2. \tag{2.74}$$

So for any given driving term the solution can be found. Note that this equation *filters* the driving field:

$$\sqrt{a_n^2 + b_n^2} = \frac{\sqrt{A_n^2 + B_n^2}}{D} \tag{2.75}$$

For large n, equation 2.74 gives $D \propto n^4$, so the high frequency harmonics are severely damped.

Discussion: If we set $p = 0$ and $r > 0$ we see that we have a Simple Harmonic Motion problem ($z'' + \omega_0^2 z = f(t)$), with natural frequency $\omega_0 = \sqrt{r}$. p represents damping of the system. If the forcing term has a much higher frequency ($n\omega \gg \omega_0$) then D is large and the amplitude is suppressed – the system cannot respond to being driven much faster than its natural oscillation frequency. In fact we see that the amplitude is greatest if $n\omega$ is about ω_0 (if p is small) – an example of resonance.

Let us look at this in more detail. If we drive the oscillator at a single frequency, so $A_n = 1$ say, for a single n, and we make $B_n = 0$ (by choosing the origin of time). All other A_n, $B_n = 0$.

The solution is

$$a_n = \frac{1}{D}(w_0^2 - n^2\omega^2); \qquad b_n = \frac{1}{D}pn\omega \tag{2.76}$$

If the damping p is small, we can ignore b_n for our initial discussion.

So, if the driving frequency is less than the natural frequency, $n\omega < \omega_0$, then a_n and A_n have the same sign, and the oscillations are in phase with the driving force. If the driving frequency is higher than the natural frequency, $n\omega > \omega_0$, then $a_n < 0$ and the resulting motion is out of phase with the driving force.

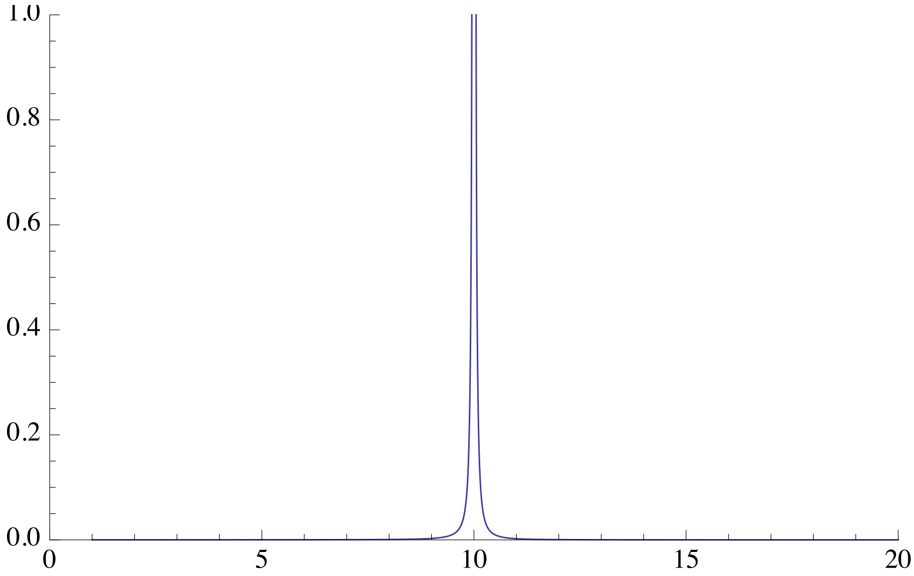

Figure 2.5: The amplitude of the response of a damped harmonic oscillator (with small damping) to driving forces of different frequencies. The spike is close to the natural frequency of the oscillator.

2.10.3 Resonance

If the driving frequency is equal to the natural frequency, $n\omega = \omega_0$, then $a_n = 0$, and all the motion is in b_n. So the motion is $\pi/2$ out of phase ($\sin(n\omega t)$ rather than the driving $\cos(n\omega t)$). Here we can't ignore p.

2.10.4 Other solutions

Finally, note that we can always add a solution to the homogeneous equation (i.e. where we set the right hand side to zero). The final solution will be determined by the initial conditions (z and dz/dt). This is because the equation is *linear* and we can superimpose different solutions.

For the present approach, this presents a problem: the undriven motion of the system will not be periodic, and hence it cannot be described by a Fourier series. This explains the paradox of the above solution, which implies that a_n and b_n are zero if A_n and B_n vanish, as they do in the homogeneous case. So apparently $z(t) = 0$ in the absence of a driving force – whereas of course an oscillator displaced from $z = 0$ will show motion even in the absence of an applied force. For a proper treatment of this problem, we have to remove the requirement of periodicity, which will be done later when we discuss the Fourier transform.

2.11 Fourier series and series expansions

We can sometimes exploit Fourier series to either give us series approximations for numerical quantities, or to give us the result of summing a series.

Consider $f(x) = x^2$, which we expanded as a Fourier series in Eqn. (2.34) above, and let's choose the expansion range to be $-\pi \to \pi$ (i.e. we'll set $L = \pi$). At $x = 0$ we have $f(x) = f_{\rm FS}(x) = 0$.

N	π_N	$\pi_N - \pi$
1	3.4641016151	0.3225089615
2	3.0000000000	−0.1415926536
3	3.2145502537	0.0729576001
4	3.0956959368	−0.0458967168
5	3.1722757341	0.0306830805
6	3.1192947921	−0.0222978615
7	3.1583061852	0.0167135316
8	3.1284817339	−0.0131109197
9	3.1520701305	0.0104774769
10	3.1329771955	−0.0086154581
100	3.1414981140	−0.0000945396
1000	3.1415916996	−0.0000009540
10000	3.1415926440	−0.0000000095
100000	3.1415926535	−0.0000000001

Table 1: A series approximation to π from Eqn. (2.79)

Substituting into Eqn. (2.34) we have

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \quad \Rightarrow \quad \frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \tag{2.77}$$

This result can be useful in two ways:

1. We solve a physics problem, and find the answer as a sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. Using the above result we can replace the sum by $\frac{\pi^2}{12}$.
2. We need a numerical approximation for π. We can get this by truncating the sum at some upper value $n = N$ [as in Eqn. (2.84)] and adding together all the terms in the sum.

   $$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots \tag{2.78}$$

   Let's call this approximation π_N:

   $$\pi_N \equiv \sqrt{12 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2}} \tag{2.79}$$

   Table 1 shows how π_N approaches π as we increase N.

We can get different series approximations by considering different values of x in the same Fourier series expansions. For instance, consider $x = \pi$. This gives:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}(-1)^n \quad \Rightarrow \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \equiv \zeta(2) \tag{2.80}$$

This is an example of the Riemann zeta function $\zeta(s)$ which crops up a lot in physics. It has limits:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \to \begin{cases} 1 & \text{as } s \to \infty \\ \infty & \text{as } s \to 1 \end{cases} \tag{2.81}$$