A THEOREM OF SYLVESTER AND SCHUR

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The theorem in question asserts that, if $n > k$, then, in the set of integers $n, n+1, n+2, \ldots, n+k-1$, there is a number containing a prime divisor greater than k. If $n = k+1$, we obtain the well-known theorem of Chebyshev. The theorem was first asserted and proved by Sylvester† about forty-five years ago. Recently Schur‡ has rediscovered and again proved the theorem.

The following proof is shorter and more elementary than the previous ones. We shall not use Chebyshev's results, so that we shall also prove Chebyshev's theorem§.

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* Received 22 March, 1934; read 20 April, 1934.

† J. J. Sylvester, "On arithmetical series", *Messenger of Math.*, 21 (1892), 1–19, 87–120; and *Collected mathematical papers*, 4 (1912), 687–731.

‡ J. Schur, "Einige Sätze über Primzahlen mit Anwendung auf Irreduzibilitätsfragen" *Sitzungsberichte der preussischen Akademie der Wissenschaften, Phys. Math. Klasse*, 23 (1929), 1–24.

§ P. Erdös, "Beweis eines Satzes von Tschebyschef", *Acta Lit. ac Sci. Regiae Universitatis Hungaricae Franscisco-Josephinae*, 5 (1932), 194–198.

We first express the theorem in the following form:

If $n \geqslant 2k$, then $\binom{n}{k}$ contains a prime divisor greater than k.

We shall prove first the following lemma:

If $\binom{n}{k}$ is divisible by a power of a prime p^α, then $p^\alpha \leqslant n$.

The expression
$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$
contains the prime p with the exponent

$$\left(\left[\frac{n}{p}\right]-\left[\frac{k}{p}\right]-\left[\frac{n-k}{p}\right]\right)+\left(\left[\frac{n}{p^2}\right]-\left[\frac{k}{p^2}\right]-\left[\frac{n-k}{p^2}\right]\right)$$
$$+\dots\left(\left[\frac{n}{p^\alpha}\right]-\left[\frac{k}{p^\alpha}\right]-\left[\frac{n-k}{p^\alpha}\right]\right),$$

where
$$p^\alpha \leqslant n < p^{\alpha+1}.$$

It is immediately clear that each of these α terms has the value 0 or 1. Hence the highest power of p which can divide $\binom{n}{k}$ is α.

1. Let $\pi(k)$ denote the number of primes less than or equal to k. It is clear that, for $k \geqslant 8$, $\pi(k) \leqslant \frac{1}{2}k$. Hence, if $\binom{n}{k}$ had no prime factors greater than k, we should have, from the lemma, $\binom{n}{k} \leqslant n^{\frac{1}{2}k}$. On the other hand, it is evident that

$$\binom{n}{k} = \frac{n}{k}\,\frac{n-1}{k-1}\,\frac{n-2}{k-2}\cdots\frac{n-k+1}{1} > \left(\frac{n}{k}\right)^k,$$

consequently
$$\left(\frac{n}{k}\right)^k < n^{\frac{1}{2}k},$$

i.e.
$$n^{\frac{1}{2}k} < k^k,$$

which evidently does not hold when $k \leqslant \sqrt{n}$. Thus we have proved the theorem for
$$8 \leqslant k \leqslant \sqrt{n}.$$

It may be observed that we have also proved incidentally that, if $n > 2$, there is always a prime number between $\sqrt{n}$ and n.

Since $\pi(k) < \frac{1}{3}k$ for $k > 37$* (we see the validity of this proposition by considering the number of integers less than k and not divisible by 2, 3, and 5) we can prove the theorem, just as in § 1, for

$$37 < k \leqslant n^{\frac{2}{3}}.$$

2. Now we consider the general case, *i.e.* $k > n^{\frac{2}{3}}$. We suppose that $k > 37$. If $\binom{n}{k}$ contains no prime divisor exceeding k, then

$$\binom{n}{k} < \prod_{p \leqslant k} p \prod_{p \leqslant \sqrt{n}} p \prod_{p \leqslant \sqrt[3]{n}} p \ldots.$$

This inequality is an immediate consequence of the lemma about the prime power divisors of $\binom{n}{k}$.

First we shall prove† that

$$4^n > \prod_{p_i \leqslant n} p_i \prod_{p_k < \sqrt{n}} p_k \prod_{p_l < \sqrt[3]{n}} p_l \ldots. \qquad (1)$$

For this purpose, we analyse the prime factors of the binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}.$$

It is evident that $\binom{2n}{n}$ contains every prime p such that $n < p \leqslant 2n$, since the numerator is divisible by p and the denominator is not.

Further, $\binom{2n}{n}$ is divisible by any prime p such that

$$\sqrt[\alpha]{n} < p \leqslant \sqrt[\alpha]{(2n)}$$

for any positive integer α. For the prime p occurs in $n!$ to the power

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots \left[\frac{n}{p^{\alpha-1}}\right],$$

and in $(2n)!$ to the power

$$\left[\frac{2n}{p}\right] + \left[\frac{2n}{p^2}\right] + \ldots \left[\frac{2n}{p^{\alpha-1}}\right] + 1,$$

since

$$\left[\frac{2n}{p^\alpha}\right] = 1.$$

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* Schur, *ibid.*, 7.

† P. Erdös, "Egy Kürschák féle elemi számelméleti tétel általánositása," *Math. és Phys. Lapok*, 39 (1932), 19–22.

Consequently the prime p is contained in $\binom{2n}{n}$ to the power

$$1+\left[\frac{2n}{p}\right]+\left[\frac{2n}{p^2}\right]+\dots\left[\frac{2n}{p^{\alpha-1}}\right]-2\left[\frac{n}{p}\right]-2\left[\frac{n}{p^2}\right]-\dots-2\left[\frac{n}{p^{\alpha-1}}\right]\geqslant 1,$$

since
$$\left[\frac{2n}{p^k}\right]\geqslant 2\left[\frac{n}{p^k}\right].$$

Let us now denote by $\{x\}$ the least integer greater than or equal to x, and put

$$a_1=\left\{\frac{n}{2}\right\},\quad a_2=\left\{\frac{n}{2^2}\right\},\quad \dots,\quad a_k=\left\{\frac{n}{2^k}\right\},$$

Then
$$a_1\geqslant a_2\geqslant a_3\dots\geqslant a_k\geqslant\dots.$$

Further,
$$a_k<\frac{n}{2^k}+1=\frac{2n}{2^{k+1}}+1\leqslant 2a_{k+1}+1,$$

and so, since a_k and a_{k+1} are integers,

$$a_k\leqslant 2a_{k+1}. \tag{2}$$

If now m is the first exponent for which $n/2^m\leqslant 1$, then $a_m=1$. Since $2a_1\geqslant n$, it is evident from (2) that the interval $1<y\leqslant n$ is completely covered by the intervals

$$a_m<y\leqslant 2a_m,\quad a_{m-1}<y\leqslant 2a_{m-1},\quad \dots,\quad a_1<y\leqslant 2a_1.$$

It is easily seen from (2) that the interval

$$1<y\leqslant[\sqrt[k]{n}]$$

is completely covered by the intervals

$$[\sqrt[k]{a_m}]<y\leqslant[\sqrt[k]{(2a_m)}],\quad [\sqrt[k]{a_{m-1}}]<y\leqslant[\sqrt[k]{(2a_{m-1})}],\quad \dots$$

for any integer $k\geqslant 1$.

From all this, it follows that

$$\prod_{p_i\leqslant n}p_i\prod_{p_k\leqslant\sqrt{n}}p_k\prod_{p_l\leqslant\sqrt[3]{n}}p_l\dots\leqslant\binom{2a_1}{a_1}\binom{2a_2}{a_2}\dots\binom{2a_m}{a_m}, \tag{3}$$

the right side being a multiple of the left.

Now we show that

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < 4^n, \tag{4}$$

which in combination with (3) establishes (1).

We easily prove by simple arithmetic that (4) holds for any number n less than or equal to 10.

Suppose now that $n \geqslant 10$ and that (4) holds for any integer less than n.

Now
$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < \binom{2a_1}{a_1} 4^{2a_2-1}, \tag{5}$$

which we obtain by applying (4) with $n = 2a_2-1$, for it is easily seen that

$$\{\tfrac{1}{2}(2a_2-1)\} = a_2, \quad \{\tfrac{1}{4}(2a_2-1)\} = a_3, \quad \ldots$$

We easily obtain by induction that, for any $n \geqslant 5$,

$$\binom{2n}{n} < 4^{n-1}.$$

Hence, from (5),

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < 4^{a_1-1+2a_2-1},$$

and, since $2a_1 \leqslant n+1$, $2a_2 \leqslant a_1+1$, the exponent

$$a_1+2a_2-2 \leqslant 2a_1-1 \leqslant n,$$

which evidently establishes (4).

Now
$$\prod_{p\leqslant k} p \prod_{p\leqslant \sqrt{k}} p \prod_{p\leqslant \sqrt[3]{k}} p \ldots < 4^k, \tag{1'}$$

and, in particular, taking $k = \sqrt{n}$,

$$\prod_{p\leqslant \sqrt{n}} p \prod_{p\leqslant \sqrt[4]{n}} p \prod_{p\leqslant \sqrt[6]{n}} p \ldots < 4^{\sqrt{n}}.$$

If now $k > n^{\frac{2}{3}}$, then $\sqrt[l]{k} \geqslant \sqrt[2l-1]{n}$ for $l \geqslant 2$, and so, from (1'),

$$\prod_{p\leqslant k} p \prod_{p\leqslant \sqrt[3]{n}} p \prod_{p\leqslant \sqrt[5]{n}} p \ldots < 4^k,$$

and so
$$\prod_{p\leqslant k} p \prod_{p\leqslant \sqrt{n}} p \prod_{p\leqslant \sqrt[3]{n}} p \ldots < 4^{k+\sqrt{n}}. \tag{6}$$

Hence $\binom{n}{k} < 4^{k+\sqrt{n}}$, and this leads, as we shall now prove, to a contradiction.

Suppose first that $n \geqslant 4k$, then $\binom{4k}{k} \leqslant \binom{n}{k}$. On the other hand,

$$\binom{4k}{k} = \binom{2k}{k} \frac{4k(4k-1)\dots(3k+1)}{2k(2k-1)\dots(k+1)} > \frac{4^k \cdot 2^k}{2k} = \frac{8^k}{2k},$$

since

$$\binom{2k}{k} \geqslant \frac{4^k}{2k},$$

and thus

$$\frac{8^k}{2k} < 4^{k+\sqrt{n}},$$

i.e.

$$\frac{2^k}{2k} < 4^{\sqrt{n}}.$$

We can easily prove by induction that $2^{\frac{1}{3}k} > 2k$, *i.e.* $2^{k-3} > k^3$, for $k > 20$ (which does not need any new restriction, for in this paragraph $k > 37$). Thus $2^{2k/3} < 2^{2\sqrt{n}}$, *i.e.* $k < 3\sqrt{n}$.

But, by hypothesis, $k > n^{\frac{2}{3}}$, whence $n < 3^6 = 729$, which means a contradiction for $n > 729$.

3. Suppose next that

$$\tfrac{5}{2}k < n < 4k.$$

Then

$$\binom{n}{k} \geqslant \binom{2k}{k} \frac{\{\frac{5}{2}k\}(\{\frac{5}{2}k\}-1)\dots(\{\frac{5}{2}k\}-k+1)}{2k(2k-1)\dots(k+1)} > \frac{4^k}{2k}\left(\frac{5}{4}\right)^k.$$

If $\binom{n}{k}$ contains again no prime divisor greater than or equal to k, then

$$\frac{4^k}{2k}\left(\frac{5}{4}\right)^k < 4^{k+\sqrt{n}},$$

i.e.

$$\frac{(\frac{5}{4})^k}{2k} < 4^{\sqrt{n}}.$$

Since

$$n < 4k, \quad (\tfrac{5}{4})^k < 2k \cdot 2^{4\sqrt{k}},$$

and

$$2^n > 2n+1 \quad \text{for} \quad n \geqslant 3,$$

i.e.

$$2^{[\sqrt{k}]} > 2[\sqrt{k}]+1 \quad \text{for} \quad k \geqslant 9,$$

and thus $$2^{2[\sqrt{k}]} > 2k,$$

it follows that $$(\tfrac{5}{4})^k < 2^{6\sqrt{k}}.$$

Since $(\frac{5}{4})^4 > 2$, $2^{\frac{1}{4}k} < 2^{6\sqrt{k}}$, which is an evident contradiction for $k > 576$. This means that our theorem holds for $n > 2304$.

4. Finally, we have to consider the case

$$2k \leqslant n \leqslant [\tfrac{5}{2}k].$$

Then $$\binom{n}{k} \geqslant \binom{2k}{k} > \frac{4^k}{2k}.$$

But in this case, when $p \leqslant k$ is a divisor of $\binom{n}{k}$, $p \leqslant \frac{1}{3}n$; for in the denominator of $n!/[k!\,(n-k)!]$, $p \leqslant k \leqslant n-k$ occurs to the second power, consequently it must occur in the numerator to at least the third power, and so $p < \frac{1}{3}n$.

Since $\frac{1}{3}n > n^{\frac{2}{3}}$ for $n > 27$, we have $\sqrt[l]{(\frac{1}{3}n)} \geqslant \sqrt[2l-1]{n}$, and from this, taking $k = \frac{1}{3}n$ in (1′), we have, just as in (6),

$$\frac{4^k}{2k} < 4^{\frac{1}{3}n+\sqrt{n}},$$

i.e. $$\frac{4^k}{2k} < 4^{\frac{5}{6}k+\sqrt{n}}.$$

From this we obtain $$\frac{4^{\frac{1}{6}k}}{2k} < 4^{\sqrt{n}}.$$

Now we have $$2^{[\sqrt{n}]} > [\sqrt{n}]+1,$$

i.e. $$4^{[\sqrt{n}]} > n \geqslant 2k,$$

and so $$4^{\frac{1}{6}k} < 4^{2\sqrt{n}}, \quad \text{or} \quad k < 12\sqrt{n}.$$

Since $k > \frac{1}{3}n$, this cannot be true if $n > 1296$. Thus we have proved the theorem when $k \geqslant 7$, with the exception of a finite number of other cases.

The case $k \leqslant 7$ may be easily settled by a simple discussion and the other exceptions by means of tables of primes.

I close by taking the opportunity of expressing my great indebtedness to Prof. L. J. Mordell for his kind assistance.

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