Normed Spaces and the Change of Base for Enriched Categories

by

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Abstract

In this thesis, we study two related concepts: a generalization of normed spaces to a categorical setting, and a study of the change of base for enriched categories. After describing the first idea, we will show how it leads to a desire to further understand the change of base. This, in turn, leads to an interesting comparison between bicategories and (pseudo) double categories.

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Chapter 1

Introduction

This thesis is about two things: a desire to understand the concept of "normed space" in a wider categorical context, and, as a result of this, a study of the change of base for enriched categories. Neither of these projects is completed in this thesis. However, we hope that the results and ideas that have come from these investigations will be helpful in future research in these areas.

The desire to understand the concept of "normed space" comes from wanting to further Lawvere's work on metric spaces. In his 1972 paper ([33]), Lawvere noted an interesting connection between metric spaces and category theory. He found that metric spaces (slighly generalized) and non-expansive mappings are $\mathbf{V}$-enriched categories and functors for a certain monoidal category $\mathbf{V}$. This discovery led to ideas from analysis being used in category theory, and ideas from category theory being used in analysis. One can further develop the theory by asking the following question: if metric spaces correspond to enriched categories, what do normed spaces correspond to? As we shall see, one answer is that normed spaces correspond to monoidal functors from compact closed categories to a monoidal category $\mathbf{V}$. Another answer, also due to Lawvere, is that they are $\mathbf{V}$-compact $\mathbf{V}$-categories. Understanding the similiarities and differences between these two viewpoints requires a study of the change of base for enriched categories.

The definition of this change of base dates as far back as Eilenberg and Kelly's original work on enriched categories ([16]). However, for the purposes for which we wish to use it, we need to understand more about this change of base. Specifically, we need to know whether it preserves monoidal and compact enriched categories. Determining whether the change of base preserves enriched monoidal categories is fully

explored in this thesis, and has some additional benefits in addition to the actual result itself.

Determining whether the change of base preserves autonomous monoidal categories is rather more difficult. Day and Street have defined a type of lax functor that preserves autonomous objects, and we show that the change-of-base has some of the properties required by their definition. However, the elements of the definition which fail to hold for the change-of-base appear to hold on a different level. What is needed is an alternative viewpoint: rather than viewing enriched profunctors as arrows in a bicategory, we need to view them as vertical arrows in a double category. As a result of this, an alternative view of the change of base for enriched categories is presented. In this alternative view, we view enriched categories as objects of a double category, rather than as objects of a 2-category or bicategory.

After working out this alternative change of base, we see that viewing enriched categories as the objects of a double category is a much more natural point of view. Because of this, and the earlier discussions of structured bicategories, we see that the most natural way forward is to build a theory of structured double categories. This program is touched on at the end of the thesis, but is a large undertaking, and will require further work.

We believe that the ideas presented in the thesis allow for many interesting areas for future investigation, in both the areas of analysis and of enriched category theory.

1.1   Chapter Overview

As a guide to the reader, here is an overview of the chapters of the thesis.

In Chapter 2, we review the basic concepts which are seen throughout the thesis: monoidal categories, monoidal functors, and enriched categories. The chapter is useful to read even if one is familiar with the concepts, as the ideas are presented with a view towards considering enriched categories as metric spaces and monoidal functors

as norms. This chapter is also used to fix terminology for both monoidal and enriched categories.

In Chapter 3, we expand upon the idea of norms as monoidal functors. Lawvere's idea of normed spaces is also given. The comparison between these requires an understanding of monoidal functors. Thus, this chapter serves as motivation for why one would be interested in change of base questions. Some of the concepts that come from this investigation are also interesting in their own right. In particular, the idea of a normed module, where one has a sub-scalar invariance rather than the usual strict scalar invariance, is discussed as a possible alternative to the idea of normed vector space.

In Chapter 4, we review the classical change of base theory. In some sense, the chapter is review, but it also slightly differs from the Eilenberg and Kelly ([16]) work, as here we are focused on monoidal categories rather than closed categories. In addition, it presents the idea that the change of base $(-)_*$ is itself a 2-functor, a result that is generally known, but not stated in the original paper. Finally, many of the proofs have been simplified through the use of an idea of applying a monoidal functor "monoidally".

In Chapter 5, we present the first major result: change of base preserves monoidal categories. The proof itself has many side benefits, not least of which is giving an interesting persepective on the monoidal structure of **V-cat**. We also show that it is not neccesary that **V** be braided for **V-cat** to have monoidal structure.

In Chapter 6, we present the change of base for enriched profunctors. This requires a review of the idea of enriched profunctors, as well as a review of the relatively recent idea of modules between lax functors.

In Chapter 7, we discuss the difficulties in using the existing notions of structured bicategory when attempting to prove results about the change of base for enriched

categories. One of the most interesting aspects of this chapter is the appearance of special "squares" of functors and profunctors. These squares form the essential basis for wishing to consider the change of base as a double functor.

In Chapter 8, we discuss the idea of viewing the change of base as a double functor between double categories. This idea perhaps has the greatest potential for future work. It shows why one might want to view **V-cat** as a double category rather than as a 2-category or bicategory. We also discuss Verity's previous work on the subject, and begin the study of structured double categories.

In the final chapter, we review the results presented, discuss open questions, and point ways to future research.

Chapter 2

Monoidal Categories and Enriched Categories

In this chapter, we will give a brief introduction to the neccesary background elements of the thesis. The main purpose of this chapter is to fix terminology, as well as present key examples of monoidal and enriched categories that will be used throughout the thesis. For those looking for more detail, the book of Kelly ([27]) is a useful reference.

2.1 Monoidal Categories

A monoidal category is a category, together with a multiplication that is associative and unital to within coherence natural isomorphisms. Specifically, we have the following:

Definition A *monoidal category* is a category $\mathbf{C}$, together with a functor $\mathbf{C} \times \mathbf{C} \xrightarrow{\otimes} \mathbf{C}$ and an object $I \in \mathbf{C}$. In addition, for any objects $A, B, C \in \mathbf{C}$, we have natural associativity isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, and natural unital isomorphisms $r_A : A \otimes I \cong A$, $l_A : I \otimes A \cong A$. Furthermore, these isomophisms need to satisfy certain coherence conditions. Specifically, we need the following pentagon to commute:

$$
\begin{array}{ccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{a_{A,B,C} \otimes 1_D} & (A \otimes (B \otimes C)) \otimes D \\
{\scriptstyle a_{A \otimes B, C, D}}\Big\downarrow & & \Big\downarrow{\scriptstyle a_{A, B \otimes C, D}} \\
(A \otimes B) \otimes (C \otimes D) & & A \otimes ((B \otimes C) \otimes D) \\
{\scriptstyle a_{A,B,C \otimes D}}\searrow & & \swarrow{\scriptstyle 1_A \otimes a_{B,C,D}} \\
& A \otimes (B \otimes (C \otimes D)) &
\end{array}
$$

as well as the triangle

$$(A \otimes I) \otimes B \xrightarrow{\quad a_{A,I,B} \quad} A \otimes (I \otimes B)$$

$$r_A \otimes 1_B \searrow \qquad \swarrow 1_A \otimes l_B$$

$$A \otimes B$$

In future, for ease of use, we will omit the subscripts from the isomorphisms a, l, r.

There are numerous examples of monoidal categories; here we will name the ones that are most important for our work.

Example 2.1.1. The category **set** of sets, with product of sets as tensor product; the unit is a chosen set with a single element. More generally, any category with finite products can be made into a monoidal category, with $\otimes$ given by $\times$, and I by 1.

Example 2.1.2. As another example of using categorical product as monoidal product, the category of small categories **cat**, equipped with the product of categories and the unit category.

Example 2.1.3. The category **ab** of abelian groups, with the usual tensor product $\otimes$; the unit is the integers under addition.

Example 2.1.4. The category $\mathbf{vec}_k$ of vector spaces over a field k, with the usual tensor product $\otimes$; the unit is the base field k.

Example 2.1.5. The category $\mathbf{ban}_k$ of Banach spaces over a field k (in this case, k is either the real or complex numbers), with the projective tensor product $\otimes$; the unit is the base field k with its usual norm $|\cdot|$.

Example 2.1.6. The extended positive real numbers $\mathbf{R}_+ = [0, \infty]$, with arrows given by $\geq$, tensor being addition of real numbers, and the unit as 0. This example is very important for the link between analysis and enriched category theory, as we shall see below.

Example 2.1.7. Any bounded lattice considered as a category, where the arrows are the instances of inequalities. Here the tensor can be taken to be the sup or inf operation, with the unit being the bottom element or the top element, respectively. Two examples of this are $\mathbf{2} = (0 \leq 1, \wedge, 1)$ and $([0, \infty], \vee, 0)$.

Example 2.1.8. Any monoid $(M, \cdot, 1_M)$ can be considered as monoidal category $\mathbf{M}$, by taking the category to be the discrete category on the set M, and taking $I = 1$, $\otimes = \cdot$. The coherence axioms are automatically satisfied since the category is discrete.

Example 2.1.9. For a monoid $(M, \cdot, 1_M)$, one can also make the power-set of M into a monoidal category $\mathbf{M_P}$. Here, the arrows are instances of $\subseteq$, and for subsets A, B,

$$A \otimes B = \{a \cdot b : a \in A, b \in B\} \text{ and } I = \{1_M\}$$

For future use, we record a basic result about monoidal categories.

Proposition 2.1.10. *In any monoidal category* $(\mathbf{V}, \otimes, I)$, $l_I = r_I$.

Proof. See Joyal and Street ([24], pg. 23). ∎

2.1.1 Monoidal Functors

One might initially suppose that the most natural arrow between two monoidal categories would be one that preserves the unit and tensor to within specified isomorphisms. That is, a functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$ such that $FA \otimes FB \cong F(A \bullet B)$ and $FI \cong J$. However, while such functors do exist in nature, an even looser version exists that does arise in numerous examples, while remaining strong enough to discuss change of base questions. These monoidal functors merely involve a comparison between the tensor products.

Definition A *monoidal functor* between monoidal categories $(\mathbf{C}, \bullet, \mathbf{I})$ and $(\mathbf{D}, \otimes, \mathbf{J})$, is a functor $\mathbf{C} \xrightarrow{N} \mathbf{D}$, together with natural transformations, called comparison arrows,

$$NA \otimes NB \xrightarrow{\widetilde{N}_{A,B}} N(A \bullet B)$$

$$J \xrightarrow{N_0} NI$$

for which the following coherence diagrams commute, for every $A, B, C \in \mathbf{C}$:

$$
\begin{array}{ccc}
(NA \otimes NB) \otimes NC & \xrightarrow{a} & NA \otimes (NB \otimes NC) \\
\downarrow{\scriptstyle \widetilde{N} \otimes 1} & & \downarrow{\scriptstyle 1 \otimes \widetilde{N}} \\
N(A \bullet B) \otimes NC & & NA \otimes (NB \bullet NC) \\
\downarrow{\scriptstyle \widetilde{N}} & & \downarrow{\scriptstyle \widetilde{N}} \\
N((A \bullet B) \bullet C) & \xrightarrow{N(a)} & N(A \bullet (B \bullet C))
\end{array}
$$

$$
\begin{array}{ccc}
NA \otimes J & & \\
\downarrow{\scriptstyle 1 \otimes N_0} & \searrow{\scriptstyle r} & \\
NA \otimes NI \xrightarrow{\widetilde{N}} N(A \bullet I) & \xrightarrow{N(r)} & NA
\end{array}
$$

$$
\begin{array}{ccc}
J \otimes NA & & \\
\downarrow{\scriptstyle N_0 \otimes 1} & \searrow{\scriptstyle l} & \\
NI \otimes NA \xrightarrow{\widetilde{N}} N(I \bullet A) & \xrightarrow{N(l)} & NA
\end{array}
$$

If the comparison arrows are isomorphisms, then the monoidal functor is said to be *strong*; if the comparisons are identities, it is said to be *strict*.

Example 2.1.11. For any monoidal $\mathbf{V}$, there is always a monoidal functor from $\mathbf{V}$ to the base category **set**, given by homming out of the monoidal unit: $N = \mathbf{C}(I, -)$. This innocent-looking functor is actually quite important for most monoidal categories. For example, when $\mathbf{V} = \mathbf{ab}$, homming out of $I = Z$ gives the forgetful functor to **set**. Similarly, homming out of the base field for vector spaces also gives the forgetful functor. Not all examples are so straightforward, however. For example, homming out of the base field in the category of Banach spaces gives the unit ball functor. When $\mathbf{V}$ is graded R-modules, only a small amount of the original information is contained in this "forgetful" functor: the functor takes a graded R-module M to it's 0th component.

Example 2.1.12. For well-behaved $\mathbf{V}$, there is a "sub-object" monoidal functor from $\mathbf{V}$ to **set**, which gives the set of subgroups, or subspaces, or closed subspaces

in the categories of abelian groups, vector spaces, and Banach spaces, respectively. See Niefield ([36], pg 170) for futher details. In Niefield and Rosenthal [37], this assignment is extended to a monoidal functor from **V** to the monoidal category of sup lattices **sup**.

Example 2.1.13. For any $x, y \in [0, \infty]$, $x + y \geq x \vee y$, so the identity function is a monoidal functor from $([0, \infty], \geq, \vee)$ to $\mathbf{R}_+$.

Example 2.1.14. If M and N are monoids, thought of as discrete monoidal categories **M** and **N**, then a monoidal functor between them is a monoid homomorphism.

Example 2.1.15. Suppose that we have a monoid homomorphism $M \xrightarrow{f} N$. This then induces a pair of functors

$$\mathbf{M_P} \xleftarrow[f^{-1}]{\overset{Pf}{\frown}} \mathbf{N_P}$$

with $Pf \dashv f^{-1}$. Both of these are monoidal, with Pf strong.

Example 2.1.16. Both Tannakian categories (Deligne and Milne [15]) and Topological Quantum Field Theories (see Atiyah [1] and the reformulation in Kock [31]) can be described as strong monoidal functors to **vec**.

There are two more examples which will be important for the next chapter: ordered abelian groups, and normed abelian groups. We begin by reviewing the concept of ordered abelian groups, then show that ordered abelian groups are examples of monoidal functors.

Definition An ordered abelian group $(G, \leq)$ is an abelian group G, together with a preorder $\leq$ on G such that for all $g, h, x \in G$,

$$g \leq h \Rightarrow g + x \leq h + x$$

An alternative way of defining an ordered abelian group is by giving its "positive cone".

Proposition 2.1.17. *An order $\leq$ on an abelian group G is equivalent to giving a submonoid P of G (known as the positive cone of G).*

Proof. Suppose we have an order $\leq$ on G. Define $P = \{g \in G : 0 \leq g\}$. We need to show that P is a submonoid of G. Since $0 \leq 0$, $0 \in P$. If $g, h \in P$, then $0 \leq h \leq h + g$, so $h + g \in P$. This P is a submonoid.

Conversely, suppose we have a submonoid P of G. Define $\leq$ by $g \leq h$ if $h - g \in P$. Since $g - g = 0$, $\leq$ is reflexive. If we have $g \leq h \leq k$, then $(k-h)+(h-g) = k - g \in P$, so $\leq$ is transitive. If we have $g \leq h$, then $(h+x)-(g+x) = h - g \in P$, so $g + x \leq h + x$. Thus $\leq$ is an order on G. ∎

Note that the preorder is an order if the submonoid P is a "strict" submonoid, that is, $a \in P$ and $-a \in P$ implies $a = 0$.

With this characterization, we can show how monoidal functors are the same as orders on an abelian group.

Proposition 2.1.18. *If G is an abelian group, considered as a discrete monoidal category $\mathbf{G}$, then giving an order on G is equivalent to giving a monoidal functor from $\mathbf{G}$ to $\mathbf{2} = (0 \leq 1, \wedge, 1)$.*

Proof. Suppose that we have a monoidal functor $\mathbf{G} \xrightarrow{\ N\ } \mathbf{2}$. Define $P = N^{-1}\{1\}$. The unit comparison for the monoidal functor N gives $1 \leq N(0)$. Thus $N(0) = 1$, so $0 \in P$. The tensor comparison for N gives $Ng \wedge Nh \leq N(g + h)$. If $g, h \in P$, then $Ng = Nh = 1$. This forces $N(g + h) = 1$, so $g + h \in P$. Thus P is a submonoid of G, and so defines an order on G.

Conversely, suppose that we have an ordered abelian group G, together with its positive cone P. Define a monoidal functor N by $Ng = 1$ if $g \in P$, 0 otherwise. Since $\mathbf{G}$ is discrete, this defines a functor. Since $0 \in P$, we have $1 \in N(0)$, giving a unit comparison for N. To show that we have a tensor comparison, note that $Ng \wedge Nh = 1$ only when $g, h \in P$. In this case, $g + h \in P$, so $N(g + h) = 1$ also. If we have $Ng \wedge Nh = 0$, then we automatically have $Ng \wedge Nh \leq N(g + h)$. This gives the neccesary comparisons for N. Finally, since $\mathbf{2}$ is a poset, all diagrams are automatically satisfied, and so N is a monoidal functor. ∎

Thus, we have another important example of a monoidal functor. Our final example will be described in more detail in the next chapter, but is similar to our ordered abelian group example:

Example 2.1.19. A norm on an abelian group G is equivalent to giving a monoidal functor from $\mathbf{G}$ to $\mathbf{R}_+$.

2.2 Enriched Category Theory

Having monoidal structure on a category allows one to "enrich" in that category. In an enriched category, the homs $\mathbf{C}(C, D)$ are now objects of a monoidal category $\mathbf{V}$, rather than being sets. The category $\mathbf{V}$ is required to be monoidal[1] so that one can define composition and identities of these enriched categories.

Definition A $\mathbf{V}$-enriched category $\mathbf{C}$ consists of the following data: a set of objects C, together with, for any $a, b \in C$, an object $\mathbf{C}(a, b) \in \mathbf{V}$. In addition, the enriched category has composition arrows

$$\mathbf{C}(A, B) \otimes \mathbf{C}(B, C) \xrightarrow{C_{(A,B,C)}} \mathbf{C}(A, C)$$

and identity arrows:

$$I \xrightarrow{1_A} \mathbf{C}(A, A)$$

(note that these arrows are in $\mathbf{V}$). The composition must be associative, so that the following diagram commutes:

$$
\begin{array}{ccc}
(\mathbf{C}(a, b) \otimes \mathbf{C}(b, c)) \otimes \mathbf{C}(c, d) & \xrightarrow{a} & \mathbf{C}(a, b) \otimes (\mathbf{C}(b, c) \otimes \mathbf{C}(c, d)) \\
\downarrow{c \otimes 1} & & \downarrow{1 \otimes c} \\
\mathbf{C}(a, c) \otimes \mathbf{C}(c, d) & & \mathbf{C}(a, b) \otimes \mathbf{C}(b, d) \\
& \searrow{c} \quad \swarrow{c} & \\
& \mathbf{C}(a, d) &
\end{array}
$$

---

[1]The Eilenberg-Kelly notion of a *closed* category is also sufficient to be able to enrich in. However, in this thesis, we will only consider monoidal $\mathbf{V}$ categories, as they are more commonly used.

and unitary, so that the following diagrams commute:

$$
\begin{array}{ccc}
I \otimes \mathbf{C}(a,b) & & \\
\downarrow \scriptstyle{1_A \otimes 1} & \searrow \scriptstyle{l} & \\
\mathbf{C}(a,a) \otimes \mathbf{C}(a,b) & \xrightarrow{\ c\ } & \mathbf{C}(a,b)
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{C}(a,b) \otimes I & & \\
\downarrow \scriptstyle{1 \otimes 1_A} & \searrow \scriptstyle{r} & \\
\mathbf{C}(a,b) \otimes \mathbf{C}(a,a) & \xrightarrow{\ c\ } & \mathbf{C}(a,b)
\end{array}
$$

For most of the monoidal categories mentioned above, the categories enriched in them are quite familiar.

Example 2.2.1. A **set**-category is a locally small category.

Example 2.2.2. If we take **1** to be the 1-object, 1 (identity) arrow category, with the trivial tensor product, then a **1**-category is a set.

Example 2.2.3. A **cat**-category is a 2-category.

Example 2.2.4. An **ab**-category is known as a pre-additive category in the literature. There are numerous examples; some common ones are **ab** itself, $\mathbf{vec_k}$, and the category of finite dimensional representations of an algebraic group. A one-object **ab**-category is a ring (see Proposition 3.1.6 for proof of this).

Example 2.2.5. $\mathbf{vec_k}$ is itself a $\mathbf{vec_k}$-category. In addition, a one-object $\mathbf{vec_k}$-category is a k-algebra.

Example 2.2.6. As for vector spaces, **ban** is itself a **ban**-category, and one-object **ban**-categories are Banach algebras. For another example, the category of all Hilbert spaces and bounded linear maps between them is a **ban**-category.

Example 2.2.7. For a monoid M, a M_P-category can be thought of as the dynamics of a non-deterministic automata (see, for example, Kasangian and Rosebrugh [25]). The objects of a M_P-category $\mathbf{X}$ are thought of as the states of the automata, the elements of M the inputs, and the homs $\mathbf{X}(x,y)$ are the set of inputs which take state x to state y.

Example 2.2.8. A **2**-category is a partially ordered set (which is not neccesarily anti-symmetric).

Example 2.2.9. An $\mathbf{R}_+$-category is a slightly generalized metric space. On the other hand, a $([0, \infty], \geq, \vee)$-category is an ultrametric space.

Let us expand slightly on this idea of $\mathbf{R}_+$ categories being metric spaces, as it is important for the motivation. A category enriched over $([0, \infty], \geq, +, 0)$ is a set X, together with a function $X \times X \xrightarrow{\ d\ } [0, \infty]$ such that:

1. $d(x, x) = 0$ (the identity arrow)

2. $d(x, y) + d(y, z) \geq d(x, z)$ (the composition arrow)

It differs from the classical metric spaces in three ways:

1. $d(x, y) = 0 \nRightarrow x = y$ (isomorphic objects are not neccesarily equal)

2. d can take the value ∞ (completeness of the base category)

3. $d(x, y) \neq d(y, x)$ (non-symmetry)

In his paper, Lawvere gives good reasons why this version of metric space should be preferred to the classical version. As an example, if one wishes one's metric to be the amount of work it takes to walk in a mountainous region, it should be non-symmetric. In addition, the fact that Lawvere's metric spaces are non-symmetric will be important for us in the next chapter.

2.2.1   V-Functors

In addition to enriched categories, one can also formulate the notion of functor between enriched categories.

Definition A $\mathbf{V}$-functor F between $\mathbf{V}$-categories $\mathbf{C}, \mathbf{D}$ consists of a function $C \xrightarrow{\ F\ } D$, as well as arrows

$$\mathbf{C}(a, b) \xrightarrow{\ F(a,b)\ } \mathbf{D}(Fa, Fb)$$

in **V**, sometimes called the "effect of F on homs" or simply the "strength" of F. These assignments must preserve composition:

$$\begin{array}{ccc}
\mathbf{C}(a,b) \otimes \mathbf{C}(b,c) & \xrightarrow{\ F \otimes F\ } & \mathbf{D}(Fa,Fb) \otimes \mathbf{D}(Fb,Fc) \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\
\mathbf{C}(a,c) & \xrightarrow{\quad F \quad} & \mathbf{D}(Fa,Fc)
\end{array}$$

and identities:

$$\begin{array}{ccc}
I & & \\
{\scriptstyle 1_a}\downarrow & \searrow{\scriptstyle 1_{Fa}} & \\
\mathbf{C}(a,a) & \xrightarrow{\ F\ } & \mathbf{D}(Fa,Fa)
\end{array}$$

The maps $F(a,b)$ will usually simply be written as F.

For most of the enriched categories mentioned above, the enriched functors are as to be expected; however, there are a few interesting examples.

Example 2.2.10. **set**-functors are ordinary functors.

Example 2.2.11. **1**-functors are functions.

Example 2.2.12. **cat**-functors are 2-functors.

Example 2.2.13. **ab**-functors are functors which preserve the addition of the arrows. They include some well known mathematical constructs: if R is a ring, thought of as one-object **ab**-category **R**, then an **ab**-functor from **R** to **ab** is the same as an R-module (for proof, see Proposition 3.1.6). If R and S are both rings, then an **ab**-functor between **R** and **S** is the same as a ring homomorphism from R to S.

Example 2.2.14. A **2**-functor between ordered sets is an order-preserving function.

Example 2.2.15. A $\mathbf{R}_+$-functor F between metric spaces (X, d_X) and (Y, d_Y) is a contractive (non-expansive) function: $d_Y(fx, fy) \leq d_X(x, y)$.

2.2.2 V-Natural Transformations

In an enriched category, we cannot choose individual arrows. Thus, enriched natural transformations are slightly more complicated to define than ordinary natural transformations. We must use use the idea that arrows in $\mathbf{V}$ from I to $C(a,b)$ take the place of morphisms from a to b.

Definition Given $\mathbf{V}$ functors $\mathbf{C} \xrightarrow{F,G} \mathbf{D}$, a $\mathbf{V}$-*natural transformation* $F \xrightarrow{\sigma} G$ consists of a family of C-indexed maps $I \xrightarrow{\sigma_c} D(Fc, Gc)$. These are to satisfy the following $\mathbf{V}$-naturality condition:

$$
\begin{array}{ccc}
\mathbf{C}(c,d) & \xrightarrow{\;\;l^{-1}\;\;} & I \otimes \mathbf{C}(c,d) \\
{\scriptstyle r^{-1}}\downarrow & & \downarrow{\scriptstyle \sigma_c \otimes G} \\
\mathbf{C}(c,d) \otimes I & & \mathbf{D}(Fc,Gc) \otimes \mathbf{D}(Gc,Gd) \\
{\scriptstyle F \otimes \sigma_d}\downarrow & & \downarrow{\scriptstyle c} \\
\mathbf{D}(Fc,Fd) \otimes \mathbf{D}(Fd,Gd) & \xrightarrow{\;\;c\;\;} & \mathbf{D}(Fc,Gd)
\end{array}
$$

2.3 The 2-Category V-cat

Taken together, $\mathbf{V}$-categories, $\mathbf{V}$-functors, and $\mathbf{V}$-natural transformations form a 2-category. We will describe each of the composites; the proof that these form a 2-category is found in ([16], pg. 466).

The composition and identities of $\mathbf{V}$-functors are relatively straightforward:

Definition Given the following $\mathbf{V}$-functors

$$
\mathbf{C} \xrightarrow{\;\;F\;\;} \mathbf{D} \xrightarrow{\;\;G\;\;} \mathbf{E}
$$

their composite GF is defined as being the composite function on objects, together

with the following strength:

$$
\begin{array}{ccc}
\mathbf{C}(c,d) & & \\
\Big\downarrow{\scriptstyle F(c,d)} & \searrow{\scriptstyle (GF)(c,d)} & \\
\mathbf{D}(Fc,Fd) & \xrightarrow{\ G(Fc,Fd)\ } & \mathbf{E}(GFc,GFd)
\end{array}
$$

The identity $\mathbf{V}$-functor is the identity on objects, and it's strength is the identity arrow.

The horizontal and vertical composites of $\mathbf{V}$-natural transformations are only slightly more complicated.

Definition Given the following $\mathbf{V}$-categories, $\mathbf{V}$-functors, and $\mathbf{V}$-natural transformations:

$$
\mathbf{C} \xrightarrow{\ \ G\ \ } \mathbf{D}
$$

the *vertical composite* of σ_1 and σ_2 is a $\mathbf{V}$-natural transformation from F to H, and has the following components:

$$
\begin{array}{ccc}
I \cong I \otimes I & & \\
\Big\downarrow{\scriptstyle (\sigma_1)_c \otimes (\sigma_2)_c} & \searrow{\scriptstyle (\sigma_2 \sigma_1)_c} & \\
\mathbf{D}(Fc,Gc) \otimes \mathbf{D}(Gc,Hc) & \xrightarrow{\ c\ } & \mathbf{D}(Fc,Hc)
\end{array}
$$

Definition Given the following $\mathbf{V}$-categories, $\mathbf{V}$-functors, and $\mathbf{V}$-natural transformations:

$$
\mathbf{C} \quad \Downarrow \sigma_1 \quad \mathbf{D} \quad \Downarrow \sigma_2 \quad \mathbf{E}
$$

the *horizontal composite* of σ_1 and σ_2 is a $\mathbf{V}$-natural transformation from $F_2 F_1$ to $G_2 G_1$, and has the following components:

$$I \cong I \otimes I$$

$$\downarrow {\scriptstyle (\sigma_1)_c \otimes (\sigma_2)_{G_1 c}}$$

$$\mathbf{D}(F_1 c, G_1 c) \otimes \mathbf{E}(F_2 G_1 c, G_2 G_1 c) \qquad {\scriptstyle (\sigma_2 \sigma_1)_c}$$

$$\downarrow {\scriptstyle F_2 \otimes 1}$$

$$\mathbf{E}(F_2 F_1 c, F_2 G_1 c) \otimes \mathbf{E}(F_2 G_1 c G_2 G_1 C) \xrightarrow{\quad c \quad} \mathbf{E}(F_2 F_1 c, G_2 G_1 c)$$

This is $\sigma_2 G_1 \cdot F_2 \sigma_1$. There is a similar description of the horizontal composite as $G_2 \sigma_1 \cdot \sigma_2 F_1$; these two descriptions are the same.

Chapter 3

Normed Spaces

In this chapter, we will look at two ways of generalizing the notion of "normed space" to a wider categorical context. As we saw in the previous chapter, one can think of metric spaces as enriched categories, allowing for an interplay of ideas between the two subjects. Michael Barr ([2]), and Walter Tholen, Maria Clementino, and others ([10], [42], [41]) have extended this idea by showing that as metric spaces correspond to $\mathbf{V}$-categories, so topological spaces correspond to a new notion: (T, V)-algebras. In this chapter, we generalize in the the opposite direction, by trying to determine what normed spaces should correspond to. Two answers are given: normed spaces as monoidal functors, and normed spaces as compact $\mathbf{V}$-categories. After investigating these two ideas, we will see that a comparison of the two requires a deeper understanding of the change-of-base functor for enriched categories, which leads us to the rest of the thesis.

3.1 Norms as Monoidal Functors

In this section, we will look at one way of generalizing normed spaces, via monoidal functors. The key idea is the relationship between normed vector spaces and a weakened version of them, normed abelian groups.

3.1.1 Normed Vector Spaces and Normed Abelian Groups

We begin by stating a (slightly modified) definition of normed vector space.

Definition A *normed vector space* is a vector space A (over $\mathbb{R}$ or $\mathbb{C}$), together with a function $A \xrightarrow{\|\cdot\|} [0, \infty]$ such that

1. $\|0\| = 0$

2. $\|a\| + \|b\| \geq \|a + b\|$

3. $\|\alpha a\| = |\alpha| \|a\|$

We have modified the usual definition in two ways: we are allowing the norm to take value ∞ (as in Lawvere metric spaces) as well as only requiring a semi-norm (that is, we do not require that $\|a\| = 0$ implies $a = 0$).

Now, we cannot directly categorify the idea of normed vector space. However, what we can do is show that the idea of a normed vector space is contained in a weaker notion, namely that of a normed abelian group. As we shall see, the idea of normed abelian group is more amenable to categorifcation. Our source for the idea of normed abelian groups is Grandis' article [20].

Definition A *normed abelian group* is an abelian group G, together with a norm $G \xrightarrow{\|\cdot\|} [0, \infty]$ which satisfies 1. and 2. for normed vector spaces. Say the norm is *symmetric* if $\|a\| = \| - a\|$.

Here are a few examples of normed abelian groups, including a non-symmetric one:

Example 3.1.1. The abelian group of integers $\mathbb{Z}$, with the usual absolute value.

Example 3.1.2. The abelian groups of integers $\mathbb{Z}$, but with the following (non-symmetric) norm:
$$\|a\| = \begin{cases} a & \text{if } a \geq 0; \\ \infty & \text{if } a < 0. \end{cases}$$
This example is important for the category of normed abelian groups (see Proposition 3.1.7).

Example 3.1.3. The abelian groups of real numbers and of complex numbers, with their usual absolute value.

Example 3.1.4. Any normed vector space has an underlying normed abelian group.

With the notable recent exception of Marco Grandis' work on normed homology [19] (where instead of a sequence of abelian groups, one assigns a sequence of normed abelian groups to a space), normed abelian groups have not often been considered by the mathematics community. However, one reason that they may not have been considered is that the metric they define $d(a, b) := \|b - a\|$ is not a metric in the classical sense - it is not symmetric unless the norm is itself symmetric. It does, however, define one of the more general Lawvere metrics.

Proposition 3.1.5. *If $(A, \| \cdot \|)$ is a normed abelian group, then $d(a, b) := \|b - a\|$ defines a (Lawvere) metric on A. The metric it defines is symmetric if and only if the norm is symmetric.*

Proof. The triangle inequality follows 2. for the norm:

$$d(a, b) + d(b, c) = \|b - a\| + \|c - b\| \geq \|b - a + c - b\| = \|c - a\| = d(a, c)$$

while the unit axiom follows from 1.:

$$d(a, a) = \|a - a\| = \|0\| = 0$$

Thus d is a metric on A.

If the norm is symmetric, then

$$d(a, b) = \|b - a\| = \|a - b\| = d(b, a),$$

so that the metric is symmetric, while if the metric is symmetric, then

$$\| - a\| = d(0, a) = d(a, 0) = \|a\|$$

shows that the norm is symmetric. ∎

Now we would like to show how one can recover the idea of normed vector space from normed abelian group. The following shows how one can recover mere vector spaces (more generally, modules) from abelian groups:

Proposition 3.1.6. ab *is itself an* **ab***-category, and the following holds:*

*1. A one-object **ab**-category is a ring.*

*2. An **ab**-functor from a ring R to **ab** is an R-module.*

Proof. Since **ab** is closed, it is itself an **ab**-category. For 1., suppose that we have a one-object **ab**-category $\mathbf{C}$. Let its one object be *, and let $\mathbf{C}(*, *) := R$, so that R is an abelian group. Then the composition is a single bi-linear map $R \otimes R \xrightarrow{\times} R$, while the unit $\mathbb{Z} \longrightarrow R$ is simply an element $1 \in R$. The associativity and unitary axioms for a ring are exactly the associativity and unitary axioms for the composition and unit of this **ab**-category.

Suppose that F is an **ab**-functor from a one-object **ab**-category $\mathbf{R}$ to **ab**. Let M denote $F(*)$. Then the effect of F on homs gives a group homomorphism

$$R \xrightarrow{\quad -\cdot-\quad} \mathbf{ab}(M, M),$$

defining an action of R on M. Then each of the axioms for an R-module exactly are the same as saying F is an **ab**-functor:

- $r(m + n) = rm + rn$ is equivalent to asking that $F(r)$ be an **ab**-morphism,

- $(r + s)m = rm + sm$ is equivalent to asking that F is itself an **ab**-morphism,

- $(rs)m = r(sm)$ is equivalent to asking that F preserves composition,

- $1m = m$ is equivalent to asking that F preserves units.

Thus, giving an **ab**-functor from $\mathbf{R}$ to **ab** is the same as giving an R-module. ∎

Thus, the idea of modules and vector spaces is contained in the category **ab** and categories enriched in **ab**. Next, we show that we can do the same thing with normed vector spaces and normed abelian groups. In other words, we will show that normed modules and vector spaces are contained in the category of normed abelian groups **normab** and categories enriched in **normab**.

Definition Let **normab** be the category with objects normed abelian groups, and maps group homomorphisms f which are also linear contractions ($\|fa\| \leq \|a\|$).

First, however, we must define the monoidal categorical structure of normed abelian groups, which is simply an extension of the usual projective tensor product for normed linear spaces.

Proposition 3.1.7. *The following defines a norm on $A \otimes B$:*

$$\|z\| := \bigwedge \left\{ \sum \|a_i\| \|b_i\| : z = \sum a_i \otimes b_i \right\}$$

This then defines a tensor product on **normab**, *such that with unit object that of Example 3.1.2,* **normab** *becomes a monoidal category.*

Proof. See Grandis [20], pgs 10-11. ∎

We also need to define the idea of normed ring and normed module:

Definition A *normed ring R* is a ring whose underlying abelian group has a norm $|\cdot|$ on it, and has the additional axioms

$$|ab| \leq |a||b| \text{ and } |1_R| \leq 1$$

A *normed module M* over a normed ring $(R, |\cdot|)$ is an R-module, whose underlying abelian group has a norm $\|\cdot\|$ on it, and has the additional axiom

$$\|ra\| \leq |r| \|a\|$$

Example 3.1.8. Both $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ are normed rings.

Example 3.1.9. The finite field $\mathbb{Z}_p$ is a normed ring when given the norm $\|[a]\| := a$.

Example 3.1.10. Any Banach algebra is a normed ring. For example, the set of bounded linear operators on a Hilbert space $B(H)$, or the set of continous functions on a compact set $C(X)$.

We can now prove a result that parallels Proposition 3.1.6.

Proposition 3.1.11. normab *is itself a* **normab**-*category, and the following holds:*

    1. A one-object **normab**-*category is a normed ring $(R, |\cdot|)$.*

2. An **normab**-*functor from* $(R, |\cdot|)$ *to* **normab** *is a* $(R, |\cdot|)$-*normed module.*

Proof. Equipping **normab** with the operator norm makes it into a **normab**-category. We have already seen in Proposition 3.1.6 how one-object **ab**-categories are rings. However, the unit and multiplication maps are now contractions, which gives

$$\|1\| \leq 1 \text{ and } \|ab\| \leq \|a\|\|b\|$$

Thus a one-object **normab** category is a normed ring.

We know that an **ab**-functor from $\mathbf{R}$ to **ab** gives an R-module. The fact that the scalar multiplication map is now a contraction also gives that

$$\|ra\| \leq |r|\|a\|$$

so that we get a normed module. ∎

However, one may have noticed that the definition of normed module differs from that of normed vector space: it only requires an inequality for the norm of a scalar multiple, rather than the usual equality. However, if the normed ring is $(\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$, then the idea of normed module and normed vector space coincide. More generally, if the normed ring is a field, and the norm preserves inverses, then a normed module over that normed ring has the usual scalar invariance.

Definition A *normed field* k is a normed ring $(k, |\cdot|)$ such that for all non-zero $x \in k$, $|x^{-1}| = |x|^{-1}$.

Example 3.1.12. Both $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ are normed fields.

Example 3.1.13. The finite field $\mathbb{Z}_p$ (with norm given above) is *not* a normed field.

Proposition 3.1.14. *If k is a normed field, then a k-normed module is the same as normed vector space.*

Proof. We only need to show that the scalar invariance inequality implies the scalar invariance equality. Indeed, we have

$$\|\alpha a\| \leq |\alpha|\|a\| = |\alpha|\|\alpha^{-1}\alpha a\| \leq |\alpha||\alpha^{-1}|\|\alpha a\| = \|\alpha a\|$$

where the last equality follows from the axiom for a normed field. ∎

In summary, we have shown that just as the idea of vector space can be recovered from the category of abelian groups, so the idea of normed vector space can be recovered from the category of normed abelian groups: a normed vector space is simply a **normab** functor. In addition, we have defined a number of interesting new concepts such as normed module, which extends the usual notion of normed vector space by only requiring sub-scalar invariance.

Our task now is to categorify the notion of normed abelian group.

3.1.2 Norms as Monoidal Functors

In this section, we will show how the axioms for a normed abelian group can be expressed in a more general categorical form. To do this, we note that an instance of $\geq$ is really an arrow in $[0, \infty]$, the addition is tensor, and 0 is the identity I. Thus the axioms for a norm on an abelian group become, if we write $\| \cdot \|$ as N:

1. $N(a) \otimes N(b) \longrightarrow N(a + b)$

2. $I \longrightarrow N(0)$

These two axioms are exactly the neccesary comparison arrows for a monoidal functor (see Definition 2.1.1). In other words, if we make the abelian group G into a discrete category $\mathbf{G}$, with $+$ as $\otimes$, and 0 as I, then a monoidal functor from $\mathbf{G}$ to $[0, \infty]$ is a norm on G (note that the coherence axioms follow automatically since the codomain category is a poset).

Of course, there is one additional piece of information that is not considered in this analysis, namely the fact that G is not just a monoid, but is actually a group. To make use of this, we note that $\mathbf{G}$ considered as a monoidal category is actually compact closed, with $* = -$. Thus we have that a normed abelian group consists of an compact closed category $\mathbf{G}$, together with a monoidal functor to $\mathbf{R}_+$. We thus generalize to make the following definition:

Definition For **V** a monoidal category, a **normed space over V** is a compact closed category **C**, together with a monoidal functor **C** $\xrightarrow{N}$ **V**.

Before we proceed, let us determine if this makes sense. We claim that as metric space is to category enriched over **V**, so normed abelian group is to normed space over **V**. However, as we have seen earlier (Proposition 3.1.5), every normed abelian group defines a metric space via $d(a, b) := N(b - a)$. So, by analogy, if this were to make sense, every normed space over **V** should define a category enriched over **V**, in the same way that a normed space defined a metric space.

Proposition 3.1.15. *Let* $(\mathbf{C}, N)$ *be a normed space over* **V**. *Then we can define a* **V**-*categorical structure on* **C**, *with homs given by* $\mathbf{C}(c, d) := N([c, d])$.

Proof. Since **C** is compact closed, it is closed, and so enriched over itself. Since N is a monoidal functor, it preserves enrichment (see Proposition 4.2.1), and so just as $d(a, b) := N(a - b)$ defines a metric space, so $\mathbf{C}(c, d) := N([c, d])$ defines an enrichment of **C** over **V**. ∎

In turn, this allows one to generalize the notion of Banach space:

Definition Suppose that $(\mathbf{C}, N)$ is a normed space over **V**. Say that $(\mathbf{C}, N)$ is a *Banach space over* **V** if the **V**-category $N_*\mathbf{C}$ is Cauchy complete.

There are a number of interesting examples of normed spaces over a monoidal category.

Example 3.1.16. Recall that a category enriched over $\mathbf{2} := (0 \leq 1, \wedge, 1)$ is a partially ordered set. If G is an abelian group (considered as a discrete monoidal category **G**) then, by Proposition 2.1.18, a monoidal functor from **G** to **2** is an ordered abelian group. So ordered abelian groups are examples of normed spaces over **2**.

Example 3.1.17. Every compact closed category **C** is normed over **set**, via $N(-) := \mathbf{C}(I, -)$. For example, the norm of a finite vector space is its underlying set. If we extend our definition of normed spaces to include merely closed categories with a

monoidal functor, then the norm of a Banach space would be its unit ball, which agrees with ideas from functional analysis.
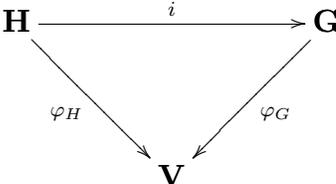
Example 3.1.18. Tannakian categories (see [15]) are strong monoidal functors from a compact monoidal category to **vec**, so are examples of normed spaces over **vec**.

In summary, we have generalized the notion of normed abelian group (which contains the notion of normed vector space) to a general categorical context, in which there are a number of other interesting examples.

3.1.3 Subgroups and Quotient Stuctures

In this section, we will look at how one can extend a few of the other elements of normed/ordered abelian groups to the wider categorical context.

Suppose that H is a subgroup of a normed/ordered group (G, φ_G), and i the inclusion map. Then H inherits an ordered or normed stucture in the obvious way:

$$
\begin{array}{ccc}
\mathbf{H} & \xrightarrow{\ i\ } & \mathbf{G} \\
& \searrow \varphi_H \quad \varphi_G \swarrow & \\
& \mathbf{V} &
\end{array}
$$

That is, $\varphi_H := \varphi_G \circ i$. In the case of normed groups, the norm on H is simply the restriction of the norm from G. In the case of ordered groups, the positive cone of H is the intersection of the positive cone of G with H.

Subgroups thus present no difficulty. The picture is a little more complicated with quotient groups, however. Again, let H be a normal subgroup of (G, φ), and let $[\cdot]$ be the quotient map. Then we have the following picture:

$$
\begin{array}{ccc}
\mathbf{G} & \xrightarrow{\ [\cdot]\ } & \mathbf{G/H} \\
& \searchdownarrow{\varphi} & \\
& \mathbf{V} &
\end{array}
$$
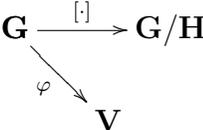
As one can see, there is no direct composition that gives a potential order/norm on G/H. However, since our **V** in either case is co-complete, we can try the left Kan

extension of φ along $[\cdot]$, and see what that gives us in each case. The formula for the left Kan extension (call it L), applied to this case, gives

$$L([x]) = \int^{g \in G} \mathbf{G}/\mathbf{H}([g], [x]) \cdot \varphi(g)$$

The co-end in $[0, \infty]$ is simply the inf. Since the category $\mathbf{G}/\mathbf{H}$ is discrete, the hom-set $\mathbf{G}/\mathbf{H}([g], [x])$ is only non-trivial if $[g] = [x]$. Thus the above reduces to

$$L([x]) = \bigwedge \{\varphi(g) : [g] = [x]\}$$

Alternatively, this can be re-written as

$$L([x]) = \bigwedge \{\varphi(x + h) : h \in H\}$$

This last expression is in fact the usual quotient norm (Conway [12], pg. 70).

The same idea applied to ordered groups also gives the correct structure. In the case of ordered groups, the left Kan extension becomes

$$L([x]) = \bigvee \{\varphi(g) : [g] = [x]\}$$

Thus $[x]$ is in the positive cone of G/H when $[x] = [p]$ for some p in the positive cone of G. That is, $P(G/H) = [P(G)]$. As for normed groups, this is the standard ordered structure on the quotient (Blyth [4], pg. 147).

Thus, both subgroups and quotients of ordered and normed groups have general categorical expressions which reduce to the familiar notions in both cases. This gives an example of why the idea of normed spaces as monoidal functors has potential as an interesting general theory. In addition, it shows that the theories of ordered structures and normed structures are more closely related than may at first appear.

3.2 Norms as Enriched Compact Spaces

Let us now investigate a slightly different point of view. Returning to Lawvere's paper on metric spaces as enriched categories, we find the following:

"...although for any given **V** we could consider "arbitrary" **V**-valued structures, there is one type of such structure which is of first importance, namely for **V** respectively truth-values [**V** = **2**], quantities [**V** = **R**$_+$], abstract sets, abelian groups, the structure of respectively poset, metric space, category, additive category...is the generally useful first approximation possible with **V**-valued logic for analyzing various problems; it even seems that there is a natural second approximation, namely the structure of "[**V**-compact closed] **V**-category" which in the four cases mentioned specializes *roughly* to partially ordered abelian group, normed abelian group, [compact closed] category, and (in the additive case) to a common generalization of the category of [projective] modules on an algebraic space and the category of finite-dimensional representations of an algebraic group. Detailed discussion of this second approximation awaits further investigation..."

What Lawvere is saying is that as metric spaces are to enriched categories, so normed abelian groups are to **V**-compact closed **V**-categories. In other words, his version of a normed space is a **V**-compact closed **V**-category (for the definition of a **V**-compact closed **V**-category, see Definition 7.1). Before we see how a **V**-compact closed **V**-category could be thought of as a normed abelian group, we need to do a little preliminary work. To begin with, with introduce a non-symmetric version of one of the standard metric space axioms.

Definition Say that a **R**$_+$-category **X** has identity of indiscernibles (IOI) if, for all $x, y \in \mathbf{X}$,

$$d(x, y) = 0 = d(y, x) \Rightarrow x = y$$

These metric spaces have the following useful property:

Lemma 3.2.1. *Suppose* **Y** *has IOI. Then for any* **R**$_+$*-functors* $\mathbf{X} \xrightarrow{F,G} \mathbf{Y}$*, if* F *is naturally isomorphic to* G*, then* $Fx = Gx$ *for all* $x \in \mathbf{X}$*.*

Proof. Since we have a **V**-natural transformation $F \longrightarrow G$, we get a family of maps $I \longrightarrow \mathbf{Y}(Fx, Gx)$. Since $I = 0$ and arrows are $\geq$, this implies $0 = \mathbf{Y}(Fx, Gx)$.

Similarly, the existence of a **V**-natural transformation $G \longrightarrow F$ gives $0 = \mathbf{Y}(Gx, Fx)$. Thus since **Y** has IOI, $Fx = Gx$ for all $x \in \mathbf{X}$. ∎

The following then shows how normed abelian groups relate to **V**-compact closed **V**-categories.

Proposition 3.2.2. *A R_+-compact closed R_+-category with IOI is a normed abelian group.*

Proof. First, suppose that **X** is a symmetric monoidal **V**-category. This gives **V**-functors

$$\mathbf{X} \otimes \mathbf{X} \xrightarrow{\quad + \quad} \mathbf{X}, \mathbb{I} \xrightarrow{\quad 0 \quad} \mathbf{X}$$

together with the natural associtivity and unit isomorphisms. Since **X** has IOI, this forces $x + 0 = x = 0 + x$ and $x + (y + z) = (x + y) + z$. The symmetry forces x + y = y + x.

If we now assume that **X** is compact closed, then we have an equivalence $\mathbf{X}^{\mathrm{op}} \xrightarrow{\quad - \quad} \mathbf{X}$ such that

$$\mathbf{X}(x, y + (-z)) \cong \mathbf{X}(x + y, z)$$

Taking $x = 0, y = x$, and $z = x$, we get $\mathbf{X}(0, x + (-x)) \cong \mathbf{X}(x, x) = 0$. Similarly, taking $x = -x, y = x, z = 0$ gives $0 = \mathbf{X}(-x, -x) \cong \mathbf{X}(-x + x, 0)$. Thus, since **X** has IOI, $x + (-x) = 0$. So **X** is an abelian group.

Finally, we can define a norm on this stucture by $\|x\| := \mathbf{X}(0, x)$. The triangle axiom for a norm follows by the **V**-functoriality of $+$. ∎

Thus while not all $\mathbf{R}_+$-compact closed $\mathbf{R}_+$ categories are normed abelian groups, those which have one of the standard metric space axioms are. The notion of **V**-compact closed **V**-category is thus another generalization of the concept of normed abelian group.

3.3   Comparison

We have now seen two versions of $\mathbf{V}$-normed space: a normed space as a compact closed category with a monoidal functor to $\mathbf{V}$, and a normed space as a $\mathbf{V}$-compact closed $\mathbf{V}$-category. Each of these represents different aspects of a normed space: the first, the aspect of being a space with a norm; the second, a metric space with additional structure.

We must now attempt to determine how similar they are for general $\mathbf{V}$, or at the very least if we can transfer one such structure to the other, and vice versa.

So, begin by supposing that we have a $\mathbf{V}$-compact closed category with unit J. For any monoidal $(\mathbf{V}, \otimes, I)$, the functor

$$\mathbf{V} \xrightarrow{\;\mathbf{V}(I,-)\;} \mathbf{set}$$

is monoidal, so it induces a change-of-base from $\mathbf{V}\text{-cat}$ to $\mathbf{cat}$. If the change-of-base functor preserves compact categories, then the resulting (ordinary) category $\mathbf{X}_0$ will also be compact. Moreover, we can also apply the change-of-base to the $\mathbf{V}$-monoidal functor $\mathbf{X} \xrightarrow{\;\mathbf{X}(J,-)\;} \mathbf{V}$ (recall from above that this was the norm of $\mathbf{X}$) to get a monoidal functor $\mathbf{X}_0 \longrightarrow \mathbf{V}$. Thus, from a compact closed $\mathbf{V}$-category, we have formed a normed space over $\mathbf{V}$.

Now suppose that we have a normed space over $\mathbf{V}$, $(\mathbf{C}, N)$. Since $\mathbf{C}$ is compact, it is closed, and so is itself a $\mathbf{C}$-category. We then apply the change-of-base N_* to $\mathbf{C}$ to get a $\mathbf{V}$-category. If the change of base preserves compact categories, this will be $\mathbf{V}$-compact.

As one can see, transferring these structures back and forth requires knowing more about the change-of-base functor; in particular, we need to know if it preserves compact closed categories. In general, a deeper investigation of the change-of-base is required.

Chapter 4

Classical Change of Base for Enriched Categories

In this chapter, we will describe the classical change-of-base functor. Most of the ideas in this chapter can be found in Eilenberg and Kelly's original article on the subject, "Closed Categories" [16]. The proofs given here, however, are original. They simplify the original proofs by defining and using the idea of applying a monoidal functor "monoidally". The first section describes this idea.

In addition, there are some new results in this chapter. Specifically, in Eilenberg and Kelly's article, they only show that the change of base 2-functor $(-)_*$ is a 2-functor between monoidal categories and categories. Here, we show a more general result, namely that $(-)_*$ can be seen as a 2-functor between monoidal categories and 2-categories. We will also give an example of how one could apply this result.

4.1   Coherence Theorems for Monoidal Functors

Before we begin proving results about change of base, it will be very helpful to prove several key lemmas regarding monoidal functors, as well as describe notation that we will use throughout. When working with monoidal functors, and in particular change of base, one often needs to apply a monoidal functor N "monoidally". There are two ways to apply a monoidal functor monoidally:

Definition Let $(\mathbf{V}, \otimes, \mathbf{I}) \xrightarrow{\ N\ } (\mathbf{W}, \bullet, \mathbf{J})$ be a monoidal functor. Given an arrow of the following type in $\mathbf{V}$:

$$A \otimes B \xrightarrow{\ f\ } C$$

we *apply N monoidally to f* to get

$$NA \bullet NB \xrightarrow{\ Nf\ } NC,$$

defined as the following composite:

$$NA \bullet NB$$

$$\downarrow \widetilde{N} \qquad \searrow \underline{N}f$$

$$N(A \otimes B) \xrightarrow{\quad N(f) \quad} NC$$

Given an arrow of the following type in $\mathbf{V}$:

$$I \xrightarrow{\ g\ } A$$

we *apply N monoidally to g* to get

$$J \xrightarrow{\ Ng\ } NA,$$

defined as the following composite:

$$J$$

$$\downarrow N_0 \qquad \searrow \underline{N}(g)$$

$$NI \xrightarrow{\quad N(g) \quad} NA$$

We will now prove a number of technical lemmas. Most of them say that given a commuting diagram in $\mathbf{V}$, one can apply N monoidally to many of the arrows and get a commuting diagram in $\mathbf{W}$ (similar to how one can apply a functor to a commuting diagram and still get a commuting diagram). These will be very useful for us later, as many of the proofs about change of base require an application of one of these lemmas. For each of the lemmas below, we begin with a monoidal functor $(\mathbf{V}, \otimes, \mathbf{I}) \xrightarrow{\ N\ } (\mathbf{W}, \bullet, \mathbf{J})$.

Lemma 4.1.1. *If the following diagram commutes in $\mathbf{V}$:*

$$I \otimes A$$

$$\downarrow f \otimes g \qquad \searrow l_A$$

$$B \otimes C \xrightarrow{\quad h \quad} A$$

then the following diagram commutes in **W***:*

$$
\begin{array}{ccc}
J \bullet NA & & \\
\downarrow{\scriptstyle \underline{Nf}\bullet Ng} & \searrow{\scriptstyle l_{NA}} & \\
NB \bullet NC & \xrightarrow{\underline{Nh}} & NA
\end{array}
$$

The above is also true with the I, J *on the right, and* l *replaced by* r*.*

Proof. Expanding the second diagram gives:

$$
\begin{array}{ccccc}
J \bullet NA & & & & \\
\downarrow{\scriptstyle N_0\bullet 1} & & & \searrow{\scriptstyle l_{NA}} & \\
NI \bullet NA & \xrightarrow{\widetilde{N}} & N(I \otimes A) & & \\
\downarrow{\scriptstyle Nf\bullet Ng} & & \downarrow{\scriptstyle N(f\otimes g)} & \searrow{\scriptstyle N(l_A)} & \\
NB \bullet NC & \xrightarrow{\widetilde{N}} & N(B \otimes C) & \xrightarrow{Nh} & NA
\end{array}
$$

The top region commutes by the coherence of N, the square by naturality of $\widetilde{N}$, and the bottom right triangle is N applied to the original diagram.

∎

Lemma 4.1.2. *If the following diagram commutes in* **V***:*

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{f\otimes g} & C \otimes D \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} \\
E & \xrightarrow{l} & F
\end{array}
$$

the the following diagram commutes in **W***:*

$$
\begin{array}{ccc}
NA \bullet NB & \xrightarrow{Nf\bullet Ng} & NC \bullet ND \\
\downarrow{\scriptstyle \underline{Nk}} & & \downarrow{\scriptstyle \underline{Nh}} \\
NE & \xrightarrow{Nl} & NF
\end{array}
$$

Proof. Expanding the second diagram gives:

$$
\begin{array}{ccc}
NA \bullet NB & \xrightarrow{\;Nf \bullet Ng\;} & NC \bullet ND \\
\Big\downarrow{\scriptstyle \widetilde{N}} & & \Big\downarrow{\scriptstyle \widetilde{N}} \\
N(A \otimes B) & \xrightarrow{\;N(f \otimes g)\;} & N(C \otimes D) \\
\Big\downarrow{\scriptstyle Nk} & & \Big\downarrow{\scriptstyle Nh} \\
NE & \xrightarrow{\;Nl\;} & NF
\end{array}
$$

The top region commutes by naturality of $\widetilde{N}$, and the bottom region is N applied to the original diagram. ∎

Lemma 4.1.3. *If the following diagram commutes in* **V***:*

$$
\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\;a\;} & A \otimes (B \otimes C) \\
\Big\downarrow{\scriptstyle f \otimes 1} & & \Big\downarrow{\scriptstyle 1 \otimes g} \\
D \otimes C & & A \otimes E \\
& \searrow{\scriptstyle h} \quad \swarrow{\scriptstyle k} & \\
& F &
\end{array}
$$

then the following diagram commutes in **W***:*

$$
\begin{array}{ccc}
(NA \bullet NB) \bullet NC & \xrightarrow{\;a\;} & NA \bullet (NB \bullet NC) \\
\Big\downarrow{\scriptstyle \underline{N}f \bullet 1} & & \Big\downarrow{\scriptstyle 1 \bullet \underline{N}g} \\
ND \bullet NC & & NA \bullet NE \\
& \searrow{\scriptstyle \underline{N}h} \quad \swarrow{\scriptstyle \underline{N}k} & \\
& NF &
\end{array}
$$

Proof. Expanding the second diagram gives:

$$
\begin{array}{c}
(NA \bullet NB) \bullet NC \xrightarrow{\quad a \quad} NA \bullet (NB \bullet NC)
\end{array}
$$

Diagram with vertices:
- $(NA \bullet NB) \bullet NC \xrightarrow{a} NA \bullet (NB \bullet NC)$
- Left edge: $\widetilde{N} \bullet 1$ down to $N(A \otimes B) \bullet NC$
- Right edge: $1 \bullet \widetilde{N}$ down to $NA \bullet N(B \otimes C)$
- $N(A \otimes B) \bullet NC \xrightarrow{\widetilde{N}} N((A \otimes B) \otimes C) \xrightarrow{Na} N(A \otimes (B \otimes C))$
- $NA \bullet N(B \otimes C) \xrightarrow{\widetilde{N}} N(A \otimes (B \otimes C))$
- $Nf \bullet 1$ down to $ND \bullet NC$
- $1 \bullet Ng$ down to $NA \bullet NE$
- $ND \bullet NC \xrightarrow{\widetilde{N}} N(D \otimes C)$
- $N((A \otimes B) \otimes C) \xrightarrow{N(f \otimes 1)} N(D \otimes C)$
- $N(A \otimes (B \otimes C)) \xrightarrow{N(1 \otimes g)} N(A \otimes E)$
- $NA \bullet NE \xrightarrow{\widetilde{N}} N(A \otimes E)$
- $N(D \otimes C) \xrightarrow{Nh} NF \xleftarrow{Nk} N(A \otimes E)$

The top region is by coherence of N, the two parallelograms by naturality of N, and the bottom region is N applied to the original diagram.

∎

Lemma 4.1.4. *If the following diagram commutes in* **V***:*

$$
\begin{array}{c}
A \otimes I \xrightarrow{b_1 \otimes c_1} B_1 \otimes C_1 \\
A \xrightarrow{r_A^{-1}} \qquad \xrightarrow{f_1} D \\
A \xrightarrow{l_A^{-1}} I \otimes A \xrightarrow{b_2 \otimes c_2} B_2 \otimes C_2 \xrightarrow{f_2} D
\end{array}
$$

then the following diagram commutes in **W***:*

$$
\begin{array}{c}
NA \bullet J \xrightarrow{Nb_1 \bullet Nc_1} NB_1 \bullet NC_1 \\
NA \xrightarrow{r_{NA}^{-1}} \qquad \xrightarrow{\underline{N}f_1} ND \\
NA \xrightarrow{l_{NA}^{-1}} J \bullet NA \xrightarrow{\underline{N}b_2 \bullet Nc_2} NB_2 \bullet NC_2 \xrightarrow{\underline{N}f_2} ND
\end{array}
$$

Proof. Expanding the second diagram gives:

The top left and bottom left sections commute by coherence of N, the top right and bottom right by naturality of N, and the hexagon in the middle is N applied to the original hexagon. ∎

It would be an interesting exercise to try to describe all diagrams in $\mathbf{V}$ to which applying N monoidally to appropriate arrows gives a commuting diagram in $\mathbf{W}$, but this is beyond the scope of this thesis.

The final lemma shows that applying functors monoidally is "functorial".

Lemma 4.1.5. *Given monoidal functors* $V \xrightarrow{\ N\ } W \xrightarrow{\ M\ } Z$, *if f is of the form*

$$A \otimes B \xrightarrow{\ f\ } C \ or\ I \xrightarrow{\ f\ } A,$$

then $\underline{(MN)}f = \underline{M}(\underline{N}f)$.

Proof. If f is of the form $A \otimes B \xrightarrow{\ f\ } C$, then

$$\underline{(MN)}f = MN(f) \circ M(\widetilde{N}) \circ \widetilde{M}$$

while

$$\underline{M}(\underline{N}f) = M(N(f) \circ \widetilde{N}) \circ \widetilde{M},$$

so the two are equal since M is a functor.

Similarly, if f is of the form $I \xrightarrow{f} A$, then

$$\underline{(MN)}f = MN(f) \circ M(N_0) \circ M_0$$

while

$$\underline{M}(\underline{N}f) = M(N(f) \circ N_0) \circ M_0,$$

again equal since M is a functor. $\blacksquare$

4.2 Change of Base N_*

We now wish to describe, for $\mathbf{V} \xrightarrow{N} \mathbf{W}$ monoidal, the "change of base" 2-functor $\mathbf{V}\text{-cat} \xrightarrow{N_*} \mathbf{W}\text{-cat}$. As such, we will need to describe it's action on V-categories, on V-functors, and on V-natural transformations. The results in this section are all due to Eilenberg-Kelly ([16]). However, the lemmas given above will make most of the we give here much more straightforward than their original versions.

Proposition 4.2.1. *Let N be as above, and $\mathbf{X}$ a $\mathbf{V}$-category. Then the following defines a $\mathbf{W}$-category $N_*\mathbf{X}$:*

- *$N_*\mathbf{X}$ has the same objects as those of $\mathbf{X}$,*

- *the hom objects are defined by $(N_*\mathbf{X})(x, y) := N(\mathbf{X}(x, y))$,*

- *the composition is N applied monoidally to the composition in $\mathbf{X}$,*

- *the identities are N applied monoidally to the identities in $\mathbf{X}$.*

Proof. That the composition is associative follows from Lemma 4.1.3, and the composition is unital follows from Lemma 4.1.1. $\blacksquare$

Proposition 4.2.2. *Let N be as above, $\mathbf{X}$ and $\mathbf{Y}$ are $\mathbf{V}$-categories, and $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ a $\mathbf{V}$-functor between them. Then we define a $\mathbf{W}$-functor $N_*\mathbf{X} \xrightarrow{N_*F} N_*\mathbf{Y}$ by:*

- *N_*F acts on objects as F does,*

- *the strength of N_*F is $(N_*F)(x, y) := N[F(x, y)]$.*

Proof. The preservation of composition follows from naturality of N, and the identity axiom follows directly. ∎

Proposition 4.2.3. *Let N be as above, and σ a $\mathbf{V}$-natural transformation between $\mathbf{C} \xrightarrow{F,G} \mathbf{D}$. Then applying N monoidally to the components of σ gives a $\mathbf{W}$-natural transformation $N_*\sigma : N_*F \longrightarrow N_*G$.*

Proof. The $\mathbf{W}$-naturality of $N_*\sigma$ follows directly from Lemma 4.1.3. ∎

Now that we have described the actions of N_*, we need to show that it defines a 2-functor.

Theorem 4.2.4. *Let N be as above. Then defining N_* on objects, arrows, and 2-cells as above, $\mathbf{V}\text{-cat} \xrightarrow{N_*} \mathbf{W}\text{-cat}$ is a 2-functor.*

Proof. First, we need to show that N_* preserves composition of $\mathbf{V}$-functors. For this, we need to show that if we have the following $\mathbf{V}$-functors:

$$\mathbf{C} \xrightarrow{\;\;F\;\;} \mathbf{D} \xrightarrow{\;\;G\;\;} \mathbf{E}$$

then $N_*(GF) = N_*(G)N_*(F)$. These two $\mathbf{W}$-functors clearly have equal action on objects, since N_* does not change how a functor acts on objects. For strengths, they are equal since N is a functor, and so preserves composition:

$$
\begin{aligned}
N_*[GF(c,d)] &= N[F(c,d) \circ G(Fc,Fd)] \\
&= NF(c,d) \circ NG(Fc,Fd) \\
&= N_*F(c,d) \circ N_*G(Fc,Fd) \\
&= (N_*G \circ N_*F)(c,d)
\end{aligned}
$$

N_* clearly preserves identity functors, since it preserves identity arrows.

Next, we need to show that N_* preserves horizontal and vertical composition of natural transformations. To begin, we suppose we have $\mathbf{V}$-natural transformations

We need to show that $N_*(\sigma_2\sigma_1) = N_*(\sigma_1)N_*(\sigma_2)$, so we need to show that their components are equal at an $x \in X$. If we expand the x-component of $N_*(\sigma_2\sigma_1)$ along the left side, and the x-component of $N_*(\sigma_1)N_*(\sigma_2)$ on the right, we get:

$$
\begin{array}{ccccc}
J & \xrightarrow{\;l^{-1}\;} & J \otimes J & \xrightarrow{\;N_0 \otimes N_0\;} & NI \otimes NI \\[1em]
\downarrow{\scriptstyle N_0} & & & {\scriptstyle \widetilde{N}} & \downarrow{\scriptstyle N\sigma_2 \otimes N\sigma_1} \\[1em]
NI & & & & N\mathbf{X}(Fx, Gx) \otimes N\mathbf{X}(Gx, Hx) \\[1em]
\downarrow{\scriptstyle N(l^{-1})} & & & & \downarrow{\scriptstyle \widetilde{N}} \\[1em]
N(I \otimes I) & & & & N(\mathbf{X}(Fx, Gx) \otimes \mathbf{X}(Gx, Hx)) \\[1em]
\downarrow{\scriptstyle N(\sigma_1 \otimes \sigma_2)} & & {\scriptstyle 1} & & \downarrow{\scriptstyle N(c)} \\[1em]
N(\mathbf{X}(Fx, Gx) \otimes \mathbf{X}(Gx, Hx)) & & \xrightarrow{\;N(c)\;} & & N\mathbf{X}(Fx, Hx)
\end{array}
$$

The top region commutes by coherence of N, the middle region by naturality of N, and the bottom region is an identity.

Finally, we need to show that N_* preserves horizontal composition. Suppose we have **V**-natural transformations:

$$
\mathbf{X} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \;{\Downarrow \sigma_1}\; \mathbf{Y} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \;{\Downarrow \sigma_2}\; \mathbf{Z}
$$

We need to show that $N_*(\sigma_2\sigma_1) = N_*(\sigma_1)N_*(\sigma_2)$, so we need to show that their component at an x is equal. If we expand the x-component of $N_*(\sigma_2\sigma_1)$ along the left

side, and the x-component of $N_*(\sigma_1)N_*(\sigma_2)$ on the right, we get:

$$
\begin{array}{ccccc}
J & \xrightarrow{\quad l^{-1} \quad} & J \bullet J & \xrightarrow{\quad N_0 \bullet N_0 \quad} & NI \bullet NI
\end{array}
$$

The top region commutes by coherence of N, the middle two regions by naturality of N, and the bottom region is an identity. Thus N_* preserves horizontal composition, and is a 2-functor.

∎

Before describing the full change of base $(-)_*$, it will be helpful to look at examples of the change of base N_* for some of the monoidal functors given in Section 2.1.1.

Example 4.2.5. As we saw in Example 2.1.11, for any monoidal $(\mathbf{V}, \otimes, I)$, there is always a monoidal functor $\mathbf{V} \xrightarrow{\ \mathbf{V}(I,-)\ } \mathbf{set}$. This induces a change-of-base 2-functor which Kelly ([27], pg. 10) calls $(-)_0$: $\mathbf{V\text{-}cat} \longrightarrow \mathbf{cat}$. This 2-functor $(-)_0$ could be described as the "underlying category" functor. For example, with $\mathbf{V} = \mathbf{ab}$, an additive category gets sent to its underlying category. As another example, the forgetful functor $\mathbf{2\text{-}cat} \longrightarrow \mathbf{cat}$ simply forgets the 2-cells. Of course, the less information $\mathbf{V}(I, -)$ retains, the less $(-)_0$ does as well. For example, with $\mathbf{V} = $ graded R-modules, very little information is retained in this forgetful 2-functor.

Example 4.2.6. When G is an abelian group, so that a monoidal $G \xrightarrow{\ N\ } \mathbf{2}$ makes

G into an ordered abelian group, the change-of-base $\mathbf{G\text{-}cat} \xrightarrow{N_*} \mathbf{ord}$ sends $\mathbf{G}$ (considered as a $\mathbf{G}$-category) to its underlying poset.

Example 4.2.7. Similarly to above, when G is an abelian group, with a monoidal $\mathbf{G} \xrightarrow{N} \mathbf{R}_+$ making G into a normed abelian group, the change-of-base $\mathbf{G\text{-}cat} \xrightarrow{N_*} \mathbf{metr}$ sends the normed abelian group G (considered as a $\mathbf{G}$-category) to its underlying metric space. An elementary version of this result was given in Proposition 3.1.5.

Example 4.2.8. The monoidal functor $([0, \infty], \geq, \vee) \longrightarrow \mathbb{R}_+$ (which is the identity on objects) shows that every ultrametric space has the structure of a metric space.

Example 4.2.9. The free abelian group functor $\mathbf{set} \xrightarrow{F} \mathbf{ab}$ is monoidal. An example of the change-of-base F_* is when the $\mathbf{set}$-category is merely a group G. In this case, the change of base category F_*G is the group ring of G. Similarly, if F is the free functor from $\mathbf{set}$ to $\mathbf{vec_k}$, then F_*G is the group algebra $k[G]$. Taking this idea further suggests that the "groupoid ring" and the "groupoid algebra" of a groupoid G should also be F_*G, so that the groupoid ring is an $\mathbf{ab}$-enriched category, and the groupoid algebra is a $\mathbf{vec_k}$-enriched category.

4.3   Change of Base as a 2-functor $(-)_*$

In some sense, proving that N_* is a 2-functor is only the first step in understanding the change of base. The next step, very important for our proof that N_* preserves enriched monoidal categories, is to understand $(-)_*$ *itself* as a 2-functor from monoidal categories to 2-categories. To begin, we need to define the cells between monoidal functors: monoidal natural transformations.

Definition Given monoidal functors $(\mathbf{V}, \otimes, I) \xrightarrow{N, M} (\mathbf{W}, \bullet, J)$, a *monoidal natural transformation* $N \xrightarrow{\alpha} M$ is a natural transformation from N to M which preserves the tensor comparisons $\widetilde{N}, \widetilde{M}$:

$$
\begin{array}{ccc}
NA \bullet NB & \xrightarrow{\ \widetilde{N}\ } & N(A \otimes B) \\
{\scriptstyle \alpha \bullet \alpha} \downarrow & & \downarrow {\scriptstyle \alpha} \\
MA \bullet MB & \xrightarrow[\ \widetilde{M}\ ]{} & M(A \otimes B)
\end{array}
$$

and the unit comparisons:

$$
\begin{array}{ccc}
& J & \\
{\scriptstyle N_0}\swarrow & & \searrow{\scriptstyle M_0} \\
NI & \xrightarrow{\quad\alpha\quad} & MI
\end{array}
$$

We also need to define how monoidal functors compose.

Definition Let $\mathbf{V} \xrightarrow{\ N\ } \mathbf{W} \xrightarrow{\ M\ } \mathbf{Z}$ be monoidal functors. Their composite is also a monoidal functor, with comparisons

$$\widetilde{MN} := \widetilde{M}M(\widetilde{N}) \text{ and } (MN)_0 := M_0 M(N_0)$$

Monoidal natural transformation compose just as natural transformations do, both horizontally and vertically.

Definition Let **moncat** denote the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations. Let **2-cat** denote the 2-category of 2-categories, 2-functors, and 2-natural transformations.

We wish to show that we can define a 2-functor $\mathbf{moncat} \xrightarrow{\ (-)_*\ } \mathbf{2\text{-}cat}$, where $(-)_*$ on an object $\mathbf{V}$ is $\mathbf{V}\text{-}\mathbf{cat}$. In the previous section, we showed how $(-)_*$ would act on monoidal functors: this is the usual change of base 2-functor N_*. We now need to show how $(-)_*$ acts on monoidal natural transformations; it should define a 2-natural transformation.

Proposition 4.3.1. *Let $N \xrightarrow{\ \alpha\ } M$ be a monoidal natural transformation, as above. Given $\mathbf{X} \in \mathbf{V}\text{-}\mathbf{cat}$, define $\alpha_*(\mathbf{X})$ as the following $\mathbf{W}$-functor from $N_*(\mathbf{X}) \longrightarrow M_*(\mathbf{X})$:*

- *The objects of $N_*(\mathbf{X})$ and $M_*(\mathbf{X})$ are the same, so we can define $\alpha_*(\mathbf{X})$ to be the identity function on objects.*

- *The strength is then a $\mathbf{W}$-arrow from $N\mathbf{X}(x, y)$ to $M\mathbf{X}(x, y)$: define it to be α at the component $\mathbf{X}(x, y)$.*

Then $\alpha_*(X)$ does define a $\mathbf{W}$-functor, and with these as its components, α_* is a 2-natural transformation from N_* to M_*.

Proof. To show that $\alpha_*(X)$ preserves composition, we only need add an α in the middle of the diagram:

$$
\begin{array}{ccc}
N\mathbf{X}(x,y) \bullet N\mathbf{X}(y,z) & \xrightarrow{\;\alpha\bullet\alpha\;} & M\mathbf{X}(x,y) \bullet M\mathbf{X}(y,z) \\
\downarrow\widetilde{N} & & \downarrow\widetilde{M} \\
N(\mathbf{X}(x,y) \otimes \mathbf{X}(y,z)) & \xrightarrow{\;\alpha\;} & M(\mathbf{X}(x,y) \otimes \mathbf{X}(y,z)) \\
\downarrow N(c) & & \downarrow M(c) \\
N\mathbf{X}(x,z) & \xrightarrow{\;\alpha\;} & M\mathbf{X}(x,z)
\end{array}
$$

The top rectangle follows from monoidal coherence of α, and the bottom from naturality of α. The preservation of identities is similar:

$$
\begin{array}{ccc}
 & J & \\
{\scriptstyle N_0}\downarrow & \searrow^{M_0} & \\
NI & \xrightarrow{\;\alpha\;} & MI \\
{\scriptstyle N(1)}\downarrow & & \searrow^{M(1)} \\
N\mathbf{X}(x,x) & \xrightarrow{\;\alpha\;} & M\mathbf{X}(x,x)
\end{array}
$$

The triangle is monoidal coherence of α, and the bottom region is naturality of α. Thus $\alpha_*(X)$ is a $\mathbf{W}$-functor.

Next, we need to show that α_* is natural. That is, given $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ a $\mathbf{V}$-functor, we need to show that

$$
\begin{array}{ccc}
N_*(\mathbf{X}) & \xrightarrow{\;\alpha_*(\mathbf{X})\;} & M_*(\mathbf{X}) \\
\downarrow{\scriptstyle N_*(F)} & & \downarrow{\scriptstyle M_*(F)} \\
N_*(\mathbf{Y}) & \xrightarrow{\;\alpha_*(\mathbf{Y})\;} & M_*(\mathbf{Y})
\end{array}
$$

commutes. Since this is equality of $\mathbf{W}$-functors, we need to show that the functions on objects are equal, and that the strengths are equal. Clearly the functions on objects

are equal: since $\alpha_*(X)$ and $\alpha_*(Y)$ are identity on objects, the two composite's function on objects are both F. That the strengths are equal is exactly due to naturality of α:

$$
\begin{array}{ccc}
N\mathbf{X}(x,y) & \xrightarrow{\ \ \alpha\ \ } & M\mathbf{X}(x,y) \\
{\scriptstyle NF}\downarrow & & \downarrow{\scriptstyle MF} \\
N\mathbf{Y}(x,y) & \xrightarrow{\ \ \alpha\ \ } & M\mathbf{Y}(x,y)
\end{array}
$$

Thus α_* is natural.

Finally, we need to show that α_* is 2-natural. That is, given the following **V**-natural transformation:

$$
\mathbf{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{Y} \qquad \Downarrow \sigma
$$

we need to show that

$$
\begin{array}{ccc}
N_*\mathbf{X} & \xrightarrow{\ \alpha_*\mathbf{X}\ } & M_*\mathbf{X} \\
{\scriptstyle N_*F}\left(\underset{N_*\sigma}{\Downarrow}\right){\scriptstyle N_*G} & & {\scriptstyle M_*F}\left(\underset{M_*\sigma}{\Downarrow}\right){\scriptstyle M_*G} \\
N_*\mathbf{Y} & \xrightarrow{\ \alpha_*\mathbf{Y}\ } & M_*\mathbf{Y}
\end{array}
$$

To show that these two composite natural transformations are equal, we need to show that their components $J \longrightarrow M\mathbf{Y}(Fx, Gx)$ are equal. Expanding each component, with the right arrows being the right composite, and the left arrows the left composite, we get:

$$
\begin{array}{ccc}
J & \xrightarrow{\ \ M_0\ \ } & MI \\
{\scriptstyle N_0}\downarrow & \nearrow{\scriptstyle \alpha} & \downarrow \\
NI & & {\scriptstyle M\sigma} \\
{\scriptstyle N\sigma}\downarrow & & \downarrow \\
N\mathbf{Y}(Fx, Gx) & \xrightarrow{\ \ \alpha\ \ } & M\mathbf{Y}(Fx, Gx)
\end{array}
$$

The triangle commutes by monoidal coherence of α, and the bottom region commutes by naturality of α. Thus α_* is 2-natural. ∎

Now that we have gathered all the neccesary components, we can prove that $(-)_*$ is itself a 2-functor. Note that this result was not shown in the original Eilenberg-Kelly paper ([16]).

Theorem 4.3.2. *Define* **moncat** $\xrightarrow{(-)_*}$ **2-cat** *on a monoidal category* **V** *as* **V-cat**, *on monoidal functors as in Theorem 4.2.4, and on monoidal natural transformations as in Proposition 4.3.1. Then* $(-)_*$ *is a 2-functor.*

Proof. We need to show that $(-)_*$ preserves 1-cell composition, and horizontal and vertical composition of 2-cells.

To show that $(-)_*$ preserves 1-cell composition, we need to show that given monoidal functors $V \xrightarrow{N} W \xrightarrow{M} Z$, we have $(MN)_* = M_*N_*$. Since these are 2-functors, we need to show that they are equal on objects (**V**-categories), 1-cells (**V**-functors) and 2-cells (**V**-natural transformations).

Given a **V**-category **X**, $(MN)_*(\mathbf{X})$ and $M_*N_*(\mathbf{X})$ both have the same set of objects, namely those of **X**. They also have the same hom-objects, given by $MN(\mathbf{X}(x,y))$. Since the compositions and identities of the categories are due to applications of $\underline{MN}$ and $(\underline{M})(\underline{N})$, respectively, they are equal due to Lemma 4.1.5.

To show that $(MN)_*$ and M_*N_* are equal on a **V**-functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$, we need to show they are equal as **Z**-functors. However, this is quite straightforward: they are both F on objects, while the strengths are $(MN)(F(x,y))$ and $M(N(F(x,y)))$, respectively.

To show that $(MN)_*$ and M_*N_* are equal on a **V**-natural transformation σ is again a direct application of Lemma 4.1.5, as their components at x as **Z**-natural transformations are $\underline{MN}(\sigma_x)$ and $\underline{M}(\underline{N}\sigma_x)$, respectively. This completes the proof that $(-)_*$ preserves 1-cell composition.

Next, we need to show that $(-)_*$ preserves vertical composition of 2-cells. Let the

following be monoidal natural transformations:

$$
\begin{array}{ccc}
& N & \\
& \Downarrow \alpha_1 & \\
\mathbf{V} \xrightarrow{\quad M \quad} & & \mathbf{W} \\
& \Downarrow \alpha_2 & \\
& P &
\end{array}
$$

We need to show that $(\alpha_2\alpha_1)_* = (\alpha_2)_*(\alpha_1)_*$ as 2-natural transformations. Thus, we need to show that their components (**W**-functors) are equal. Because of their definition 4.3.1, both are the identity on objects. As for their strengths, their composite in both cases is merely the composite arrows.

Finally, we need to show that $(-)_*$ preserves horizontal composition of 2-cells. This follows exactly as for the vertical composition, as the horizontal composition of monoidal natural transformations and 2-natural transformations are both the usual horizontal composition. ∎

4.4 Adjunctions in Moncat

To conclude this chapter, we make some brief remarks on why knowing that $(-)_*$ is a 2-functor is important. One major application will be seen in the next chapter, when we apply $(-)_*$ to a monoidal natural transformation to get a 2-natural transformation. Another application, however, deals with monoidal adjunctions. Since $(-)_*$ is a 2-functor, it preserves adjunctions. That is, if $F \dashv U$ is an adjunction in **moncat**, then $F_* \dashv U_*$ is an adjunction in 2-cat. This tells us that if the initial monoidal categories have an adjoint relationship, then so will categories enriched over those monoidal categories.

This can be useful, if we can find monoidal adjunctions. Fortunately, a result of Max Kelly shows finding monoidal adjunctions is no more difficult than finding adjunctions.

Theorem 4.4.1. *Suppose that F is a monoidal functor. Then F has a right adjoint U in* **moncat** *if and only if it is both strong monoidal and has a right adjoint as a functor.*

Proof. See Kelly ([26], pg. 264). ■

Since many of the free functors F between monoidal categories are strong, this shows that most of the usual adjunctions between monoidal categories extend to 2-adjunctions between categories enriched between them.

Example 4.4.2. The free functor $\mathbf{set} \xrightarrow{F} \mathbf{ab}$ is strong, and hence the adjunction $F \dashv U$ between abelian groups and sets is monoidal. Thus we have a 2-adjunction $F_* \dashv U_*$ between the 2-categories of additive categories and locally small categories.

Chapter 5

Change of Base and Enriched Monoidal Categories

In this chapter, we will prove that the change of base 2-functor N_* preserves enriched monoidal categories. In other words, if $\mathbf{C}$ is a $\mathbf{V}$-category which has a monoidal structure, relative to $\mathbf{V}$, then when transfered by N_* to $\mathbf{W}$, it gets a monoidal structure relative to $\mathbf{W}$. In the first section, we will present an overview of the idea behind the proof. All the major ideas are presented, without any of the details. The next few sections then fill in the details of the proof.

There were two additional benefits that came as a result of the techniques used in this proof. First of all, an alternative way of defining the tensor product of $\mathbf{V}$-categories was found. This alternate definition (see section 5.5) is interesting in its own right, and the idea behind it has the potential to be useful when defining the tensor products of higher-dimensional enriched categories: see, for example, Leinster ([35]), and Forcey ([17]). Secondly, in this chapter we demonstrate that $(-)_*$ is itself monoidal; this in turn shows that it is not neccesary that $\mathbf{V}$ be braided for $\mathbf{V\text{-cat}}$ to be monoidal.

5.1 Idea of the Proof

Let us begin with the basic definition of a monoidal $\mathbf{V}$-category. Structurally, it is the same as an ordinary monoidal category, but with functors replaced by $\mathbf{V}$-functors, the product of categories replaced with the tensor product of $\mathbf{V}$-categories, and an object of $\mathbf{X}$ replaced with a $\mathbf{V}$-functor from the unit $\mathbf{V}$-category.

Definition A *monoidal $\mathbf{V}$-category* is a $\mathbf{V}$-category $\mathbf{X}$, together with $\mathbf{V}$-functors

$$\mathbf{X} \otimes \mathbf{X} \xrightarrow{\otimes \mathbf{x}} \mathbf{X}, \mathbb{I} \xrightarrow{I\mathbf{x}} \mathbf{X}$$

together with associativity and unit 2-cells, which satisfy the equations for a monoidal category given in section 2.1.

Looking at the above definition, one may think that proving that an N_* preserves **V**-monoidal **V**-categories is relatively straightforward. In fact, it is, but for one point: there is an enormous amount of detail that needs to be checked. None of these details are difficult; once one looks at what needs to be checked for each particular detail, the result follows almost immediately. However, this indicates that perhaps there should be a more straightforward way to prove the desired result. The answer, as is usually the case, is to generalize, and look for higher structure that may prove useful.

The key is to look at the structure of the 2-category **V-cat**. It itself has a monoidal structure, given certain conditions on **V**. In the usual 2-category **cat** = **set-cat**, the monoidal structure is given by product. However, in the general case of **V-cat**, one defines the tensor product of **V**-categories using the tensor product in **V**. This does require a certain commutativity condition on the base **V** (a "braiding"), but this symmetry is present in most of the familiar **V** (in fact, as we will show, one needs a condition slightly less restrictive than a braiding).

So, **V-cat** becomes a monoidal 2-category. Now, just as one can define the notion of a monoid in a monoidal category, one can similarly define the notion of a pseudomonoid in a monoidal bicategory (and hence in a monoidal 2-category). In the particular case we are interested in, pseudomonoids in the monoidal 2-category **V-cat** are exactly the monoidal **V**-categories.

Now, just as there are monoidal functors between monoidal categories, so there are monoidal lax (or pseudo) functors between monoidal bicategories. As one would expect, these preserve pseudomonoids, just as monoidal functors preserve monoids. So, if we can show that N_* is a monoidal 2-functor, this shows that N_* preserves pseudomonoids in **V-cat**; in other words, N_* preserves monoidal **V**-categories. Since our base **V** needs to be braided, we may assume our N should be as well. Our result, then, will be the following: if N is a braided monoidal functor, then N_* is a monoidal

2-functor [1]. This will show that N_* preserves monoidal **V**-categories.

Now, a monoidal pseudofunctor has a number of complicated elements to it, as one can see from the definition in section 5.6. If $\mathcal{M} \xrightarrow{T} \mathcal{N}$ is a lax functor between monoidal bicategories, for it to have monoidal structure requires a pseudonatural transformation of the following form:

$$
\begin{array}{ccc}
M \times M & \xrightarrow{\otimes_M} & M \\
{\scriptstyle T \times T}\downarrow & \underset{\Rightarrow}{\chi} & \downarrow{\scriptstyle T} \\
N \times N & \xrightarrow[\otimes_N]{} & N
\end{array}
$$

So, in our particular case of $T = \mathbf{V\text{-}cat} \xrightarrow{N_*} \mathbf{W\text{-}cat}$, we will need a pseudonatural transformation

$$
\begin{array}{ccc}
\mathbf{V\text{-}cat} \times \mathbf{V\text{-}cat} & \xrightarrow{\otimes} & \mathbf{V\text{-}cat} \\
{\scriptstyle N_* \times N_*}\downarrow & \underset{\Rightarrow}{\chi} & \downarrow{\scriptstyle N_*} \\
\mathbf{W\text{-}cat} \times \mathbf{W\text{-}cat} & \xrightarrow[\otimes]{} & \mathbf{W\text{-}cat}
\end{array}
$$

Looking at the definition of χ, one can see that χ is the comparison map for N_*, just as $\widetilde{N}$ is the comparison map for N. So if the χ for N_* was to exist, it would surely have to come from the $\tilde{N}$'s. Just as we raise the dimension of N to get N_*, we must also raise the dimension of $\tilde{N}$ to get χ. In keeping with the nomenclature, this would make $\chi = (\tilde{N})_*$. That is, we must apply the 2-functor $(-)_*$ to $\tilde{N}$ to get χ.

To do this, we need that $\tilde{N}$ be not just a natural transformation, but a monoidal natural transformation. After all, $(-)_*$ only applies to monoidal categories, monoidal functors, and monoidal natural transformations. But we will show (Proposition 5.3.6) that if V and N are braided, then indeed $\tilde{N}$ is monoidal. So when N is braided, we have the following diagram of monoidal categories, monoidal functors, and monoidal natural transformations:

---

[1]Note that N_* cannot itself be braided, since **V-cat** is not itself braided unless **V** is symmetric: see Day and Street [13].

$$\begin{array}{ccc}
\mathbf{V} \times \mathbf{V} & \xrightarrow{\otimes_{\mathbf{V}}} & \mathbf{V} \\
{\scriptstyle N \times N}\downarrow & \overset{\tilde{N}}{\Longrightarrow} & \downarrow{\scriptstyle N} \\
\mathbf{W} \times \mathbf{W} & \xrightarrow[\otimes_{\mathbf{W}}]{} & \mathbf{W}
\end{array}$$

We apply $(-)_*$ to the entire diagram to get:

$$\begin{array}{ccc}
(\mathbf{V} \times \mathbf{V})\text{-}\mathbf{cat} & \xrightarrow{(\otimes_V)_*} & \mathbf{V}\text{-}\mathbf{cat} \\
{\scriptstyle (N \times N)_*}\downarrow & \overset{(\tilde{N})_*}{\Longrightarrow} & \downarrow{\scriptstyle N_*} \\
(\mathbf{W} \times \mathbf{W})\text{-}\mathbf{cat} & \xrightarrow[(\otimes_W)_*]{} & \mathbf{W}\text{-}\mathbf{cat}
\end{array}$$

Unfortunately, this is not quite what we require. If we look at the diagram for χ, we see that we need the top-left object to be $\mathbf{V}\text{-}\mathbf{cat} \times \mathbf{V}\text{-}\mathbf{cat}$, not $(\mathbf{V} \times \mathbf{V})\text{-}\mathbf{cat}$. Fortunately, we can add these objects to our diagram, with the help of the 2-functor $N_* \times N_*$, and the tensor products of $\mathbf{V}\text{-}\mathbf{cat}$ and $\mathbf{W}\text{-}\mathbf{cat}$:

If we can fill in the dotted arrows, and fill in two-cells (or equalities) in each of the regions, then by pasting the 2-cells together, we will have our desired χ. Filling in the two dotted arrows is not difficult, but it is an interesting link, as one can see that it is *itself* a monoidal comparison for the two-functor $(-)_*$, making $(-)_*$ into a monoidal

2-functor! Moreover, the top and bottom regions show an interesting factorization for the tensor products on **V-cat** and **W-cat**.

Thus, to prove our desired result, we need to show three things: (1) for N braided, $\widetilde{N}$ is monoidal, (2) that $(-)_*$ is itself monoidal, which gives the dotted arrows and the equality of the left cell, and (3) the nature of the tensor product of **V**-categories, and why it factors as shown in the diagram above. In addition, we will need to define the concepts of monoidal bicategory, pseudomonoids in a monoidal bicategory, and monoidal pseudofunctors.

We begin by reviewing the definition of braided monoidal categories.

5.2 Braided Monoidal Categories

Definition Suppose $(\mathbf{V}, \otimes, I)$ is a monoidal category. A *braiding* on **V** is a natural isomorphism

$$X \otimes Y \xrightarrow{\ \sigma(x,y)\ } Y \otimes X,$$

such that the following two diagrams commute:

$$
\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\ a\ } & A \otimes (B \otimes C) \\
\downarrow{\scriptstyle \sigma \otimes 1} & & \downarrow{\scriptstyle \sigma} \\
(B \otimes A) \otimes C & & (B \otimes C) \otimes A \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\
B \otimes (A \otimes C) & \xrightarrow{\ 1 \otimes \sigma\ } & B \otimes (C \otimes A)
\end{array}
\qquad
\begin{array}{ccc}
A \otimes (B \otimes C) & \xrightarrow{\ a^{-1}\ } & (A \otimes B) \otimes C \\
\downarrow{\scriptstyle 1 \otimes \sigma} & & \downarrow{\scriptstyle \sigma} \\
A \otimes (C \otimes B) & & C \otimes (A \otimes B) \\
\downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle a^{-1}} \\
(A \otimes C) \otimes B & \xrightarrow{\ \sigma \otimes 1\ } & (C \otimes A) \otimes B
\end{array}
$$

A braiding is a *symmetry* if it is its own inverse:

$$
\begin{array}{ccc}
& A \otimes B & \\
{\scriptstyle \sigma}\downarrow & & \searrow{\scriptstyle 1} \\
B \otimes A & \xrightarrow{\ \sigma\ } & A \otimes B
\end{array}
$$

All of the examples considered in section 2.1 have braidings. In all cases the braiding is quite straightforward; in **ab**, for example, the braiding is the usual switch isomorphism of abelian groups $A \otimes B \cong B \otimes A$ defined by $a \otimes b \mapsto b \otimes a$. In fact, all of the braidings in the examples mentioned are symmetric. However, all of our results only require braided monoidal categories, so we will only assume braidings rather than symmetries.

Before we go on, we should note that the two axioms for braided monoidal categories imply a number of other basic results.

Proposition 5.2.1. *In any braided monoidal category* $(\mathbf{V}, \otimes, I, \sigma)$, *the following diagrams commute:*

$$
\begin{array}{ccc}
A \otimes I & & \\
\downarrow{\scriptstyle\sigma} \quad \searrow{\scriptstyle r} & & \\
I \otimes A \xrightarrow{\;l\;} A &
\end{array}
\qquad
\begin{array}{ccc}
I \otimes A & & \\
\downarrow{\scriptstyle\sigma} \quad \searrow{\scriptstyle l} & & \\
A \otimes I \xrightarrow{\;r\;} A &
\end{array}
$$

Proof. See Joyal and Street ([24], pg. 34). ∎

Unlike the case of monoidal categories, not every diagram built out of the isomorphisms a, l, r, σ commutes. However, there is a powerful technique for determining when such diagrams commute. The idea, found in Joyal and Street's paper on braided monoidal categories ([24], pgs. 34-45), is to build the two composites in the free braided tensor category. The free braid category consists of strings between points in the plane, and an appearance of the arrow σ is a crossing of two strings. For example, $A \otimes B \xrightarrow{\;\sigma\;} B \otimes A$ would appear as follows:

$$\text{\reflectbox{}}$$

$A \quad B$

If the two string diagrams are "braid isotopic" - in other words, if one can be deformed into the other without passing the strings though each other, then the two composites are equal. For example, the first axiom for a braided monoidal category asserts the equality of the following two braids, with the left composite being the left

braid, and the right composite being the right braid:
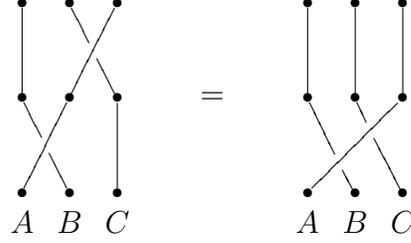
which are indeed braid isotopic. We will use this technique to prove the commutativity of several diagrams in the next section.

Before we move on, however, we need to define braided monoidal functors.

Definition Suppose $(\mathbf{V}, \otimes, I, \sigma_V) \xrightarrow{N} (\mathbf{W}, \bullet, J, \sigma_W)$ is a monoidal functor between braided monoidal categories. N is a *braided monoidal functor* if the following diagram commutes for each $X, Y \in \mathbf{V}$:

$$
\begin{array}{ccc}
NX \bullet NY & \xrightarrow{\;\sigma_W\;} & NY \bullet NX \\
\Big\downarrow{\scriptstyle \widetilde{N}} & & \Big\downarrow{\scriptstyle \widetilde{N}} \\
N(X \otimes Y) & \xrightarrow[N(\sigma_V)]{} & N(Y \otimes X)
\end{array}
$$

A *symmetric monoidal functor* is simply a braided monoidal functor between symmetric monoidal categories.

The first example of a monoidal functor we gave, example 2.1.11, is always braided if $\mathbf{V}$ is so:

Proposition 5.2.2. *For any braided monoidal category $(\mathbf{V}, \otimes, I, \sigma)$, the monoidal functor*

$$
\mathbf{V} \xrightarrow{\;\mathbf{V}(I,-)\;} \mathbf{set}
$$

is braided.

Proof. We need to show that for any $X, Y \in \mathbf{V}$, the diagram

$$
\begin{array}{ccc}
\mathbf{V}(I,X) \times \mathbf{V}(I,Y) & \xrightarrow{\;\sigma_{\mathbf{set}}\;} & \mathbf{V}(I,Y) \times \mathbf{V}(I,X) \\
\Big\downarrow{\scriptstyle \mathbf{V}(I,\otimes)} & & \Big\downarrow{\scriptstyle \mathbf{V}(I,\otimes)} \\
\mathbf{V}(I, X \otimes Y) & \xrightarrow[\mathbf{V}(I,\sigma_V)]{} & \mathbf{V}(I, Y \otimes X)
\end{array}
$$

commutes. So, we need to show that the for arrows $I \xrightarrow{f} X, I \xrightarrow{g} Y$, the composites

$$I \xrightarrow{l^{-1}} I \otimes I \xrightarrow{f \otimes g} X \otimes Y \xrightarrow{\sigma} Y \otimes X$$

and

$$I \xrightarrow{l^{-1}} I \otimes I \xrightarrow{g \otimes f} Y \otimes X$$

are equal. Now, by Proposition 2.1.10, $l_I^{-1} = r_I^{-1}$, and then by applying Proposition 5.2.1, we can write the second composite as

$$I \xrightarrow{l^{-1}} I \otimes I \xrightarrow{\sigma} I \otimes I \xrightarrow{g \otimes f} Y \otimes X$$

The two composites are then equal by naturality of σ. ∎

All other examples given in section 2.1.1 are also braided.

In the next section, we will see what we can prove about the elements of a braided monoidal category.

5.3   Properties of the Natural Isomorphisms

In this section, we would like to investigate some properties of the natural isomorphisms a, l, r, and for a monoidal functor, $\widetilde{N}$. We would like to show that for braided $\mathbf{V}$, each of these natural transformations is in fact a monoidal natural transformation. This will be useful for us, as it allows us to apply the 2-functor $(-)_*$ to them. Of course, to prove that each of these natural transformations are monoidal, we need to prove that the categories and functors that they go between are also themselves monoidal. The main reason we need $\mathbf{V}$ braided is to make $\otimes$ into a monoidal functor.

Interestingly, to prove that l and r are monoidal natural does not require braiding, while to show that a and $\widetilde{N}$ are does. We will begin with the results that do not require braiding on $\mathbf{V}$.

Proposition 5.3.1. *If $(\mathbf{V}, \otimes, I)$ is a monoidal category, then the functor*

$$\mathbf{V} \xrightarrow{- \otimes I} \mathbf{V}$$

is a monoidal functor, when equipped with unit comparison

$$I \otimes I \xrightarrow{\ r\ } I$$

and tensor comparison

$$(X \otimes I) \otimes (Y \otimes I) \xrightarrow{\ r \otimes 1\ } X \otimes (Y \otimes I) \xrightarrow{\ a^{-1}\ } (X \otimes Y) \otimes I$$

Similarly,

$$\mathbf{V} \xrightarrow{\ I \otimes -\ } \mathbf{V}$$

is also monoidal.

Proof. In this case, the diagrams for a monoidal functor all involve instances of a, l, r, a^{-1}, so the result follows from the coherence theorem for monoidal categories (Mac Lane [32], pg. 165). ∎

Proposition 5.3.2. *For any monoidal category $(\mathbf{V}, \otimes, I)$, the coherence natural transformations l and r are monoidal natural transformations.*

Proof. We know, by the previous result, that the functors $I \otimes -$ and $- \otimes I$ are monoidal, so the question of whether l and r can be monoidal makes sense. Just as in the previous proof, the required diagram for monoidal natural transformations involves only instances of a, l, r, a^{-1}, so the result again follows from Mac Lane's coherence theorem. ∎

Proposition 5.3.3. *Suppose that $(\mathbf{V}, \otimes, I)$ and $(\mathbf{W}, \bullet, J)$ are monoidal categories. Then the product category $\mathbf{V} \times \mathbf{W}$ is also a monoidal category, with unit object (I, J) and multiplication*

$$(x_1, y_1) \star (x_2, y_2) := (x_1 \otimes x_2, y_1 \bullet y_2)$$

Proof. The proof is entirely straightforward, and merely relies on the fact that $\times$ is a functor. ∎

The rest of our results in this section require braided $\mathbf{V}$.

Proposition 5.3.4. *Suppose that* $(\mathbf{V}, \otimes, I, \sigma)$ *is a braided monoidal category. Then the functor* $\mathbf{V} \times \mathbf{V} \xrightarrow{\;\otimes\;} \mathbf{V}$ *is monoidal, when equipped with unit comparison*
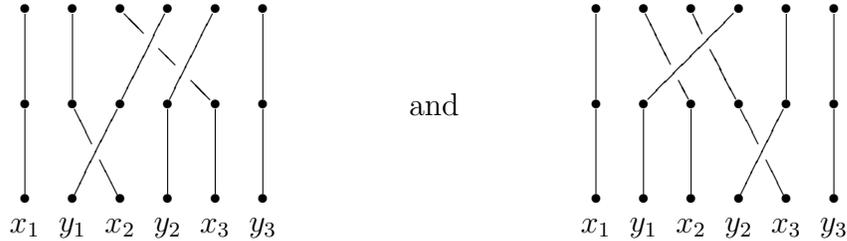
$$I \xrightarrow{\;r^{-1}\;} I \otimes I$$

and tensor comparison

$$(x_1 \otimes y_1) \otimes (x_2 \otimes y_2) \xrightarrow{\;(a^{-1})(1 \otimes a)\sigma(1 \otimes a^{-1})(a)\;} (x_1 \otimes x_2) \otimes (y_1 \otimes y_2).$$

Proof. The first diagram, without associativities, and with juxtaposition for $\otimes$, is as follows:

$$
\begin{array}{ccc}
x_1 y_1 x_2 y_2 x_3 y_3 & \xrightarrow{\;111\sigma1\;} & x_1 y_1 x_2 x_3 y_2 y_3 \\
\downarrow{\scriptstyle 1\sigma111} & & \downarrow{\scriptstyle 1\sigma11} \\
x_1 x_2 y_1 y_2 x_3 y_3 & \xrightarrow{\;11\sigma1\;} & x_1 x_2 x_3 y_1 y_2 y_3
\end{array}
$$

The braid diagrams for the two composites are then:

and

which, upon inspection, are isotopic. The unit axioms similarly follow.

∎

Note that the inverse braiding, σ^{-1}, could also be used. Thus, when we say that the natural isomorphisms are monoidal natural transformations, we mean with respect to this particular monoidal structure on $\otimes$, as there could be others.

Proposition 5.3.5. *Suppose that* $(\mathbf{V}, \otimes, \mathbf{I}, \sigma)$ *is a braided monoidal category. Then the coherence isomorphism a is a monoidal natural transformation.*

Proof. First, let F and G be the functors $F(a_1, a_2, a_3) = (a_1 \otimes a_2) \otimes a_3$, $G(a_1, a_2, a_3) = a_1 \otimes (a_2 \otimes a_3)$. Both of these are monoidal by the previous proposition. Now, to show

that a is monoidal natural, we need to show that the following diagram commutes:

$$
\begin{array}{ccc}
FA \otimes FB & \xrightarrow{\ a \otimes a\ } & GA \otimes GB \\
\Big\downarrow{\scriptstyle \tilde{F}} & & \Big\downarrow{\scriptstyle \tilde{G}} \\
F(A \otimes B) & \xrightarrow[\ a\ ]{} & G(A \otimes B)
\end{array}
$$

Expanded, with $A = (a_1, a_2, a_3), B = (b_1, b_2, b_3)$, this means that the following must commute (to save space, we write tensor as concatenation):

$$
\begin{array}{ccc}
a_1 a_2 a_3 b_1 b_2 b_3 & \xrightarrow{\ 1\sigma 11\ } & a_1 b_1 a_2 a_3 b_2 b_3 \\
\Big\downarrow{\scriptstyle 11\sigma 1} & & \Big\downarrow{\scriptstyle 111\sigma 1} \\
a_1 a_2 b_1 b_2 a_3 b_3 & \xrightarrow[\ 1\sigma 111\ ]{} & a_1 b_1 a_2 b_2 a_3 b_3
\end{array}
$$

Translated into braids:

and

which, upon inspection, are isotopic. ∎

We have shown that for $\mathbf{V}$ braided, a, l, and r are monoidal natural. However, it is not true that σ itself is monoidal natural, unless the braiding is symmetric. Indeed, its monoidal naturality diagram is as follows:

$$
\begin{array}{ccc}
A_1 A_2 B_1 B_2 & \xrightarrow{\ \sigma\sigma\ } & A_2 A_1 B_2 B_1 \\
\Big\downarrow{\scriptstyle 1\sigma 1} & & \Big\downarrow{\scriptstyle 1\sigma 1} \\
A_1 B_1 A_2 B_2 & \xrightarrow[\ \sigma\ ]{} & A_2 B_2 A_1 B_1
\end{array}
$$

Translated into braids:

and

$$A_1 \quad A_2 \quad B_1 \quad B_2 \qquad\qquad\qquad A_1 \quad A_2 \quad B_1 \quad B_2$$
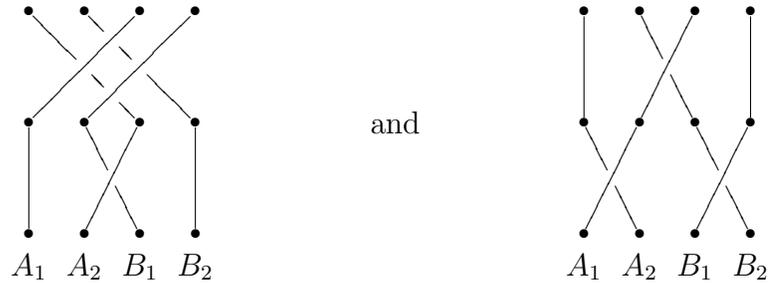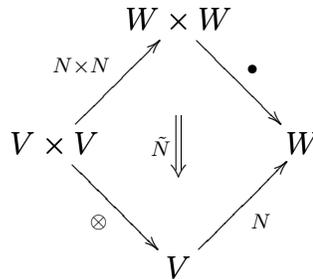
Which, upon inspection, are *not* isotopic, showing that, in general, σ is not a monoidal natural transformation.

Finally, our most important result for this section:

Proposition 5.3.6. *Suppose that* $(\mathbf{V}, \otimes, \mathbf{I}, \sigma_{\mathbf{V}}) \xrightarrow{N} (\mathbf{W}, \bullet, \mathbf{J}, \sigma_{\mathbf{W}})$ *is a braided monoidal functor. Then* $\widetilde{N}$ *is a monoidal natural transformation.*

Proof. The comparison $\widetilde{N}$ goes between the following functors:

$$
\begin{array}{ccc}
 & W \times W & \\
\nearrow^{N \times N} & & \searrow^{\bullet} \\
V \times V \quad \tilde{N}\big\Downarrow & & W \\
\searrow_{\otimes} & & \nearrow_{N} \\
 & V &
\end{array}
$$

We have shown that for $\mathbf{V}$ braided, the tensor product for $\mathbf{V}$ is itself a monoidal functor, so that the functors and categories above are all monoidal. We now wish to show that in addition, $\widetilde{N}$ is itself monoidal.

If we let $F = \bullet(N \times N)$ and $G = N(\otimes)$, then we need to show that the following commutes:

$$
\begin{array}{ccc}
FA * FB & \xrightarrow{\widetilde{N} * \widetilde{N}} & GA * GB \\
\Big\downarrow^{\widetilde{F}} & & \Big\downarrow^{\widetilde{G}} \\
F(A \otimes B) & \xrightarrow[\widetilde{N}]{} & G(A \otimes B)
\end{array}
$$

We now expand the above diagram, with $A = (x_1, y_1), B = (x_2, y_2)$. Because of space constraints, we will leave out the associativity isomorphisms, and write both tensors as juxtaposition.

The top and bottom regions commute by coherence of N, the middle region by naturality of $\widetilde{N}$, the top left parallelogram since N is braid monoidal, and the bottom right parallelogram again by naturality of $\widetilde{N}$. ∎

5.4 $(-)_*$ is Monoidal

In this section, we will show how $(-)_*$ itself has monoidal structure.

Proposition 5.4.1. *The category* **moncat** *is cartesian and hence monoidal.*

Proof. Earlier in this chapter, we defined the product of monoidal categories, in proposition 5.3.3. The product of monoidal functors is also readily seen to be monoidal. If $V_1 \xrightarrow{N} W_1$ and $V_2 \xrightarrow{M} W_2$ are monoidal functors, then their product comparison is given by $\widetilde{N} \times \widetilde{M}$, and the axioms hold since the functor $\times$ preserves the diagrams showing that $\widetilde{N}$ and $\widetilde{M}$ are both themselves monoidal. The unit monoidal category is the category **1**, which has one object and one (identity) arrow. ∎

We now have that both **moncat** and **2-cat** are monoidal 2-categories. We can then ask whether $(-)_*$, as a 2-functor between them, is itself monoidal. The answer

is yes, though, somewhat surprisingly, given the nature of the two categories, it is not a strong or strict monoidal functor.

Proposition 5.4.2. *The functor* **moncat** $\xrightarrow{(-)_*}$ **2-cat** *can be equipped with comparisons which make it into a monoidal 2-functor.*

Proof. For the unit comparison, we need a functor from the unit category **1** to **1-cat**. Since **1-cat** = **set**, define this functor by sending the single object of **1** to the 1-point set $\{*\}$. For the tensor comparison, we need a 2-functor

$$\mathbf{V}\text{-}\mathbf{cat} \times \mathbf{W}\text{-}\mathbf{cat} \xrightarrow{\widetilde{(-)_*}} (\mathbf{V} \times \mathbf{W})\text{-}\mathbf{cat}.$$

Let $\mathbf{X} \in \mathbf{V}\text{-}\mathbf{cat}$ and $\mathbf{Y} \in \mathbf{W}\text{-}\mathbf{cat}$. Define the $(\mathbf{V} \times \mathbf{W})$ category $\widetilde{(-)_*}(\mathbf{X}, \mathbf{Y})$ to have object set $X \times Y$, and homs

$$[\widetilde{(-)_*}](\mathbf{X}, \mathbf{Y})((x_1, y_1), (x_2, y_2)) := (\mathbf{X}(x_1, x_2), \mathbf{Y}(y_1, y_2)).$$

Given a $\mathbf{V}$-functor $\mathbf{X_1} \xrightarrow{F} \mathbf{X_2}$ and a $\mathbf{W}$-functor $\mathbf{Y_1} \xrightarrow{G} \mathbf{Y_2}$, define $\widetilde{(-)_*}(F, G)$ to have effect on objects given by $F \times G$, with effect on homs also given by product:

$$(\mathbf{X_1}(x_1, x_2), \mathbf{Y_1}(y_1, y_2)) \xrightarrow{(F(x_1, x_2), G(y_1, y_2))} (\mathbf{X_2}(Fx_1, Fx_2), \mathbf{Y_2}(Gx_1, Gx_2))$$

The axioms for a $\mathbf{V}$-functor are then easy to check. The definition of the product of natural transformations is similarly easy to define and check.

For the tensor comparison axiom, we need the following diagram to commute:

$$
\begin{array}{ccc}
\mathbf{V_*W_*Z_*} & \xrightarrow{1(\tilde{-})_*} & \mathbf{V_*(WZ)_*} \\
\downarrow{\scriptstyle (\tilde{-})_*1} & & \downarrow{\scriptstyle (\tilde{-})_*} \\
\mathbf{(VW)_*Z_*} & \xrightarrow{(\tilde{-})_*} & \mathbf{(VWZ)_*}
\end{array}
$$

This is easy to check. For $\mathbf{X_1} \in \mathbf{V}\text{-}\mathbf{cat}, \mathbf{X_2} \in \mathbf{W}\text{-}\mathbf{cat}, \mathbf{X_3} \in \mathbf{Z}\text{-}\mathbf{cat}$, both composites have object set $X_1 \times X_2 \times X_3$, and both have homs $(\mathbf{X_1}(a_1, b_1), \mathbf{X_2}(a_2, b_2), \mathbf{X_3}(a_3, b_3))$.

For naturality, we need the following diagram to commute, for $\mathbf{V_1} \xrightarrow{N} \mathbf{V_2}$ and $\mathbf{W_1} \xrightarrow{M} \mathbf{W_2}$ monoidal functors:

$$
\begin{array}{ccc}
\mathbf{V_1}\text{-cat} \times \mathbf{W_1}\text{-cat} & \xrightarrow{[\tilde{()}]_\star} & (\mathbf{V_1} \times \mathbf{W_1})\text{-cat} \\
\downarrow{\scriptstyle N_\star \times M_\star} & & \downarrow{\scriptstyle (N \times M)_\star} \\
\mathbf{V_2}\text{-cat} \times \mathbf{W_2}\text{-cat} & \xrightarrow[{[\tilde{()_\star}]}]{} & (\mathbf{V_2} \times \mathbf{W_2})\text{-cat}
\end{array}
$$

For objects $x_1, x_2 \in \mathbf{X}$, $y_1, y_2 \in \mathbf{Y}$, the left composite is given by

$$
\begin{aligned}
& \widetilde{(-)}_*[N_* \times M_*(\mathbf{X}, \mathbf{Y})]((x_1, y_1), (x_2, y_2)) \\
=\ & \widetilde{(-)}_*(N_*\mathbf{X}, M_*\mathbf{Y}((x_1, y_1), (x_2, y_2)) \\
=\ & (N_*\mathbf{X}(x_1, x_2), M_*\mathbf{Y}(y_1, y_2)) \\
=\ & (N\mathbf{X}(x_1, x_2), M\mathbf{Y}(y_1, y_2)),
\end{aligned}
$$

while the right composite is

$$
\begin{aligned}
& (N \times M)_*(\widetilde{(-)}_*)(\mathbf{X} \times \mathbf{Y})((x_1, y_1), (x_2, y_2)) \\
=\ & (N \times M)_*(\mathbf{X}(x_1, x_2), \mathbf{Y}(y_1, y_2)) \\
=\ & (N\mathbf{X}(x_1, x_2), M\mathbf{Y}(y_1, y_2)).
\end{aligned}
$$

Thus, the diagram commutes, and $\widetilde{(-)}_*$ is natural.

Showing that the unit axioms hold is similar.

$\blacksquare$

Before we move on, we should note a nice bonus of this proposition: it gives a direct proof that for $\mathbf{V}$ braided, $\mathbf{V}\text{-cat}$ is a monoidal 2-category. Since $(-)_*$ is monoidal, it takes pseudomonoids to pseudomonoids, and so takes a pseudomonoid in $\mathbf{moncat}$ to a pseudomonoid in $\mathbf{2-cat}$; that is, a monoidal 2-category.

In fact, this tells us that one doesn't need $\mathbf{V}$ braided for $\mathbf{V}\text{-}\mathbf{cat}$ to be monoidal. One only needs $\mathbf{V}$ to have pseudomonoidal structure in $\mathbf{moncat}$. This is a more general object than a braided monoidal category. Specifically, it is a monoidal category $(\mathbf{V}, \otimes, I)$, equipped with additional monoidal functors $\mathbf{V} \times \mathbf{V} \xrightarrow{\star} \mathbf{V}$ and $\mathbf{1} \xrightarrow{J} \mathbf{V}$. The comparison for $\star$ gives a middle-four comparison

$$(x_1 \star x_2) \otimes (x_3 \star x_4) \longrightarrow (x_1 \otimes x_3) \star (x_2 \otimes x_4)$$

and the unit comparison gives us

$$I \longrightarrow J$$

satisfying coherence axioms. If $\mathbf{V}$ is braided, one can take $\otimes = \star, I = J$; the middle-four comparison is then an instance of the braiding.

5.5   Tensor Product of V-categories

In the previous sections, we have shown that for a braided monoidal category $(\mathbf{V}, \otimes, I, \sigma)$, the tensor product $\otimes$ is a monoidal functor, and we have demonstrated that $(-)_*$ is a monoidal 2-functor. We now combine these two things to define the tensor prodcut of $\mathbf{V}$-categories.

Definition   Let $(\mathbf{V}, \otimes, I, \sigma)$ be a braided monoidal category. Define the tensor product of $\mathbf{V}$ categories by the composite

$$\mathbf{V}\text{-}\mathbf{cat} \times \mathbf{V}\text{-}\mathbf{cat} \xrightarrow{\widetilde{(-)_*}} (\mathbf{V} \times \mathbf{V})\text{-}\mathbf{cat} \xrightarrow{\otimes_*} \mathbf{V}\text{-}\mathbf{cat}$$

Defining the tensor product of $\mathbf{V}$-categories in this way has several advantages. The biggest advantage is that we require no additional work to show that our tensor product of $\mathbf{V}$-categories is again a $\mathbf{V}$-category. If one defines the tensor product directly, this does in fact take a lot of work to prove: see, for example, Forcey ([17], pgs. 11-19). The second advantage is that knowing that the tensor product of $\mathbf{V}$-categories factors as above is a neccesary ingredient for our proof that N_* preserves monoidal $\mathbf{V}$-categories. The final advantage is that we have gained a greater understanding of the tensor product construction. Interestingly, the first factor of the tensor product

always exists, whether the base $\mathbf{V}$ is braided or not. It is only the second factor that requires braiding. In some sense, then, the map

$$\mathbf{V}\text{-}\mathbf{cat} \times \mathbf{V}\text{-}\mathbf{cat} \xrightarrow{\;(\tilde{-})_*\;} (\mathbf{V} \times \mathbf{V})\text{-}\mathbf{cat}.$$

is a "pre"tensor product for $\mathbf{V}$-categories which always exists, regardless of braiding considerations.

Before moving on, let us expand the definition of the tensor product of $\mathbf{V}$-categories (as given above) to show that it agrees with the usual definition of the tensor product of $\mathbf{V}$-categories.

Proposition 5.5.1. *Let* $(\mathbf{V}, \otimes, I, \sigma)$ *be a braided monoidal category, and let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be* $\mathbf{V}$*-categories. Then the tensor product of* $\mathbf{V}$*-categories* $\mathbf{X} \otimes \mathbf{Y}$ *has objects* $X \times Y$, *homs given by*

$$(\mathbf{X} \otimes \mathbf{Y})[(x_1, y_1), (x_2, y_2)] := \mathbf{X}(x_1, x_2) \otimes \mathbf{Y}(y_1, y_2),$$

composition given by the composite

$$\{\mathbf{X}(x_1, x_2) \otimes \mathbf{Y}(y_1, y_2)\} \otimes \{\mathbf{X}(x_2, x_3) \otimes \mathbf{Y}(y_2, y_3)\}$$

$$\downarrow {\scriptstyle (1 \otimes a^{-1})(a)}$$

$$\mathbf{X}(x_1, x_2) \otimes \{\{\mathbf{Y}(y_1, y_2) \otimes \mathbf{X}(x_2, x_3)\} \otimes \mathbf{Y}(y_2, y_3)\}$$

$$\downarrow {\scriptstyle 1 \otimes (\sigma \otimes 1)}$$

$$\mathbf{X}(x_1, x_2) \otimes \{\{\mathbf{X}(x_2, x_3) \otimes \mathbf{Y}(y_1, y_2)\} \otimes \mathbf{Y}(y_2, y_3)\}$$

$$\downarrow {\scriptstyle (a^{-1})(1 \otimes a)}$$

$$\{\mathbf{X}(x_1, x_2) \otimes \mathbf{X}(x_2, x_3)\} \otimes \{\mathbf{Y}(y_1, y_2) \otimes \mathbf{Y}(y_2, y_3)\} \xrightarrow{\;c \otimes c\;} \mathbf{X}(x_1, x_3) \otimes \mathbf{Y}(y_1, y_3)$$

and identities given by:

$$I \xrightarrow{\;r^{-1}\;} I \otimes I \xrightarrow{\;i_x \otimes i_y\;} \mathbf{X}(x, x) \otimes \mathbf{Y}(y, y).$$

This is indeed the same tensor product as given in Eilenberg and Kelly ([16], pg. 518), Joyal and Street ([23], pg. 22), or Forcey ([17], pg. 11).

Finally, let us define the **V**-category which acts as a unit for this tensor operation:

Definition Let $(\mathbf{V}, \otimes, I)$ be a monoidal category. The *unit* **V**-category $\mathbb{I}$ is the **V**-category with a single object $*$, and hom $\mathbb{I}(*, *) := I$.

5.6 Monoidal Bicategories

In the previous section, we saw an example of a 2-category with a monoidal operation: the 2-category **V-cat**. We will now give the abstract definition of a bicategory with a monoidal operation. Technically, we do not need the full generality of a monoidal *bicategory* for this chapter, since our bicategory is in fact a 2-category. However, we will need this full generality in the next chapter, when we deal with the bicategory **V-prof**.

The definition we give will actually be a Gray monoid rather than a monoidal bicategory. As Day and Street note in Monoidal Bicategories ([13], pg. 100), the coherence theorem of the monograph on tricategories ([18]), allows one to prove results in the simpler Gray monoids, then transfer to the more general monoidal bicategory. In terms of the monoidal structure, the main difference between the two is that in a general monoidal bicategory, associators are required at both the object and arrow level, while in a Gray monoid, associators are only required for the arrows.

Definition A *Gray monoid* is a 2-category $\mathcal{M}$, together with an object $I \in \mathcal{M}$, for any object $A \in \mathcal{M}$ two 2-functors

$$\mathcal{M} \xrightarrow{L_A, R_A} \mathcal{M}$$

which agree on objects (define $A \otimes B := L_A(B) = R_B(A)$), and for arrows $A \xrightarrow{f} A', B \xrightarrow{g} B'$, an invertible 2-cell

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{L_A(g)} & A \otimes B' \\
{\scriptstyle R_B(f)} \downarrow & \overset{c_{f,g}}{\Longrightarrow} & \downarrow {\scriptstyle R_{B'}(f)} \\
A' \otimes B & \xrightarrow[L_{A'}(g)]{} & A' \otimes B'
\end{array}
$$

The following axioms then must hold:

1. For all objects A, B:

$$L_I = R_I = 1_{\mathcal{M}}, \quad L_{A \otimes B} = L_A L_B, \quad R_{A \otimes B} = R_B R_A, \quad R_B L_A = L_A R_B.$$

2. If both f and g are identites, then $c_{f,g}$ is an identity 2-cell.

3. For all arrows $A \xrightarrow{f} A', B \xrightarrow{g} B', C \xrightarrow{h} C'$,

$$L(A)(c_{g,h}) = c_{L_A(g),h}, \quad c_{f,L_B(h)} = c_{R_B(f),h}, \quad R_C(c_{f,g}) = c_{f,R_C(g)}$$

4. For arrows $A \xrightarrow{f,h} A', B \xrightarrow{g,k} B'$, and 2-cells $f \xrightarrow{\alpha} h, g \xrightarrow{\beta} k$,

5. For arrows $A \xrightarrow{f} A', B \xrightarrow{g} B', A' \xrightarrow{f'} A'', B' \xrightarrow{g'} B''$,

As we have just seen, **V-cat**, for braided **V**, is an example of a Gray monoid/monoidal bicategory. The bicategory **V-prof**, to be defined in the next chapter, is also a monoidal bicategory (again for **V** braided).

Now that we have defined Gray monoids, we can define pseudomonoids therein: they are monoidal objects, moved up a dimension. In this definition, and often throughout the rest of the thesis, we shall use A^n to mean $A \otimes A \otimes \cdots \otimes A \otimes A$ (n times).

Definition A *pseudomonoid* in a Gray monoid $(\mathcal{M}, \otimes, I)$ consists of an object $A \in \mathcal{M}$, arrows

$$A^2 \xrightarrow{\mu} A, I \xrightarrow{\eta} A$$

and invertible 2-cells

$$
\begin{array}{ccc}
A^3 \xrightarrow{1\otimes\mu} A^2 & \quad & A \\
\end{array}
$$

with 2-cells α, l, r in squares:
$$
\begin{CD}
A^3 @>{1\otimes\mu}>> A^2 \\
@V{\mu\otimes1}VV @VV{\mu}V \\
A^2 @>>{\mu}> A
\end{CD}
\quad \alpha \Rightarrow
$$

$$
\begin{CD}
A @. \\
@V{\eta\otimes1}VV \searrow^{1} \\
A^2 @>>{\mu}> A
\end{CD}
\quad l \Rightarrow
\qquad
\begin{CD}
A @. \\
@V{1\otimes\eta}VV \searrow^{1} \\
A^2 @>>{\mu}> A
\end{CD}
\quad r \Rightarrow
$$

such that the following equalities are satisfied:

$$
\begin{array}{ccc}
& & \\
\text{(left 3-dimensional diagram)} & = & \text{(right 3-dimensional diagram)}
\end{array}
$$

with labels $A^4 \xrightarrow{11\mu} A^3$, $\mu11$, $1\mu1$, 1μ, $\alpha1 \Rightarrow$, $\mu1$, A^3, A^2, $1\alpha \Rightarrow$, $\alpha \Rightarrow$, μ, A on the left; and $A^4 \xrightarrow{11\mu} A^3$, $\mu11$, $c_{\mu,\mu} \Rightarrow$, $\mu1$, 1μ, A^2, $\alpha \Rightarrow$, $1u$, A^2, $\alpha \Rightarrow$, $\mu1$, μ, A on the right.

and

$$
\text{(left diagram)} = \text{(right diagram)}
$$

Left: A^2, $1\eta1$, $1l \Rightarrow$, 1, $A^3 \xrightarrow{1\mu} A^2$, $\mu1$, $\alpha \Rightarrow$, μ, $A^2 \xrightarrow{\mu} A$.

Right: A^2, $1\eta1$, 1, A^3, $r1 \Rightarrow$, 1, A^2, $\mu1$, $=$, μ, $A^2 \xrightarrow{\mu} A$.

We can now relate this to the idea of monoidal **V**-categories by the following proposition:

Proposition 5.6.1. *A pseudomonoid in the monoidal bicategory* **V-cat** *is a monoidal* **V***-category.*

Proof. See Day and Street ([13], pg. 141). ∎

We now need to find what types of morphisms of bicategories preserve pseudomonoids.

Definition A *weak monoidal homomorphism* T between monoidal bicategories $(\mathcal{M}, \otimes, I)$, $(\mathcal{N}, \bullet, J)$ is a pseudofunctor from $\mathcal{M}$ to $\mathcal{N}$, equipped with a pseudonatural transformation χ, a 1-cell ι, and modifications ω, ζ, κ. The pseudonatural transformation χ has 1-cell components:

$$TX \bullet TY \xrightarrow{\chi_{X,Y}} T(X \otimes Y)$$

and 2-cell components:

$$
\begin{array}{ccc}
TX \bullet TY & \xrightarrow{\quad \chi \quad} & T(X \otimes Y) \\
{\scriptstyle Tf \bullet Tg} \downarrow & {\scriptstyle \chi_{f,g}} \atop \Rightarrow & \downarrow {\scriptstyle T(f \otimes g)} \\
TX' \bullet TY' & \xrightarrow{\quad \chi \quad} & T(X' \otimes Y')
\end{array}
$$

The arrow ι compares the units:

$$J \xrightarrow{\ \iota\ } TI$$

The modification ω has components:

$$
\begin{array}{ccc}
TX \bullet TY \bullet TZ & \xrightarrow{\ 1 \bullet \chi\ } & TX \bullet T(Y \otimes Z) \\
{\scriptstyle \chi \bullet 1} \downarrow & {\scriptstyle \omega_{X,Y,Z}} \atop \Rightarrow & \downarrow {\scriptstyle \chi} \\
T(X \otimes Y) \bullet TZ & \xrightarrow{\quad \chi \quad} & T(X \otimes Y \otimes Z)
\end{array}
$$

The modifications ζ, κ have components:

$$
\begin{array}{ccccc}
 & & TX & & \\
 & {\scriptstyle 1 \bullet \iota}\nearrow & \downarrow {\scriptstyle 1} & \nwarrow {\scriptstyle \iota \bullet 1} & \\
 & {\scriptstyle \zeta_X} \atop \Rightarrow & & {\scriptstyle \kappa_X} \atop \Leftarrow & \\
TX \bullet TI & \xrightarrow{\ \chi\ } & TX & \xleftarrow{\ \chi\ } & TI \bullet TX
\end{array}
$$

These are to satisfy certain coherence conditions (see [13], pg. 111).

Finally, we have the following proposition:

Proposition 5.6.2. *Weak monoidal homomorphisms take pseudomonoids to pseudomonoids.*

Proof. The proof is quite straightfoward, and can again be found in Day and Street ([13], pg. 112). ∎

5.7 Proof of Result

We are now in a position to prove our main result for this chapter.

Theorem 5.7.1. *Suppose that* $(\mathbf{V}, \otimes, I, \sigma_1) \xrightarrow{\ N\ } (\mathbf{W}, \bullet, J, \sigma_2)$ *is a braided monoidal functor between braided monoidal categories. Then the change of base 2-functor* N_* *takes monoidal* $\mathbf{V}$*-categories to monoidal* $\mathbf{W}$*-categories.*

Proof. As we have seen in the previous section, to prove this result, it suffices to show that N_* has the structure of a weak monoidal homomorphism. As in the introduction, define χ by composing the following 2-cells:

We now need to define the modifications ω, ζ, κ. Since these are higher-dimesional versions of the equations for a monoidal functor, we can do as we did above: that is, make the equations for $\widetilde{N}$ into composites of two-cells, and apply $(-)_*$, adding in extra cells as neccesary, to get the modifications ω, ζ, and κ. However, it is easier

to work with them directly, as it turns out that the composites they go between are equal. Indeed, ω should be a 2-cell in the square

$$
\begin{array}{ccc}
N_*\mathbf{X} \bullet N_*\mathbf{Y} \bullet N_*\mathbf{Z} & \xrightarrow{\ 1\bullet\chi\ } & N_*\mathbf{X} \bullet N_*\mathbf{YZ} \\
\downarrow{\scriptstyle \chi\bullet 1} & & \downarrow{\scriptstyle \chi} \\
N_*\mathbf{XY} \bullet N_*\mathbf{Z} & \xrightarrow{\quad\chi\quad} & N_*\mathbf{XYZ}
\end{array}
$$

Given $x_1, x_2 \in \mathbf{X}, y_1, y_2 \in \mathbf{Y}, z_1, z_2 \in \mathbf{Z}$, the actions of these composite $\mathbf{W}$-functors are

$$
\begin{array}{ccc}
N\mathbf{X}(x_1,x_2) \bullet N\mathbf{Y}(y_1,y_2) \bullet N\mathbf{Z}(z_1,z_2) & \xrightarrow{\ 1\bullet\widetilde{N}\ } & N\mathbf{X}(x_1,x_2) \bullet N(\mathbf{Y}(y_1,y_2) \otimes \mathbf{Z}(z_1,z_2)) \\
\downarrow{\scriptstyle \widetilde{N}\bullet 1} & & \downarrow{\scriptstyle \widetilde{N}} \\
N(\mathbf{X}(x_1,x_2) \otimes \mathbf{Y}(y_1,y_2)) \bullet \mathbf{Z}(z_1,z_2) & \xrightarrow{\quad\widetilde{N}\quad} & N(\mathbf{X}(x_1,x_2) \otimes \mathbf{Y}(y_1,y_2) \otimes \mathbf{Z}(z_1,z_2))
\end{array}
$$

Thus these composites are in fact equal, by the coherence of $\widetilde{N}$. Similarly, the composites which ζ and κ need to compare are also equal. As a result, we can make ω, ζ, and κ the identity modifications. The coherence conditions for these modifications then follow automatially.

Thus, N_* can be assigned the structure of a weak monoidal homormophism, and as a result, preserves monoidal $\mathbf{V}$-categories, as required.

∎

5.8   Involutions

In our final section of this chapter, we investigate the involutive structure of $\mathbf{V}\text{-}\mathbf{cat}$. One can construct the opposite of a $\mathbf{V}$-category, provided that $\mathbf{V}$ is braided. What is interesting in the above context is that this involutive structure, namely the map which takes a $\mathbf{V}$-category and gives the opposite $\mathbf{V}$-category:

$$
(\mathbf{V}\text{-}\mathbf{cat})^{\mathrm{co}} \xrightarrow{\ ()^{\mathrm{op}}\ } \mathbf{V}\text{-}\mathbf{cat},
$$

can be construcuted similarly to the tensor product on $\mathbf{V}\text{-}\mathbf{cat}$. It too, can be factored into two parts. The first, which exists whether $\mathbf{V}$ is braided or not, can be seen as

part of the structure on the 2-functor $(-)_*$. The second is an application of $(-)_*$ to a monoidal functor going from $\mathbf{V}^{\mathrm{rev}}$ to $\mathbf{V}$, where $\mathbf{V}^{\mathrm{rev}}$ is the monoidal category with $A \otimes_{\mathrm{rev}} B := B \otimes A$.

Let us begin by formally defining this "reversed" monoidal category.

Definition Suppose that $(\mathbf{V}, \otimes, I, a, l, r)$ is a monoidal category. Define a new monoidal category $\mathbf{V}^{\mathrm{rev}}$, with the same objects as $\mathbf{V}$, with the tensor product defined as above, with $a_{\mathrm{rev}} := a^{-1}$, $l_{\mathrm{rev}} = r$, and $r_{\mathrm{rev}} = l$.

For $\mathbf{V}$ braided, there is always a monoidal functor from $\mathbf{V}^{\mathrm{rev}}$ to $\mathbf{V}$:

Proposition 5.8.1. *Suppose that $(\mathbf{V}, \otimes, I, \sigma)$ is a braided monoidal category. Then there is a monoidal functor $\mathbf{V}^{rev} \xrightarrow{\ U\ } \mathbf{V}$, defined as the identity on objects, which has the tensor comparison*
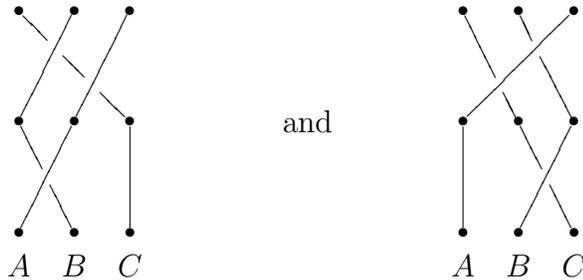
$$A \otimes B \xrightarrow{\ \sigma\ } B \otimes A$$

and unit comparison the identity on I.

Proof. To check that this is a monoidal functor, we need to show that the following diagram commutes:

$$
\begin{array}{ccc}
A \otimes B \otimes C & \xrightarrow{\ 1 \otimes \sigma\ } & A \otimes C \otimes B \\
{\scriptstyle \sigma \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\
B \otimes A \otimes C & \xrightarrow[\ \sigma\ ]{} & C \otimes B \otimes A
\end{array}
$$

Translating the composites to braids, we have

$$\text{and}$$

$$A \quad B \quad C \qquad\qquad A \quad B \quad C$$

The two are indeed braid isotopic. The unit axioms follow from proposition 5.2.1. ∎

Now, we can investigate the structure on $(-)_*$. We have seen that $(-)_*$ is a 2-functor with a weak monoidal structure. However, it has even more than that: it also compares the op structures of **moncat** and **2-cat**. We call these op structures involutions on 2-categories.

Definition An *involution* on a 2-category $\mathbf{A}$ is a 2-functor $\mathbf{A} \xrightarrow{R} \mathbf{A}$ such that R is strong monoidal, and $R^2 = 1_{\mathbf{A}}$.

For **moncat**, the op structure is given by the 2-functor $\mathbf{moncat} \xrightarrow{()^{\mathrm{rev}}} \mathbf{moncat}$. For **2-cat**, the op structure is given by $()^{\mathbf{co}}$, the reversal of 2-cells.

Now, an involutive monoidal functor between involutive monoidal categories would compare the op structures, in the sense that we should have a 2-cell:

$$
\begin{array}{ccc}
\mathbf{moncat} & \xrightarrow{\;()_*\;} & \mathbf{2\text{-}cat} \\
{\scriptstyle \mathrm{rev}} \downarrow & {\displaystyle ()^{\mathrm{rev}} \atop \Longleftarrow} & \downarrow {\scriptstyle \mathrm{co}} \\
\mathbf{moncat} & \xrightarrow[\;()_*\;]{} & \mathbf{2\text{-}cat}
\end{array}
$$

so that we get for each $\mathbf{V} \in \mathbf{moncat}$ a 2-functor

$$
(\mathbf{V\text{-}cat})^{\mathrm{co}} \xrightarrow{\;[()^{\mathbf{op}}]_{\mathbf{V}}\;} \mathbf{V}^{\mathrm{rev}}\text{-}\mathbf{cat}
$$

We can define such a functor as follows. For $\mathbf{X} \in \mathbf{V\text{-}cat}$, X^{op} will have the same objects as V, but $X^{\mathrm{op}}(x, y) := X(y, x)$, with composition similarly adjusted. Similarly, the action of a $\mathbf{V}$-functor will remain the same, but the strength will reverse. Because of these reversals, while $()^{\mathrm{op}}$ does not change the actual 2-cells, they will go in opposite directions; hence the need for $_{\mathrm{co}}$. This functor then makes $()_*$ into an involutive monoidal 2-functor.

Then, just as we did with the tensor product of $\mathbf{V}$-categories, we can define the $(-)^{\mathrm{op}}$ operation on $\mathbf{V\text{-}cat}$ as a composite.

Definition Suppose that $(\mathbf{V}, \otimes, I, \sigma)$ is a braided monoidal category. Then define the opposite $\mathbf{V}$-category operation as the composite

$$\mathbf{V}\text{-cat}^{\mathrm{co}} \xrightarrow{[()^{\mathbf{op}}]_{\mathbf{V}}} \mathbf{V}^{\mathrm{rev}}\text{-cat} \xrightarrow{U_*} \mathbf{V}\text{-cat}$$

As with our definition of the tensor product of $\mathbf{V}$-categories, this definition makes clear that there is a "pre" op-operation on $\mathbf{V}$-**cat**, and instantly demonstrates, without any further work, how an opposite $\mathbf{V}$-category is again a $\mathbf{V}$-category.

Chapter 6

Change of Base for Profunctors

To show that N_* preserved monoidal $\mathbf{V}$-categories, we showed that N_* was 2-monoidal, and so took pseudomonoids in $\mathbf{V\text{-}cat}$ to pseudomonoids in $\mathbf{W\text{-}cat}$. If we wish to look at whether N_* preserves autonomous (compact) $\mathbf{V}$-categories, we would like to do something similar: show that N_* is an "autonomous" monoidal functor, and so preserves autonomous objects.

Unfortunately, we cannot do this in the 2-category $\mathbf{V\text{-}cat}$, as the notion of an autonomous object in $\mathbf{V\text{-}cat}$ does not make sense. The problem is that to describe autonomous objects in a bicategory, the bicategory itself must be autonomous. This is analagous to the situation of monoids in $\mathbf{V\text{-}cat}$: to describe monoids in $\mathbf{V\text{-}cat}$, $\mathbf{V\text{-}cat}$ itself must have monoidal structure - which it does, if $\mathbf{V}$ is braided. On the other hand, $\mathbf{V\text{-}cat}$ does not have autonomous structure.

This problem can be rectified if one considers the bicategory $\mathbf{V\text{-}prof}$ instead. Here, the 1-cells are $\mathbf{V}$-profunctors rather than $\mathbf{V}$-functors. Then, $\mathbf{V\text{-}prof}$ does have an autonomous structure, and certain types of autonomous objects in $\mathbf{V\text{-}prof}$ are exactly the autonomous $\mathbf{V}$-categories.

Thus, if we want to do something similar to the previous chapter, what we need first of all is a change of base from $\mathbf{V\text{-}prof}$ to $\mathbf{W\text{-}prof}$. As soon as we begin investigating the change of base for profunctors, several points of interest arise. For N monoidal, the change of base we get $\mathbf{V\text{-}prof} \xrightarrow{N_\#} \mathbf{W\text{-}prof}$ is a lax, rather than pseudo, functor. In other words, it preserves composition only up to a comparison arrow. We then show that a monoidal natural transformation can be sent to a new type of 2-cell between lax functors: a module. Two recent papers ([11], [28]) have

discussed this idea, and argued that modules are the proper 2-cells for lax functors; the results of this change of base give further evidence that that is the case.

The chapter is laid out as follows. We begin by reviewing the idea of **V**-profunctors, and their associated bicategory **V-prof**. We then describe the change of base lax functor $N_\#$. Finally, we show how monoidal natural transformations can be sent to modules.

6.1 The Bicategory V-prof

In this section, we describe the basics of the bicategory of **V**-categories, **V**-profunctors, and **V**-profunctor morphisms. As we will be working directly with this bicategory, it is necessary to describe explicitly all of these notions, together with their associated compositions and identities.

6.1.1 V-profunctors

V-profunctors $\mathbf{X} \xrightarrow{P} \mathbf{Y}$ are often described as **V**-functors

$$\mathbf{Y}^{\mathrm{op}} \otimes \mathbf{X} \longrightarrow \mathbf{V}$$

However, describing **V**-profunctors in this way requires that **V** be braided (for $\mathbf{Y}^{\mathrm{op}} \otimes \mathbf{X}$ to be a **V**-category) and closed (for **V** to be a **V**-category). At least initially, we would like to work in as general a setting as possible, and describe the change of base for profunctors for any monoidal **V**. Accordingly, we will give the more elementary definition of a profunctor, in terms of a module together with actions. In fact, there is another reason for describing them in this way, not just a desire for greater generality: it is our belief that results are clearer and easier to understand when profunctors are described as modules rather than as functors.

Definition Given **V**-categories $\mathbf{X}, \mathbf{Y}$, a **V**-profunctor $\mathbf{X} \xrightarrow{P} \mathbf{Y}$ consists of families of objects of **V**, as well as left and right actions. So we have, for each $x \in \mathbf{X}$, $y \in \mathbf{Y}$, an object $P(y, x) \in \mathbf{V}$; the left action consists of a family of **V**-arrows,

$$\mathbf{Y}(y', y) \otimes P(y, x) \xrightarrow{P_l(y', y, x)} P(y', x)$$

and the right action is another family of **V**-arrows

$$P(y,x) \otimes \mathbf{X}(x,x') \xrightarrow{P_r(y,x,x')} P(y,x').$$

These actions must be unitary, so that the following diagrams commute:

$I \otimes P(y,x)$

$1_y \otimes 1$ l

$\mathbf{Y}(y,y) \otimes P(y,x) \xrightarrow{P_l} P(y,x)$

$P(y,x) \otimes I$

$1 \otimes 1_x$ r

$P(y,x) \otimes \mathbf{X}(x,x) \xrightarrow{P_r} P(y,x)$

These actions are associative in **Y**:

$$(\mathbf{Y}(y'',y') \otimes \mathbf{Y}(y',y)) \otimes P(y,x) \xrightarrow{a} \mathbf{Y}(y'',y') \otimes (\mathbf{Y}(y',y) \otimes P(y,x))$$

$c \otimes 1$ $1 \otimes P_l$

$\mathbf{Y}(y'',y) \otimes P(y,x)$ $\mathbf{Y}(y'',y') \otimes P(y',x)$

P_l P_l

$P(y'',x)$

and in **X**:

$$(P(y,x) \otimes \mathbf{X}(x,x')) \otimes \mathbf{X}(x',x'') \xrightarrow{a} P(y,x) \otimes (\mathbf{X}(x,x') \otimes \mathbf{X}(x',x''))$$

$P_r \otimes 1$ $1 \otimes c$

$P(y,x') \otimes \mathbf{X}(x',x'')$ $P(y,x) \otimes \mathbf{X}(x,x'')$

P_r P_r

$P(y,x'')$

Finally, the actions must be jointly associative:

$$(\mathbf{Y}(y'y) \otimes P(y,x)) \otimes \mathbf{X}(x,x') \xrightarrow{a} \mathbf{Y}(y'y) \otimes (P(y,x) \otimes \mathbf{X}(x,x'))$$

$P_l \otimes 1$ $1 \otimes P_r$

$P(y',x) \otimes \mathbf{X}(x,x')$ $\mathbf{Y}(y'y) \otimes P(y,x')$

P_r P_l

$P(y',x')$

Certainly, the definition of a **V**-profunctor is not as natural as that of **V**-functor. However, they turn out to be familiar objects for each of the usual monoidal categories.

Example 6.1.1. A **set**-profunctor $\mathbf{X} \xrightarrow{P} \mathbf{Y}$ is the same as a functor $\mathbf{Y}^{\mathrm{op}} \times \mathbf{X} \xrightarrow{P} \mathbf{set}$. More generally, for any closed braided **V**, a **V**-profunctor is the same as a **V**-functor $\mathbf{Y}^{\mathrm{op}} \otimes \mathbf{X} \xrightarrow{P} \mathbf{V}$.

Example 6.1.2. If R and S are rings (1-object **ab**-categories), then an **ab**-profunctor $\mathbf{R} \xrightarrow{P} \mathbf{S}$ is an (R, S)-bimodule. Thus the idea of profunctor between **ab**-categories generalizes the notion of bimodule.

Example 6.1.3. A **2**-profunctor $(X, \leq) \xrightarrow{P} (Y, \leq)$ is a relation which is down-closed in $(Y, \leq)$ and up-closed in $(X, \leq)$. Indeed, the module $P(y, x) \in \mathbf{2}$ gives a relation $\sim$, the left action gives

$$y' \leq y \sim x \Rightarrow y' \sim x,$$

and the right action gives

$$y \sim x \leq x' \Rightarrow y \sim x'.$$

Example 6.1.4. An $\mathbf{R}_+$-profunctor $(X, d) \xrightarrow{P} (Y, d)$ is the same as giving a function $Y \times X \xrightarrow{P} [0, \infty]$ such that

$$\bigwedge_{y \in Y} d(y', y) + P(y, x) \geq P(y', x)$$

and

$$\bigwedge_{x \in X} P(y, x) + d(x, x') \geq P(y, x')$$

As we shall see later (Proposition 7.1.2), these are related to Cauchy sequences.

6.1.2 V-forms

We will now define the morphisms of **V**-profunctors.

Definition A **V**-morphism of **V**-profunctors $\mathbf{X} \xrightarrow{P,Q} \mathbf{Y}$ (here also called a **V**-form) consists of a **V**-family of arrows

$$P(y, x) \xrightarrow{\psi(y, x)} Q(y, x)$$

which are compatible with the left action:

$$
\begin{array}{ccc}
\mathbf{Y}(y',y) \otimes P(y,x) & \xrightarrow{\;1\otimes\psi\;} & \mathbf{Y}(y',y) \otimes Q(y,x) \\
{\scriptstyle P_L}\big\downarrow & & \big\downarrow{\scriptstyle Q_L} \\
P(y',x) & \xrightarrow{\;\;\psi\;\;} & Q(y',x)
\end{array}
$$

and the right action:

$$
\begin{array}{ccc}
P(y,x) \otimes \mathbf{X}(x,x') & \xrightarrow{\;\psi\otimes 1\;} & Q(y,x) \otimes \mathbf{X}(x,x') \\
{\scriptstyle P_R}\big\downarrow & & \big\downarrow{\scriptstyle Q_R} \\
P(y,x') & \xrightarrow{\;\;\psi\;\;} & Q(y,x')
\end{array}
$$

6.1.3 Composition and Identities of V-profunctors

The composition of $\mathbf{V}$-profunctors $\mathbf{X} \xrightarrow{\;P\;} \mathbf{A} \xrightarrow{\;Q\;} \mathbf{Y}$ is usually described as a co-end:

$$
(QP)(y,x) := \int^{a\in\mathbf{A}} Q(y,a) \otimes P(a,x)
$$

However, this definition presumes that one has identified the profunctors with functors. As noted above, we have taken the point of view that profunctors are best viewed as modules with actions. In this case, one describes the composite module $QP(y,x)$ as the co-equalizer of

$$
\sum_{a,a'\in A} Q(y,a) \otimes A(a,a') \otimes P(a',x)
\underset{Q_L\otimes 1}{\overset{1\otimes P_R}{\rightrightarrows}}
\sum_{a\in A} Q(y,a) \otimes P(a,x)
$$

When working with these co-ends, however, it is easier to give the universal property of $\int^a Q(y,a)\otimes P(a,x)$ explicitly, without reference to any other limits or colimits. This universal property is best stated when P is merely a right (X,A) module, and Q a left (A,Y) module.

Definition Given P, Q as above, the object $\int^a Q(y,a)\otimes P(a,x)$, comes with a family of injections

$$
Q(y,a) \otimes P(a,x) \xrightarrow{\;i_a\;} \int^a Q(y,a) \otimes P(a,x)
$$

which make the following commute for each y, a, a', x:

$$(Q(y,a) \otimes A(a,a')) \otimes P(a',x) \xrightarrow{\ a\ } Q(y,a) \otimes (A(a,a') \otimes P(a',x))$$

$$Q_L \otimes 1 \downarrow \qquad\qquad\qquad\qquad 1 \otimes P_R \downarrow$$

$$Q(y,a') \otimes P(a',x) \qquad\qquad\qquad\qquad Q(y,a) \otimes P(a,x)$$

$$\searrow^{i_{a'}} \qquad\qquad i_a \swarrow$$

$$\int^a Q(y,a) \otimes P(a,x)$$

and which are universal, in that for any object C with a family of maps

$$Q(y,a) \otimes P(a,x) \xrightarrow{\ f_a\ } C$$

which make the following diagram commute:

$$(Q(y,a) \otimes A(a,a')) \otimes P(a',x) \xrightarrow{\ a\ } Q(y,a) \otimes (A(a,a') \otimes P(a',x))$$

$$Q_L \otimes 1 \downarrow \qquad\qquad\qquad\qquad 1 \otimes P_R \downarrow$$

$$Q(y,a') \otimes P(a',x) \qquad\qquad\qquad\qquad Q(y,a) \otimes P(a,x)$$

$$\searrow^{f_{a'}} \qquad\qquad f_a \swarrow$$

$$C$$

then there is unique map f such that for all a,

$$Q(y,a) \otimes P(a,x) \xrightarrow{\ i_a\ } \int^a Q(y,a) \otimes P(a,x)$$

$$\searrow_{f_a} \qquad\qquad \downarrow f$$

$$C$$

commutes.

We should make a few notes on how to work with these co-ends. To check that two maps out of a co-end are equal is easy. Indeed, suppose that we have two maps $\int^a Q(y,a) \otimes P(a,x) \xrightarrow{f,g} C$. If we show that

$$Q(y,a) \otimes P(a,x) \xrightarrow{\ i_a\ } \int^a Q(y,a) \otimes P(a,x) \xrightarrow{\ f,g\ } C$$

are equal instead, then by the universal property, this implies $f = g$. Many of our later proofs will require that we show that two maps out of co-ends, which themselves go into co-ends, will be equal. As we shall almost always define maps into co-ends by applying an injection (the i_a arrows), this will reduce our diagrams to merely looking at the actions of the arrows on the objects of the co-ends, rather than on the co-ends themselves. This calculation is shown explictly in some of our earlier proofs (an example can be found in Theorem 6.2.3), but in later proofs, we will merely show the result of pre-composing with an injection without explicity showing the calculations that reduced the co-ends to their constituent objects.

The reason for defining this composition object for mere one-sided modules is that we wish to show that this co-end respects tensoring. This proposition would be impossible to state if we looked at bimodules, as tensoring a bimodule on one side does not continue to give a bimodule (in the non-symmetric case).

Lemma 6.1.5.   *1. If Q is a left (Y, A) module, then for all $v \in V$, we can make another left (Y, A) module $v \otimes Q$ which has module objects $(v \otimes Q)(y, a) = v \otimes Q(y, a)$.*

   2. Suppose that $\otimes$ preserves colimits in each variable. Then if Q is as above, and P is a right (A, X) module, we have for all $v \in V$,

$$v \otimes \int^a Q(y, a) \otimes P(a, x) \cong \int^a (v \otimes Q)(y, a) \otimes P(a, x)$$

   A similar result holds with left replaced by right and vice versa.

Proof.   1. If we define the left action of $v \otimes Q$ as

$$(v \otimes Q(y, a)) \otimes A(a, a') \xdashrightarrow{\ (v \otimes Q)_L\ } v \otimes Q(y, a')$$

with maps a going down to $v \otimes (Q(y, a) \otimes A(a, a'))$ and $v \otimes Q_L$ going up,

then it is easy to check that this action is associative and unital.

2. Since $\otimes$ preserves colimits, the result follows.

∎

With this lemma, it is easy to define the actions of the composite module QP. By above,

$$\mathbf{Y}(y',y) \otimes \int^a Q(y,a) \otimes P(a,x) = \int^a \mathbf{Y}(y',y) \otimes Q(y,a) \otimes P(a,x),$$

so we can define the left action of QP, for $a \in \mathbf{A}$, by

$$\mathbf{Y}(y',y) \otimes Q(y,a) \otimes P(a,x) \xrightarrow{Q_L \otimes 1} Q(y',y) \otimes P(a,x) \xrightarrow{i_a} \int^a Q(y',a) \otimes P(a,x),$$

and similarly for the right action of QP.

The identity profunctor $\mathbf{X} \xrightarrow{I_{\mathbf{X}}} \mathbf{X}$ has module the hom-objects of X, $I_{\mathbf{X}}(y,x) = \mathbf{X}(y,x)$, with left and right action given by $\mathbf{X}$-composition.

Of course, to do this, we need $\mathbf{V}$ to have colimits, and $\otimes$ to preserve them. Accordingly, we make the following definition:

Definition Say that a monoidal category $\mathbf{V}$ is *cocomplete* if it has all colimits, and $\otimes$ preserves these colimits in each variable.

We have described $\mathbf{V}$-profunctors, their morphisms, their compositions, and their identities. We can now bring this all together and show that, with vertical and horizontal compositions of $\mathbf{V}$-forms, these form a bicategory. We will give the main ideas of the proof, without going into all the details, as the result is well-established in the literature (see, for example, Street [39]).

Theorem 6.1.6. *For $\mathbf{V}$ cocomplete, there is a bicategory with objects $\mathbf{V}$-categories, arrows $\mathbf{V}$-profunctors, and 2-cells $\mathbf{V}$-forms.*

Proof. The objects of $\mathbf{V}$-**prof** are $\mathbf{V}$-categories. For $\mathbf{X}, \mathbf{Y}$ $\mathbf{V}$-categories, define the category $\mathbf{V}$-**prof**$(\mathbf{X}, \mathbf{Y})$ to have objects $\mathbf{V}$-profunctors and arrows $\mathbf{V}$-forms. The

composition (vertical composition) of **V**-forms
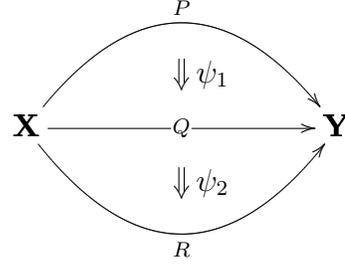
is given by simple composition:

$$(\psi_2 \circ \psi_1)(y, x) = \psi_2(y, x) \circ \psi_1(y, x).$$

Simimlarly, the identity **V**-form, for a **V**-profunctor $\mathbf{X} \xrightarrow{P} \mathbf{X}$, is simply the identity:

$$1_P(x', x) := 1_{P(x', x)}$$

Because the composition and identities are simply composition and identities in **V**, it follows immediately that **V**-prof$(\mathbf{X}, \mathbf{Y})$ is a category.

We now need to give the horizontal composition functors

$$\mathbf{V}\text{-}\mathbf{prof}(\mathbf{X}, \mathbf{Y}) \times \mathbf{V}\text{-}\mathbf{prof}(\mathbf{Y}, \mathbf{Z}) \xrightarrow{C} \mathbf{V}\text{-}\mathbf{prof}(\mathbf{X}, \mathbf{Z})$$

Above, we gave the action of this functor on objects, the co-end composition. We still need to give the action of this functor on arrows; that is, the horizontal composition of **V**-forms
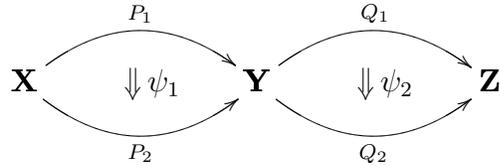
So, we need to define arrows

$$\int^y Q_1(z, y) \otimes P_1(y, x) \xrightarrow{\psi_2 * \psi_1} \int^y Q_2(z, y) \otimes P_2(y, x)$$

At a component y, define the arrows by the composite

$$Q_1(z, y) \otimes P_1(y, x) \xrightarrow{\psi_2 \otimes \psi_1} Q_2(z, y) \otimes P_2(y, x) \xrightarrow{i_y} \int^y Q_2(z, y) \otimes P_2(y, x)$$

These maps satisfy the universal property for the co-end due to the **V**-form axioms for ψ_1 and ψ_2.

Checking that this functor preserves identities is straightforward, since the vertical identities are simply identity morphisms in **V**. Checking that the functor preserves composition is the middle-4 interchange property: given **V**-forms

we need to show that

$$(\psi_4 * \psi_3) \circ (\psi_2 * \psi_1) = (\psi_4 \circ \psi_2) * (\psi_3 \circ \psi_1)$$

At $x \in \mathbf{X}, y \in \mathbf{Y}, z \in \mathbf{Y}$, the left composite and right composites are

$$(\psi_4 * \psi_3)(z, x) \circ (\psi_2)(\psi_1)(z, x) \text{ and } (\psi_4 \circ \psi_2) * (\psi_3 \circ \psi_1)(z, x)$$

respectively. By pre-composing with an injection, this reduces to:

This itself is nothing more than bifunctoriality of $\otimes$.

To check the associativity of the composition, we need to show that given **V**-profunctors

$$\mathbf{W} \xrightarrow{P} \mathbf{X} \xrightarrow{Q} \mathbf{Y} \xrightarrow{R} \mathbf{Z}$$

we need $(RQ)P \cong R(QP)$. At $z \in \mathbf{Z}, w \in \mathbf{W}$, these have actions

$$(RQ)P(z, w) = \int^x \left[\int^y R(z, y) \otimes Q(y, x) \right] \otimes P(x, w)$$

and

$$R(QP)(z, w) = \int^y R(z, y) \otimes \left[\int^x Q(y, x) \otimes P(x, w) \right]$$

To do this, first apply our earlier result about tensoring and co-ends (Proposition 6.1.5), apply the Fubini theorem to switch the co-ends, then apply the associator a.

To check the unit ismorphisms, it suffices to show that $P(y, x')$ satisfies the universal property of $(FI_X)(y, x') = \int^x F(y, x) \otimes X(x, x')$. Indeed, the required injection maps are the left actions of F, $F_L : F(y, x) \otimes X(x, x') \longrightarrow F(y, x')$, and the required universal equations are then exactly the associativity requirement for F_L. Checking that F_L satisfies the requirements for a V-form is nothing more than applying the associativity and mutual associativity axioms for F_L and F_R. Thus F_L witesses the fact that $FI_X \cong F$, showing that the hom functors are indeed the identity for profunctor composition.

∎

6.2 Change of Base as a Lax Functor

We now need to define $(-)_\#$ on monoidal functors $\mathbf{V} \xrightarrow{N} \mathbf{W}$, so that we get a lax functor $\mathbf{V\text{-}prof} \xrightarrow{N_\#} \mathbf{W\text{-}prof}$.

Since the objects of $\mathbf{V\text{-}prof}$ are the same as those of $\mathbf{V\text{-}cat}$, we define $N_\#$ on objects just as N_* was defined on objects (see Proposition 4.2.1). In particular, the hom-objects of $N_\#\mathbf{X}$ are $(N_\#\mathbf{X})(x, y) := N(\mathbf{X}(x, y))$.

Now, given $\mathbf{X} \xrightarrow{P} \mathbf{Y} \in \mathbf{V\text{-}prof}$, we need to define $N_\#\mathbf{X} \xrightarrow{N_\#P} N_\#\mathbf{Y} \in \mathbf{W\text{-}prof}$. Since this will again be a profunctor, we need to give a module, as well as a left and right action. Begin by defining the modules as

$$(N_\#P)(y, x) := N(P(y, x))$$

Then define the left action (the dotted arrow) as the composite of the two solid arrows:

$$N\mathbf{Y}(y',y) \otimes NF(y,x) \xdashrightarrow{(N_\#F)_L} NF(y',x)$$

$$\tilde{N} \searrow \qquad \nearrow N(F_L)$$

$$N(\mathbf{Y}(y',y') \otimes F(y,x))$$

Similarly, define the right action (the dotted arrow) as the composite of the two solid arrows:

$$NP(y,x) \otimes N\mathbf{X}(x,x') \xdashrightarrow{(N_\#P)_R} NP(y',x)$$

$$\tilde{N} \searrow \qquad \nearrow N(P_R)$$

$$N(P(y,x) \otimes \mathbf{X}(x,x'))$$

In other words, recalling from Chapter 4 the idea of applying a functor monoidally, the actions of $N_\#P$ are nothing more than applying N monoidally to each of P's actions. That is, we have defined

$$(N_\#P)_L := \underline{N}P_L \text{ and } (N_\#P)_r := \underline{N}P_R$$

As a result, the proof that $N_\#P$ is a $\mathbf{W}$-profunctor simply requires applying the early lemmas of Chapter 4.

Proposition 6.2.1. *With the components described above, $N_\#P$ becomes a $\mathbf{W}$-profunctor.*

Proof. The identity axioms follow from Lemma 4.1.1, while the three associativity axioms all follow from Lemma 4.1.3. ∎

Defining $N_\#$ on a $\mathbf{V}$-form $P \xrightarrow{\alpha} Q$ is similar.

Proposition 6.2.2. *Suppose that a $\mathbf{V}$-profunctor morphism ψ has components*

$$P(y,x) \xrightarrow{\psi(y,x)} Q(y,x).$$

Then

$$NP(y,x) \xrightarrow{N\psi(y,x)} NQ(y,x)$$

defines a $\mathbf{W}$-profunctor morphism $N_\#\psi$.

Proof. The axioms for a **W**-profunctor morphism both follow from Lemma 4.1.2. ∎

We would now like to see how $N_\#$ respects composition and identities. To look at composition, we will start with $\mathbf{X} \xrightarrow{P} \mathbf{A}, \mathbf{A} \xrightarrow{Q} \mathbf{Y}$, and take $x \in \mathbf{X}, y \in \mathbf{Y}$. We then have

$$(N_\# Q)(N_\# P)(y, x) = \int^{a \in \mathbf{A}} NQ(y, a) \otimes NP(a, x)$$

and

$$N_\#(QP)(y, x) = N\left(\int^{a \in \mathbf{A}} Q(y, a) \otimes P(a, x)\right)$$

Obviously, in general these are not isomorphic. However, we can get an arrow in one direction. By the universal property of co-ends, it suffices to find an arrow out of the first co-end into the second. We can find such a comparison arrow, via the following composite:

Since this is merely N applied monoidally to i_a, the universal property follows from Lemma 4.1.3, using the universal property of i_a as the original commuting diagram.

We have defined $N_\#$ on objects, arrows, and 2-cells. Now, we need to describe how it acts on the composites and identities of these morphisms. As we have just seen, it only respects composition up to a comparison arrow.. However, as we shall see, it does preserves identities exactly, so $N_\#$ is what is known as a normal lax functor.

Theorem 6.2.3. *Suppose that* $(\mathbf{V}, \otimes, I) \xrightarrow{N} (\mathbf{W}, \bullet, J)$ *is a monoidal functor. Then with actions on arrows and 2-cells as above,* $\mathbf{V}\text{-}\mathbf{prof} \xrightarrow{N_\#} \mathbf{W}\text{-}\mathbf{prof}$ *has comparison arrows that make it into a normal lax functor.*

Proof. We will follow the definition of lax functor given in Leinster [34]. We have described the action of $N_\#$ on profunctors and their morphisms, so we have given, for each $\mathbf{X}, \mathbf{Y} \in \mathbf{V}\text{-}\mathbf{prof}$, the components of a map

$$\mathbf{V}\text{-}\mathbf{prof}(\mathbf{X}, \mathbf{Y}) \xrightarrow{N_\#} \mathbf{W}\text{-}\mathbf{prof}(N_\# \mathbf{X}, N_\# \mathbf{Y})$$

The first thing we need to check is that this defines a functor; in other words, we need to show that $N_{\#}$ preserves vertical composition and identities. Suppose that we have **V**-profunctor morphisms

$$
\begin{array}{c}
P \\
\Downarrow \psi_1 \\
\mathbf{X} \xrightarrow{\quad Q \quad} \mathbf{Y} \\
\Downarrow \psi_2 \\
R
\end{array}
$$

We need to show that $N_{\#}(\psi_2 \circ \psi_1) = N_{\#}\psi_2 \circ N_{\#}\psi_1$. Recall that vertical composition of profunctor morphisms is simply given by composition: $(\psi_2 \circ \psi_1)(y, x) = \psi_2(y, x) \circ \psi_1(y, x)$. Thus, $N_{\#}$ preserves vertical composition since N is a functor:

$$
\begin{aligned}
N_{\#}(\psi_2 \circ \psi_1)(y, x) &= N((\psi_2 \circ \psi_1)(y, x)) \\
&= N(\psi_2(y, x) \circ \psi_1(y, x)) \\
&= N\psi_2(y, x) \circ N\psi_1(y, x) \\
&= (N_{\#}\psi_2 \circ N_{\#}\psi_1)(y, x)
\end{aligned}
$$

Similarly, vertical identities are simply identity morphisms, so again they are preserved since N is a functor.

Above, we defined the comparison arrows ρ for horizontal composition. We need to show that these form the components of a natural transformation

$$
\begin{array}{ccc}
\mathbf{V}\text{-}\mathbf{prof}(\mathbf{X}, \mathbf{Y}) \times \mathbf{V}\text{-}\mathbf{prof}(\mathbf{Y}, \mathbf{Z}) & \xrightarrow{N_{\#} \times N_{\#}} & \mathbf{W}\text{-}\mathbf{prof}(N_{\#}\mathbf{X}, N_{\#}\mathbf{Y}) \times \mathbf{W}\text{-}\mathbf{prof}(N_{\#}\mathbf{Y}, N_{\#}\mathbf{Z}) \\
\downarrow{\scriptstyle C} & \Downarrow \rho & \downarrow{\scriptstyle C} \\
\mathbf{V}\text{-}\mathbf{prof}(\mathbf{X}, \mathbf{Z}) & \xrightarrow[N_{\#}]{} & \mathbf{W}\text{-}\mathbf{prof}(N_{\#}\mathbf{X}, N_{\#}\mathbf{Z})
\end{array}
$$

To check naturality, we need to check that given the following profunctor morphisms:

$$
\begin{array}{ccc}
& P_1 & \quad & Q_1 \\
\mathbf{X} & \Downarrow \psi_1 \quad \mathbf{Y} & \Downarrow \psi_2 \quad \mathbf{Z} \\
& P_2 & \quad & Q_2
\end{array}
$$

and $x \in \mathbf{X}, y \in \mathbf{Y}, z \in \mathbf{Z}$, the following square must commute:

$$\int^y NP_2(z,y) \bullet NP_1(y,x) \xrightarrow{\quad\rho\quad} N(\int^y P_2(z,y) \otimes P_1(y,x))$$

$$\downarrow{\scriptstyle \int^y N\psi_2 \bullet N\psi_1} \qquad\qquad\qquad\qquad \downarrow{\scriptstyle N(\int^y \psi_2 \otimes \psi_1)}$$

$$\int^y NQ_2(z,y) \bullet NQ_1(y,x) \xrightarrow{\quad\rho\quad} N(\int^y Q_2(z,y) \otimes Q_1(y,x))$$

By pre-composing with an injection at y, we can expand the above diagram to (note that some of the instances of $\cdot$ are $\otimes$, and others are $\bullet$):

$$NP_2(z,y) \cdot NP_1(y,x) \xrightarrow{\widetilde{N}} N(P_2(z,y) \cdot P_1(y,x)) \xrightarrow{N(\psi_2 \cdot \psi_1)} N(Q_2(z,Gy) \cdot Q_1(Gy,x))$$

with intermediate maps i_y, $N(i_y)$, $N\psi_2 \cdot N\psi_1$, $\int^y NP_2(z,y) \cdot NP_1(y,x) \xrightarrow{\rho} N \int^y P_2(z,y) \cdot P_1(y,x)$, $N(i_{Gy})$, $NQ_2(z,Gy) \cdot NQ_1(Gy,x)$, $\int N\psi_1 \cdot N\psi_2$, $N(\int \psi_2 \cdot \psi_1)$, i_{Gy}, $\widetilde{N}$, $\int^y NQ_2(z,y) \cdot NP_1(y,x) \xrightarrow{\rho} N \int^y Q_2(z,y) \cdot P_1(y,x)$, $N(Q_2(z,Gy) \cdot Q_1(Gy,x)) \xrightarrow{N(i_{Gy})}$

Note that the expansion on the right is actually N of the map out of the co-end. By expanding this, we are reduced to checking that the outer diagram commutes. This does commute by naturality of $\widetilde{N}$.

For $\mathbf{V}$-profunctors

$$\mathbf{W} \xrightarrow{P} \mathbf{X} \xrightarrow{Q} \mathbf{Y} \xrightarrow{R} \mathbf{Z}$$

the hexagon coherence of $N_\#$ at a given $y \in \mathbf{Y}, x \in \mathbf{X}$ reduces to the hexagon

coherence for $\widetilde{N}$:

$$(NR(z,y) \bullet NQ(y,x)) \bullet NP(x,w) \xrightarrow{\quad a \quad} NR(z,y) \bullet (NQ(y,x) \bullet NP(x,w))$$

with the left vertical map $\widetilde{N} \otimes 1$ down to

$$N(R(z,y) \otimes Q(y,x)) \bullet NP(x,w) \qquad\qquad NR(z,y) \bullet N(Q(y,x) \otimes P(x,w))$$

the right vertical map $1 \otimes \widetilde{N}$, and left/right maps $\widetilde{N}$ down to

$$N((R(z,y) \otimes Q(y,x)) \otimes R(x,w)) \xrightarrow{\quad a \quad} N(R(z,y) \otimes (Q(y,x) \otimes R(x,w)))$$

For the normality, we wish to show that for $\mathbf{A} \in \mathbf{V\text{-prof}}$, $1_{N_\#(\mathbf{A})} = N(1_\mathbf{A})$. Recall that the identity profunctor is simply the hom of the category, so $1_{N_\#(\mathbf{A})}(a,b) = N(\mathbf{A}(a,b))$. However, N applied to a profunctor is merely N of the module, so $(N(1_\mathbf{A}))(a,b) = N(\mathbf{A}(a,b))$ also. In both cases, the actions are given by composition in $N(\mathbf{A})$ - namely, an application of $\widetilde{N}$ followed by N of composition in $\mathbf{A}$. Thus the profunctors are, in fact, equal, and so $N_\#$ is normal.

$\blacksquare$

6.3 The Bicategorical Change of Base $()_\#$

Given a monoidal functor $\mathbf{V} \xrightarrow{N} \mathbf{W}$, we have described a lax functor $\mathbf{V\text{-prof}} \xrightarrow{N_\#} \mathbf{W\text{-prof}}$. However, this is only part of the change-of-base: we also need to know what to send a monoidal natural transformation $N \xrightarrow{\alpha} M$ to. Then, just as we had

$$\mathbf{moncat} \xrightarrow{\quad N_* \quad} \mathbf{2-cat},$$

where $\mathbf{V}$ is sent to $\mathbf{V\text{-cat}}$, we would also like to have

$$\mathbf{cmoncat} \xrightarrow{\quad N_* \quad} \mathbf{bicat},$$

where $\mathbf{V}$ is sent to $\mathbf{V\text{-prof}}$, and $\mathbf{cmoncat}$ is co-complete monoidal categories (we need co-completeness to get profunctor composition).

We need to know what to send monoidal natural transformations to. Recall that lax transformations are problematic as they do not have whisker composites with profunctors. Thus, instead of sending a monoidal natural transformation to a lax natural transformation, we will define $\alpha_\#$ as a different type of cell between lax functors: a module.

To begin, we will show how $()_\#$ can be described as a functor.

6.3.1 The Change of Base $()_\#$ as a Functor

While bicategories, lax functors, and lax natural transformations do not form a 2-category, bicategories and lax functors do form a category (Benabou [3]). It thus makes sense to ask whether $()_\#$ is a functor, and indeed it is.

Proposition 6.3.1. *Let* **bicat** *be the category with objects bicategories and arrows lax functors. Then*

$$\textbf{cmoncat} \xrightarrow{\ ()_\#\ } \textbf{bicat}$$

is a functor.

Proof. We need to show that given monoidal functors $\mathbf{V} \xrightarrow{\ N\ } \mathbf{V'} \xrightarrow{\ M\ } \mathbf{V''}$, $(MN)_\# = M_\# N_\#$; in particular, we then need to check that the two are equal on 0-cells, on 1-cells, and on 2-cells.

- 0-Cells. Since $M_\#$, $N_\#$ act on objects as M_*, N_* do, given a $\mathbf{V}$-category X, the $\mathbf{V}$-categories $(MN)_\# X$ and $M_\# N_\# X$ are equal, with both having hom objects $MN(X(x,y))$.

- 1-Cells. Let $F : X \longrightarrow Y$ be a $\mathbf{V}$-profunctor. It is easy to see that the modules $(MN)_\# F$ and $(M_\# N_\#)F$ are the same, with both being $MN(F(y,x))$.

  Now, the left action of $(MN)_\# F$ consists of the solid lines in the diagram below, with the dotted lines the expansion of the diagonal arrow:

$$MNY(y',y) \otimes MNF(y,x) \xrightarrow{((MN)_\# F)_L} MNF(y',x)$$

with vertical arrow $\widetilde{M}$, diagonal arrow $\widetilde{MN}$, right vertical arrow $(MN)(F_L)$, and bottom arrow:

$$M(NY(y',y) \otimes NF(y,x)) \dashrightarrow_{M(\widetilde{N})} MN(Y(y',y) \otimes F(y,x))$$

The left action of $(M_\# N_\#)F$ is the solid lines in the next diagram, with the dotted lines the expansion of the diagonal arrow:

$$MNY(y',y) \otimes MNF(y,x) \xrightarrow{((M_\# N_\#)F)_L} MNF(y',x)$$

with $\widetilde{M}$, $M((N_\# F)_L)$, $M(N(F_L))$, and

$$M(NY(y',y) \otimes NF(y,x)) \dashrightarrow_{M(\widetilde{N})} MN(Y(y',y) \otimes F(y,x))$$

Comparing the two, one can see that the left actions are equal.

In a similar fashion, one can show that the right actions are equal as well.

- 2-Cells. Given a $\mathbf{V}$-form $\alpha : F \longrightarrow G$, it is easy to see that

$$((MN)_\# \alpha)(y,x) = MN(\alpha(y,x)) = (M_\# N_\# \alpha)(y,x)$$

so that they agree on 2-cells.

Showing that $()_\#$ preserves identities is entirely straightforward.

$\blacksquare$

6.3.2 Modules

As we mentioned earlier, lax natural transformations are problematic, in that they cannot be composed. In addressing this problem, two groups came up with a similar solution: "modules" between lax functors. The version described by Kelly et al.

[28] is in a slightly more general context, but when restricted to bicategories and lax functors, is the same as the idea presented by Cockett et al. [11].

We will first define this concept, then show how $\alpha_\#$ can be defined as a module.

Definition Given two lax functors $\mathcal{A} \xrightarrow{F,G} \mathcal{B}$, a module $F \xrightarrow{M} G$ consists of a family of functors, parametrized by the objects of A and B:

$$\mathcal{A}(a,b) \xrightarrow{M(a,b)} \mathcal{B}(Fa, Gb)$$

together with a right action:

$$
\begin{array}{ccc}
\mathcal{A}(a',a) \times \mathcal{A}(a,b) & \xrightarrow{F \times M} & \mathcal{B}(Fa', Fa) \times \mathcal{B}(Fa, Gb) \\
\Big\downarrow{\scriptstyle C} & \Big\Downarrow{\scriptstyle \widetilde{M}_L} & \Big\downarrow{\scriptstyle C} \\
\mathcal{A}(a',b) & \xrightarrow{M} & \mathcal{B}(Fa', Gb)
\end{array}
$$

and a left action:

$$
\begin{array}{ccc}
\mathcal{A}(a,b) \times \mathcal{A}(b,b') & \xrightarrow{M \times G} & \mathcal{B}(Fa, Gb) \times \mathcal{B}(Gb, Gb') \\
\Big\downarrow{\scriptstyle C} & \Big\Downarrow{\scriptstyle \widetilde{M}_R} & \Big\downarrow{\scriptstyle C} \\
\mathcal{A}(a,b') & \xrightarrow{M} & \mathcal{B}(Fa, Gb')
\end{array}
$$

These actions must satisfy left and right associativity axioms:

$$
\begin{array}{ccc}
F \times F \times M & \xrightarrow{F \times \widetilde{M}_L} & F \times M \\
\Big\downarrow{\scriptstyle \widetilde{F} \times M} \quad = & & \Big\downarrow{\scriptstyle \widetilde{M}_L} \\
F \times M & \xrightarrow{\widetilde{M}_L} & M
\end{array}
\qquad
\begin{array}{ccc}
M \times G \times G & \xrightarrow{\widetilde{M}_R \times G} & M \times G \\
\Big\downarrow{\scriptstyle M \times \widetilde{G}} \quad = & & \Big\downarrow{\scriptstyle \widetilde{M}_R} \\
M \times G & \xrightarrow{\widetilde{M}_r} & M
\end{array}
$$

(where $\widetilde{F}$ and $\widetilde{G}$ are F and G's composition comparisons), left and right unit axioms:

$$
\begin{array}{ccc}
1 \times M & & \\
\Big\downarrow{\scriptstyle F_0 \times M} & \searrow^{1} & \\
F \times M & \xrightarrow{\widetilde{M}_L} & M
\end{array}
\qquad
\begin{array}{ccc}
M \times 1 & & \\
\Big\downarrow{\scriptstyle M \times G_0} & \searrow^{1} & \\
M \times G & \xrightarrow{\widetilde{M}_R} & M
\end{array}
$$

(where F_0 and G_0 are F and G's unit comparisons), and mixed associativity:

$$
\begin{array}{ccc}
F \times M \times G & \xrightarrow{\;F \times \widetilde{M}_R\;} & F \times M \\
{\scriptstyle \widetilde{M}_L \times G}\Big\downarrow & = & \Big\downarrow{\scriptstyle \widetilde{M}_L} \\
M \times G & \xrightarrow[\;\widetilde{M}_R\;]{} & M
\end{array}
$$

Example 6.3.2. Suppose that the bicategories each have only a single object. Then the bicategories $\mathcal{A}$ and $\mathcal{B}$ are monoidal categories $(\mathbf{V}, \otimes, I), (\mathbf{W}, \bullet, J)$, and lax functors F, G between them are monoidal functors. In this case, a module between the monoidal functors F, G reduces to a functor $\mathbf{V} \xrightarrow{M} \mathbf{W}$, together with actions

$$
FA_1 \bullet MA_2 \xrightarrow{\widetilde{M}} M(A_1 \otimes A_2) \text{ and } MA_1 \bullet GA_2 \xrightarrow{\widetilde{M}} M(A_1 \otimes A_2)
$$

As Jeff Egger has noted, one-sided versions of these modules have appeared implicitly in many contexts; in particular, in the work of Kock on monoidal monads [29, 30].

6.3.3 $\alpha_\#$ as a Module

We will now show that given a monoidal natural transformation $N \xrightarrow{\alpha} M$, we can form a module $N_\# \xrightarrow{\alpha_\#} M_\#$. The first thing we need is a family of functors

$$
\mathbf{V}\text{-}\mathbf{prof}(\mathbf{X}, \mathbf{Y}) \xrightarrow{\;\alpha_\#(\mathbf{X},\mathbf{Y})\;} \mathbf{W}\text{-}\mathbf{prof}(N_\#\mathbf{X}, M_\#\mathbf{Y})
$$

Thus, given a $\mathbf{X} \xrightarrow{P} \mathbf{Y} \in \mathbf{V}\text{-}\mathbf{prof}$, we need $N_\#\mathbf{X} \xrightarrow{\alpha_\# P} M_\#\mathbf{Y} \in \mathbf{W}\text{-}\mathbf{prof}$. Define

$$
(\alpha_\# P)(y, x) := MP(y, x)
$$

We also need left and right actions. In contrast to earlier definitions of profunctors, the definitions of the two actions are quite different. The left action is given by:

$$
\begin{array}{ccc}
M\mathbf{Y}(y', y) \otimes MF(y', x) & \dashrightarrow & MF(y', x) \\
& {\scriptstyle \widetilde{M}}\searrow \qquad \nearrow {\scriptstyle M(F_L)} & \\
& M(\mathbf{Y}(y'y) \otimes F(y, x)) &
\end{array}
$$

while the right action is

$$MF(y,x) \otimes N\mathbf{X}(x,x') \dashrightarrow MF(y,x')$$

with vertical maps $1 \otimes \alpha$ (left) and $M(F_R)$ (right), and bottom

$$MF(y,x) \otimes M\mathbf{X}(x,x') \xrightarrow{\widetilde{M}} M(F(y,x) \otimes \mathbf{X}(x,x'))$$

Lemma 6.3.3. *With the components described above, $\alpha_\# P$ becomes a $\mathbf{W}$-profunctor.*

Proof. Since $(\alpha_\# P)_L = \underline{N}P_L$, the left unit and associativity axioms follow from Lemmas 4.1.1 and 4.1.3, respectively. Noting that $(\alpha_\# P)_R$ is $\underline{N}P_L$ pre-composed with α, one can show the other axioms by again using Lemmas 4.1.1 and 4.1.3, using the additional fact that α is a monoidal natural transformation.

For example, the right unit axiom is

$$MF(y,x) \otimes J$$

with arrows $1 \otimes N_0$, $1 \otimes M_0$, l, leading to

$$MF(y,x) \otimes N(I) \xrightarrow{1 \otimes \alpha} MF(y,x) \otimes M(I) \xrightarrow{\widetilde{M}} M(F(y,x) \otimes I)$$

with vertical maps $1 \otimes N(i)$, $1 \otimes M(i)$, $M(1 \otimes i)$, $M(l)$, and bottom row

$$MF(y,x) \otimes NX(x,x) \underset{1 \otimes \alpha}{\to} MF(y,x) \otimes MX(x,x) \underset{\widetilde{M}}{\to} M(F(y,x) \otimes X(x,x)) \xrightarrow{M(F_R)} MF(y,x)$$

The top left triangle is monoidal naturality of α, the square below it by naturality of α, and the rest of the diagram follows by Lemma 4.1.1. ∎

We have defined $\alpha_\#$ on profunctors; now we need to define it on their morphisms. Suppose we have a $\mathbf{V}$-form $P \xrightarrow{\sigma} Q$. We need components

$$(\alpha_\# F)(y,x) \xrightarrow{\alpha_\# \sigma} (\alpha_\# G)(y,x)$$

Define them by

$$MF(y,x) \xrightarrow{M\sigma(y,x)} MG(y,x)$$

Lemma 6.3.4. *With the components as above, $\alpha_\#\sigma$ becomes a **W**-form.*

Proof. The first axiom for a **W**-form follows directly from Lemma 4.1.2. The second requires the commutativity of

$$
\begin{array}{ccc}
MF(y,x) \bullet N\mathbf{X}(x',x) & \xrightarrow{\ M\sigma\bullet 1\ } & MG(y,x) \bullet N\mathbf{X}(x',x) \\
{\scriptstyle 1\bullet\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle 1\bullet\alpha} \\
MF(y,x) \bullet MX(x'x) & \xrightarrow{\ M\sigma\bullet 1\ } & MG(y,x) \bullet M\mathbf{X}(x',x) \\
{\scriptstyle \underline{M}F_R}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{M}G_R} \\
MF(y,x) & \xrightarrow{\qquad M\sigma\qquad} & MG(y,x)
\end{array}
$$

The top diagram commutes by naturality of α, the bottom by Lemma 4.1.2. ∎

Now that we have defined how $\alpha_\#$ acts on profunctors and their morphisms, we need to check it defines a functor.

Lemma 6.3.5. *With components as described above,*

$$
\mathbf{V}\text{-}\mathbf{prof}(\mathbf{X},\mathbf{Y}) \xrightarrow{\ \alpha_\#(\mathbf{X},\mathbf{Y})\ } \mathbf{W}\text{-}\mathbf{prof}(N_\#\mathbf{X}, M_\#\mathbf{Y})
$$

is a functor.

Proof. Recall that the identity **V**-form is given by

$$
1_P(y,x) := 1_{P(y,x)}
$$

while composition is

$$
\sigma_2\sigma_1(y,x) := \sigma_2(y,x) \circ \sigma_1(y,x)
$$

Since we defined

$$
\alpha_\#\sigma(y,x) := M\sigma(y,x)
$$

it follows that $\alpha_\#(X,Y)$ is a functor since M is. ∎

Of course, in addition to the family of functors, we also need to define the left and right actions for $\alpha_\#$ itself. That is, we would need actions:

$$\mathbf{V\text{-}prof}(X',X) \times \mathbf{V\text{-}prof}(X,Y) \xrightarrow{N_\# \times \alpha_\#} \mathbf{W\text{-}prof}(N_\#X', N_\#X) \times \mathbf{V\text{-}prof}(N_\#X, M_\#Y)$$

$$\left. C \right\downarrow \qquad (\alpha_\#)_L \big\Downarrow \qquad \left. C \right\downarrow$$

$$\mathbf{V\text{-}prof}(X',Y) \xrightarrow{\quad \alpha_\# \quad} \mathbf{W\text{-}prof}(N_\#X', M_\#Y)$$

$$\mathbf{V\text{-}prof}(X,Y) \times \mathbf{V\text{-}prof}(Y,Y') \xrightarrow{\alpha_\# \times M_\#} \mathbf{W\text{-}prof}(N_\#X, M_\#Y) \times \mathbf{V\text{-}prof}(M_\#Y, M_\#Y')$$

$$\left. C \right\downarrow \qquad (\alpha_\#)_R \big\Downarrow \qquad \left. C \right\downarrow$$

$$\mathbf{V\text{-}prof}(X,Y') \xrightarrow{\quad \alpha_\# \quad} \mathbf{W\text{-}prof}(N_\#X, M_\#Y')$$

To define the right action, let us begin by simplifying what the two composites look like, for $P \in \mathbf{V\text{-}prof}(\mathbf{X'}, \mathbf{X})$, $Q \in \mathbf{V\text{-}prof}(\mathbf{X}, \mathbf{Y})$. The top right composite reduces to the co-end

$$\int^{a \in \mathbf{X'}} MQ(y,a) \bullet NP(a,x)$$

while the bottom left composite is

$$M \left[\int^{a \in \mathbf{X'}} Q(y,a) \otimes P(a,x) \right]$$

Thus, we can define an arrow from the first to the second, via the composite

$$MG(y,a) \bullet NF(a,x) \dashrightarrow M\left(\int^{a \in \mathbf{X}} G(y,a) \otimes F(a,x) \right)$$

$$\left. 1 \bullet \sigma \right\downarrow \qquad\qquad \left\uparrow M(i_a) \right.$$

$$MG(y,a) \bullet MF(a,x) \xrightarrow{\quad \widetilde{M} \quad} M(G(y,a) \otimes F(a,x))$$

For the left action, the top right composite is

$$\int^{b \in \mathbf{Y}} MQ(y,b) \bullet MF(b,x)$$

while the bottom left composite is

$$M \left(\int^{b \in \mathbf{Y}} Q(y,b) \otimes F(b,x) \right)$$

In this case, the action arrow is then merely

$$MQ(y,b) \bullet MP(b,x) \xrightarrow{\widetilde{M}} M(Q(y,b) \otimes P(b,x)) \xrightarrow{M(i_b)} M\left(\int^{b \in \mathbf{Y}} Q(y,b) \otimes F(b,x)\right)$$

Before we show that these are actions for $\alpha_\#$ we need to show that these actions both define $\mathbf{V}$-forms.

Lemma 6.3.6. *With components as described above, both $(\alpha_\#)_L$ and $(\alpha_\#)_R$ define $\mathbf{V}$-forms.*

Proof. First, notice that the right action, $(\alpha_\#)_R$, is simply the comparison $\mathbf{V}$-form $(M_*Q)(M_*P) \longrightarrow M_*(QP)$. Since we already know this is a $\mathbf{V}$-form (see remarks before Theorem 6.2.3), it follows that $(\alpha_\#)_R$ is also a $\mathbf{V}$-form.

Showing that the left action $(\alpha_\#)_L$ is a $\mathbf{V}$-form requires a bit more work, because of its use of α. By precomposing with an injection, the compatibility with left action axiom for $(\alpha_\#)_L$ reduces to

$$
\begin{array}{ccc}
M\mathbf{Y}(y',y) \cdot MQ(y,x) \cdot NP(x,x') & \xrightarrow{1 \cdot 1 \cdot \alpha} & M\mathbf{Y}(y',y) \cdot MQ(y,x) \cdot MP(x,x') \\
\downarrow{\scriptstyle \widetilde{M} \cdot 1} & {\scriptstyle \widetilde{M} \cdot 1} & \downarrow{\scriptstyle 1 \cdot \widetilde{M}} \\
M(\mathbf{Y}(y',y) \cdot Q(y,x)) \cdot NP(x,x') & & M\mathbf{Y}(y',y) \cdot M(Q(y,x) \cdot P(x,x')) \\
\downarrow{\scriptstyle MQ_L \cdot 1} \quad {\scriptstyle 1 \cdot \alpha} \quad M(\mathbf{Y}(y',y) \cdot Q(y,x)) \cdot MP(x,x') & & \downarrow{\scriptstyle \widetilde{M}} \\
MQ(y',x) \cdot NP(x,x') & {\scriptstyle \widetilde{M}} & M(\mathbf{Y}(y',y) \cdot Q(y,x) \cdot P(x,x')) \\
\downarrow{\scriptstyle 1 \cdot \alpha} \quad {\scriptstyle M(Q_L) \cdot 1} & & \downarrow{\scriptstyle M(Q_L \cdot 1)} \\
MQ(y',x) \cdot MP(x,x') & \xrightarrow{\quad \widetilde{M} \quad} & M(Q(y',x) \cdot P(x,x'))
\end{array}
$$

The top and left regions commute by bifunctoriality of tensor, the top right by coherence of M, and the bottom right by naturality of $\widetilde{M}$. After precomposing with an

injection, the compatibility with right action axiom for $(\alpha_\#)_L$ reduces to

$$MQ(y,x) \cdot NP(x,x') \cdot NX(x',x'') \xrightarrow{\quad 1\cdot\alpha\cdot 1 \quad} MQ(y,x) \cdot MP(x,x') \cdot NX(x'x'')$$

The top region commutes by bifunctoriality of tensor, the top left region by monoidal naturality of α, the region below it by naturality of α, the top right region by bifunctoriality of tensor, the region below it by coherence of M, and the bottom region by naturality of $\widetilde{M}$. Thus $(\alpha_\#)_L$ satisfies both axioms for a **V**-form. ∎

We can now bring everything together and show that $\alpha_\#$ is a module.

Theorem 6.3.7. *Suppose that* $\mathbf{V} \xrightarrow{N,M} \mathbf{W}$ *are monoidal functors, with* $N \xrightarrow{\alpha} M$ *a monoidal natural transformation between them. Then, with the components described above,* $\alpha_\#$ *defines a module between the lax functors* $N_\#$ *and* $M_\#$.

Proof. We have shown that the components of our putative module $\alpha_\#$ are well-defined. It remains to show the five axioms of left and right associativity, left and right units, and mixed associativity. We will show the two most complicated of these (mixed associativity and left associativity); the remaining three are easy to check.

For the mixed associativity, we need to show that

$$
\begin{array}{ccc}
N_\# \times \alpha_\# \times M_\# & \xrightarrow{\ N_\# \times (\alpha_\#)_R\ } & N_\# \times \alpha_\# \\
{\scriptstyle (\alpha_\#)_L \times M_\#} \downarrow & = & \downarrow {\scriptstyle (\alpha_\#)_L} \\
\alpha_\# \times M_\# & \xrightarrow[\ (\alpha_\#)_R\ ]{} & \alpha_\#
\end{array}
$$

To expand this, we will have to abbreviate our notation. We will write $(\mathbf{W}, \mathbf{X})$ for $\mathbf{V}\text{-}\mathbf{prof}(\mathbf{W}, \mathbf{X})$, and N, M, α for $N_\#, M_\#, \alpha_\#$. Then expanding the above, this means that for $\mathbf{V}$-categories $\mathbf{W}', \mathbf{W}, \mathbf{X}, \mathbf{X}'$, this composite of $\mathbf{V}$-forms:

$$
\begin{array}{ccc}
(\mathbf{W}',\mathbf{W}) \times (\mathbf{W},\mathbf{X}) \times (\mathbf{X},\mathbf{X}') & \xrightarrow{\ N \times \alpha \times M\ } & (N\mathbf{W}',N\mathbf{W}) \times (N\mathbf{W},M\mathbf{X}) \times (M\mathbf{X},M\mathbf{X}') \\
{\scriptstyle C \times 1} \downarrow & \Downarrow\ \alpha_L \times M & \downarrow {\scriptstyle C \times 1} \\
(\mathbf{W}',\mathbf{X}) \times (\mathbf{X},\mathbf{X}') & \xrightarrow[\ \alpha \times M\ ]{} & (N\mathbf{W}',N\mathbf{X}) \times (N\mathbf{X},M\mathbf{X}') \\
{\scriptstyle C} \downarrow & \Downarrow\ \ \alpha_R & \downarrow {\scriptstyle C} \\
(\mathbf{W}',\mathbf{X}') & \xrightarrow[\ \alpha\ ]{} & (N\mathbf{W}',M\mathbf{X}')
\end{array}
$$

must be equal to this composite of $\mathbf{V}$-forms:

$$
\begin{array}{ccc}
(\mathbf{W}',\mathbf{W}) \times (\mathbf{W},\mathbf{X}) \times (\mathbf{X},\mathbf{X}') & \xrightarrow{\ N \times \alpha \times M\ } & (N\mathbf{W}',N\mathbf{W}) \times (N\mathbf{W},M\mathbf{X}) \times (M\mathbf{X},M\mathbf{X}') \\
{\scriptstyle 1 \times C} \downarrow & \Downarrow\ N \times \alpha_R & \downarrow {\scriptstyle 1 \times C} \\
(\mathbf{W}',\mathbf{W}) \times (\mathbf{W},\mathbf{X}') & \xrightarrow[\ N \times \alpha\ ]{} & (N\mathbf{W}',N\mathbf{W}) \times (N\mathbf{W},M\mathbf{X}') \\
{\scriptstyle C} \downarrow & \Downarrow\ \ \alpha_L & \downarrow {\scriptstyle C} \\
(\mathbf{W}',\mathbf{X}') & \xrightarrow[\ \alpha\ ]{} & (N\mathbf{W}',M\mathbf{X}')
\end{array}
$$

Suppose we have $\mathbf{V}$-profunctors $P \in (\mathbf{W}', \mathbf{W}), Q \in (\mathbf{W}, \mathbf{X}), R \in (\mathbf{X}, \mathbf{X}')$. Then by precomposing with an injection, the top composite reduces to the right side and the

bottom composite to the left side of:

$$MR(x',x) \cdot MQ(x,w) \cdot NP(w,w') \xrightarrow{1\cdot1\cdot\alpha} MR(x',x) \cdot MQ(x,w) \cdot MP(w,w')$$

The top left region commutes by bifunctoriality of tensor, the bottom right region by coherence of M.

Expanding the left associativity axiom, we are required to show that

$$(\mathbf{W''},\mathbf{W'}) \times (\mathbf{W'},\mathbf{W}) \times (\mathbf{W},\mathbf{X}) \xrightarrow{N\times N\times\alpha} (N\mathbf{W''},N\mathbf{W'}) \times (N\mathbf{W'},N\mathbf{W}) \times (N\mathbf{W},M\mathbf{X})$$

is equal to

$$(\mathbf{W''},\mathbf{W'}) \times (\mathbf{W'},\mathbf{W}) \times (\mathbf{W},\mathbf{X}) \xrightarrow{N\times N\times\alpha} (N\mathbf{W''},N\mathbf{W'}) \times (N\mathbf{W'},N\mathbf{W}) \times (N\mathbf{W},M\mathbf{X})$$

Suppose we have $\mathbf{V}$-profunctors $P \in (\mathbf{W''},\mathbf{W'}), Q \in (\mathbf{W'},\mathbf{W}), R \in (\mathbf{W},\mathbf{X})$. Then by precomposing with an injection, the top composite reduces to the left side and the

bottom composite to the right side of:

$$MR(x,w) \cdot NQ(w,w') \cdot NP(w',w'') \xrightarrow{\;1\cdot\alpha\cdot1\;} MR(x,w) \cdot MQ(w,w') \cdot NP(w',w'')$$

$$1\cdot\widetilde{N} \qquad\qquad 1\cdot\alpha\cdot\alpha \qquad\qquad 1\cdot1\cdot\alpha \qquad\qquad \widetilde{M}\cdot1$$

$$MR(x,w) \cdot N(Q(w,w') \cdot P(w',w'')) \qquad\qquad M(R(x,w) \cdot Q(w,w')) \cdot NP(w',w'')$$

$$1\cdot\alpha \qquad MR(x,w) \cdot MQ(w,w') \cdot MP(w',w'') \qquad 1\cdot\alpha$$

$$1\cdot\widetilde{M} \qquad\qquad \widetilde{M}\cdot1$$

$$MR(x,w) \cdot M(Q(w,w') \cdot P(w',w'')) \qquad\qquad M(R(x,w) \cdot Q(w,w')) \cdot MP(w',w'')$$

$$\widetilde{M} \qquad\qquad\qquad \widetilde{M}$$

$$M(R(x,w) \cdot Q(w,w') \cdot P(w',w''))$$

The top and right regions commute by bifunctoriality of tensor, the left region by monoidal naturality of α, and the bottom region by coherence of M.

As mentioned above, the remaining three axioms are easy to check, and so $\alpha_\#$ is indeed a module.

∎

cx

Chapter 7

Change of Base and Compact Monoidal Categories

In this chapter, we will investigate whether change of base preserves compact (autonomous) monoidal categories. As mentioned at the beginning of the last chapter, autonomous monoidal categories are best described in the bicategory **V-prof** rather than the 2-category **V-cat**. Accordingly, in the last chapter we gave a change of base for profunctors, and in this chapter we will continue the investigation of this change of base. We will show that while $N_\#$ fails to be preserve general autonomous objects, there may yet be hope that it preserves autonomous monoidal categories; we simply need to change the context in which we are working.

We wish to show just that as N_* preserved monoidal objects, so $N_\#$ preserves autonomous objects. Now, compact categories are generally thought of as being "two-sided" objects. Thus one's initial impression may be that for this to be possible, $N_\#$ must be *strong* monoidal, and this would require N itself to be strong monoidal.

However, in their investigation of autonomous objects in an autonomous bicategory, Day, McCrudden and Street [14] showed that to preserve autonomous objects, a lax functor does not need to be strong monoidal. Specifically, a monoidal lax functor preserves autonomous objects if it is itself autonomous (it preserves the autonomous structure of the bicategories it goes between with certain comparisons) and is special (it preserves adjoints and units up to isomorphism). So, a monoidal lax functor like $N_\#$ can preserve autonomous objects, provided that it has some additional structure.

So, we would like to see whether $N_\#$ is special autonomous. To show that $N_\#$ is special is relatively straightforward, as it essentially comes down to the fact that $N_\#$ is a 2-functor when restricted to **V-cat**. To show that $N_\#$ is autonomous, we need to

see how it compares the autonomous structures of **V-prof** and **W-prof**. However, the dualization in these bicategories is given by taking the opposite **V**-category, and it is not hard to show that $N_\#$ preserves opposite categories exactly: $N_\#(\mathbf{X}^{\mathrm{op}}) = (N_\#\mathbf{X})^{\mathrm{op}}$. There are additional coherence axioms to check, and to formulate these, one needs $N_\#$ to be monoidal. Surprisingly, of the three parts required to preserve autonomous monoidal objects, this is the part that fails: $N_\#$ is not monoidal, even though N_* was.

However, this does not show that $N_\#$ does not preserve autonomous monoidal categories, only that $N_\#$ does not preserve all autonomous monoidal objects in **V-prof**. The autonomous monoidal categories are only a subset of these. Specifically, since $N_\#$ is not monoidal, it will not preserve all promonoidal categories. While it does not preserve promonoidal categories, we know from Chapter 5 that it does preserve monoidal categories, and we will show that it has most of the structure of an autonomous monoidal lax functor. We will discuss how we may be able to put this knowledge to good use in the final chapter. For now, however, we would like to indicate how close $N_\#$ is to being an autonomous monoidal lax functor.

We begin by reviewing the idea of a Cauchy-complete **V**-category, due to Lawvere, then move into the definitions of autonomous monoidal bicategory and autonomous objects therein.

7.1 Cauchy Complete V-categories

Definition Say that a **V**-category **X** is *Cauchy-complete* if every profunctor $\mathbf{Y}\overset{P}{\longrightarrow}\mathbf{X}$ with a right adjoint is representable by a functor. That is, there exists a **V**-functor F so that $P(x,y) \cong \mathbf{Y}(x, Fy)$.

Proposition 7.1.1. *To show that* **X** *is Cauchy-complete, it suffices to show that every profunctor* $\mathbf{1} \overset{P}{\longrightarrow} \mathbf{X}$ *with a right adjoint is representable by a functor.*

Proof. See Bourceux ([5], pg 319). ∎

As first shown by Lawvere, this categorical notion of Cauchy complete and the classical notion of Cauchy complete coincide when we are dealing with metric spaces:

Proposition 7.1.2. *If* $\mathbf{V} = \mathbf{R}_+$*, then a metric space* $(\mathbf{X}, d)$ *is Cauchy complete in the classical sense exactly when it is Cauchy complete in the* $\mathbf{V}$*-categorical sense.*

Proof. By the previous proposition, to show that $(\mathbf{X}, d)$ is Cauchy complete in the $\mathbf{V}$-categorical sense, it suffices to show that every profunctor $\mathbf{1} \xrightarrow{P} \mathbf{X}$ with a right adjoint is representable by a functor $\mathbf{1} \xrightarrow{F} \mathbf{X}$. However, such a functor is merely a point of $\mathbf{X}$. Now, recall that a metric space is (classically) Cauchy-complete when it is isomorphic to its Cauchy completion. This is the metric space with Cauchy sequences (x_n), as points, and metric $d_C((x_n), (y_n)) = \lim_n d(x_n, y_n)$. Thus, to show our desired result, we will show that every pair of adjoint profunctors $P \dashv Q$ between $\mathbf{1}$ and $\mathbf{X}$ corresponds to a Cauchy sequence of $\mathbf{X}$, and vice versa.

Suppose we have such a pair of adjoint profunctors. These are functions $\mathbf{X} \xrightarrow{P,Q} \mathbb{R}_+$ with some additional properties (see Example 6.1.4). Since they are adjoint to one another, they are equipped with a unit $1_{\mathbf{1}} \longrightarrow gf$ and a co-unit $fg \longrightarrow 1_X$. Since we are enriched in the poset $\mathbb{R}_+$, the existence of unit reduces to the condition

$$\bigwedge_{x \in X} Px + Qx = 0$$

while the existence of a co-unit reduces to the condition

$$\forall x, y \in X, \ Py + Qx \geq d(y, x).$$

By the first condition, for each n, we can find an $x_n \in X$ so that $Px_n + Qx_n < \frac{1}{n}$. This is Cauchy since, by the second condition,

$$d(x_n, x_m) \leq Px_n + Qx_m < \frac{1}{n} + \frac{1}{m}$$

so that, given ϵ, we can take N so that $\frac{2}{N} < \epsilon$.

Conversely, suppose that we have a Cauchy sequence (x_n). Define functions

$$P(x) = d_C((x), (x_n)) \text{ and } Q(x) = d_C((x_n), (x))$$

To show the unit condition, we need to show that

$$\wedge_{x \in X} \lim_n d(x, x_n) + \lim_n d(x_n, x) = 0$$

To show this, given ϵ, since (x_n) is Cauchy, we can find N large enough so that $d(x_N, x_n) < \frac{\epsilon}{2}$. Then simply take $x = x_N$. To show the co-unit condition, we use the triangle inequality:

$$Py + Qx = \lim_n d(y, x_n) + d(x_n, x) \geq lim_n d(y, x) = d(y, x)$$

Thus $P \dashv Q$ is a pair of adjoint profunctors. ∎

By using the Cauchy complete condition, we can access the **V**-functors inside **V-prof** as the arrows with right adjoints. An example of this is the concept of a map monoidal object.

Definition A *map monoidal object* in a monoidal bicategory $\mathcal{M}$ is a monoidal object (X, p, I) such that both p and I have right adjoints.

When **X** is Cauchy complete and the monoidal bicategory is **V-prof**, map monoidal objects are familiar.

Proposition 7.1.3. *Suppose that* **X** *is a map monoidal object in* **V-prof***. If* **X** *is also Cauchy-complete, then* **X** *is a monoidal* **V***-category.*

Proof. Since the multiplication and unit arrows both have right adjoints and **X** is Cauchy-complete, they are both representable by functors, and **X** is thus a monoidal **V**-category. ∎

To determine when the change of base preserves autonomous monoidal categories, we need to see that autonomous monoidal **V**-categories are certain types of objects in **V-prof**. First, however, we need to give the definition of autonomous monoidal **V**-category.

Definition An monoidal **V**-category $(\mathbf{X}, \otimes, I)$ is *autonomous* if it is equipped with an equivalence $\mathbf{X}^{\mathrm{op}} \xrightarrow{(-)^*} \mathbf{X}$ such that there is a **V**-natural isomorphism

$$\mathbf{X}(X, Z \otimes Y^*) \cong \mathbf{X}(X \otimes Y, Z).$$

Recall that we could describe monoidal **V**-categories as pseudomonoids in **V-cat**, or, as above, as map monoidal objects in **V-prof**. To do this, however, first required that we describe the monoidal structure of **V-cat** or **V-prof**. Similarly, to describe autonomous monoidal categories as autonomous objects, we first need to describe the autonomous structure of **V-prof**.

7.2   Autonomous Structure of V-prof

The definition of autonomous bicategory was first given in Day and Street's paper [13], here we present the definition given in Street's later paper [40].

Definition Suppose that $(\mathcal{M}, \otimes, \mathcal{I})$ is a monoidal bicategory. For objects $A, B \in \mathcal{M}$, say that B is a *right bidual for A* if there exists a morphism $A \otimes B \overset{e}{\longrightarrow} I$ such that for each C, D, the functor $\mathcal{M}(C, B \otimes D) \longrightarrow \mathcal{M}(A \otimes C, D)$ which sends f to $(e \otimes 1)(1 \otimes f)$ is an equivalence of categories. This means that there is a unique-up-to-isomorphism morphism $I \overset{n}{\longrightarrow} B \otimes A$ such that $(e \otimes 1)(1 \otimes n)$ is isomorphic to the identity of A, and $(1 \otimes e)(n \otimes 1)$ is isomorphic to the identity of B. We say that the unit is n and the counit e. If every object in $\mathcal{M}$ has both a right and left bidual, then $\mathcal{M}$ is called *autonomous*. The right bidual of an object A will be denoted by A^o.

The prototypical example of an autonomous monoidal bicategory is **V-prof**; we will sketch the proof of this result. The biduals are given by taking $(-)^{\mathrm{op}}$. Note that the unit and co-unit maps can not be represented by functors, so **V-cat** itself is *not* autonomous.

Before we sketch the proof, it will be helpful to recall the Yoneda density lemma:

Lemma 7.2.1. *Suppose that* **V** *is a monoidal closed category, and* $\mathbf{X} \overset{F}{\longrightarrow} \mathbf{V}$ *is a* **V***-functor. Then*

$$Fx \cong \int^y Fy \otimes \mathbf{X}(y, x)$$

Proof. See Kelly [27], pg. 53 (3.71). ∎

Now we can show that **V-prof** has autonomous structure.

Theorem 7.2.2. *For* **V** *symmetric,* **V-prof** *has the structure of an autonomous monoidal bicategory.*

Proof. For **V** symmetric, **V-prof** is a symmetric monoidal bicategory, so that we need only show each object has a right bidual. We will show that A has right bidual A^{op}. Define

$$\mathbb{I} \xrightarrow{\ n\ } \mathbf{A}^{\mathrm{op}} \otimes \mathbf{A} \text{ by } n(a', a, *) := \mathbf{A}(a, a'),$$

with right action

$$\mathbf{A}(a, a') \otimes I \longrightarrow \mathbf{A}(a, a')$$

given by the right unit isomorphism r, and left action

$$\mathbf{A}(a'_1, a'_2) \otimes \mathbf{A}(a_2, a_1) \otimes \mathbf{A}(a_1, a'_1) \longrightarrow \mathbf{A}(a_2, a'_2)$$

given by the braiding followed by two compositions.

Similarly, define

$$A \otimes A^{\mathrm{op}} \xrightarrow{\ e\ } \mathbb{I} \text{ by } e(*, a, a') := \mathbf{A}(a', a),$$

with right action

$$\mathbf{A}(a'_1, a_1) \otimes \mathbf{A}(a_1, a_2) \otimes \mathbf{A}(a'_2, a'_1) \longrightarrow \mathbf{A}(a'_2, a_2)$$

given by braiding then composition, and left action

$$I \otimes \mathbf{A}(a', a) \longrightarrow \mathbf{A}(a', a)$$

given by by left unit isomorphism l.

To show that these form a dual pairing, consider

$$
\begin{aligned}
(e \otimes 1)(1 \otimes n)(a_1, a_2) &= \int^{a_3, a_4, a_5} (e \otimes 1)(a_1, (a_3, a_4, a_5)) \otimes (1 \otimes n)((a_3, a_4, a_5), a_2) \\
&= \int^{a_3, a_4, a_5} e(a_3, a_4) \otimes 1(a_1, a_5) \otimes 1(a_3, a_2) \otimes n(a_4, a_5) \\
&= \int^{a_3, a_4, a_5} \mathbf{A}(a_4, a_3) \otimes \mathbf{A}(a_1, a_5) \otimes \mathbf{A}(a_3, a_2) \otimes \mathbf{A}(a_5, a_4) \\
&\cong \int^{a_4, a_5} \mathbf{A}(a_4, a_2) \otimes \mathbf{A}(a_1, a_5) \otimes \mathbf{A}(a_5, a_4) \ (\text{Yoneda in } a_3) \\
&\cong \int^{a_5} \mathbf{A}(a_1, a_5) \otimes \mathbf{A}(a_5, a_2) \ (\text{Yoneda in } a_4) \\
&\cong \mathbf{A}(a_1, a_2) \ (\text{Yoneda in } a_5)
\end{aligned}
$$

as required. The other composite follows similarly. ∎

In an autonomous monoidal bicategory, we can define an autonomous monoidal object.

Definition Suppose that (A, m) is an pseudomonoid in an autonomous monoidal bicategory $\mathcal{M}$. A morphism $A^o \xrightarrow{d} A$ is called *left dualization* for A when there exist 2-cells

$$
\begin{array}{ccc}
A^o \otimes A \xrightarrow{d \otimes 1} A \otimes A & \qquad & I \xrightarrow{e} A \\
n \uparrow \quad \Downarrow \alpha \quad \downarrow m & & j \uparrow \quad \Downarrow \beta \quad \uparrow m \\
I \xrightarrow{j} A & & A \otimes A^o \xrightarrow{1 \otimes d} A \otimes A
\end{array}
$$

Satisfying two coherence conditions (see [14], pg. 2). These structures are unique up to isomorphism. When a pseudomonoid A admits both left and right dualizations, then it is called *autonomous*.

Finally, we have the following result, which tells us how to identify autonomous (compact) monoidal **V**-categories.

Proposition 7.2.3. *An autonomous monoidal* **V***-category is an autonomous pseudomonoid in* **V-prof** *in which the unit, multiplication, and dualization profunctors are all representable by* **V***-functors.*

Proof. See [14], pg. 8. ∎

7.3 $N_\#$ as Autonomous Monoidal

Now that we understand autonomous monoidal objects in an autonomous monoidal bicategory, we can look at the types of morphisms which preserve them. We first need to define two types of lax functors: special, and autonomous.

Definition A lax functor $\mathcal{M} \xrightarrow{F} \mathcal{N}$ is *special* when:

- the identity constraint is always invertible (the lax functor is "normal")

- for $A \xrightarrow{f} B$ a map, the composition constraint $F(g) \circ F(f) \longrightarrow F(g \circ f)$ is invertible.

Special is needed so that the lax functor is a pseudo-functor when restricted to the category of maps.

Definition Suppose that $\mathcal{M} \xrightarrow{F} \mathcal{N}$ is a lax monoidal functor between autonomous monoidal bicategories. F is called *autonomous* when it is equipped with a pseudo-natural family

$$(FX)^o \xrightarrow{\kappa_X} F(X^o)$$

and two modifications:

satisfying two coherence conditions (see pgs.13-14 of [14]).

Then we have the following:

Theorem 7.3.1. *If $\mathcal{M} \xrightarrow{F} \mathcal{N}$ is a special lax autonomous monoidal functor between autonomous monoidal bicategories, and X is an autonomous monoidal object in $\mathcal{M}$, then FX can be given the structure of an autonomous monoidal object in $\mathcal{N}$.*

Proof. See [14], pg. 15. ∎

7.3.1 $N_\#$ is Special

Let us now investigate whether $N_\#$ satisfies the conditions of the theorem above.

Theorem 7.3.2. *If* $(\mathbf{V}, \otimes, I) \xrightarrow{\ N\ } (\mathbf{W}, \bullet, J)$ *is a monoidal functor, then the lax functor* $N_\#$, *when restricted to the Cauchy complete* $\mathbf{V}$-*categories, is special.*

Proof. Earlier, we saw that $N_\#$ is normal (as part of theorem 6.2.3), so all that remains is to to prove the second condition. Suppose that we have $\mathbf{V}$-profunctors

$$\mathbf{X} \xrightarrow{\ P\ } \mathbf{A} \xrightarrow{\ Q\ } \mathbf{Y}$$

with P a map. Since all $\mathbf{V}$-categories under consideration are Cauchy complete, we can represent P by a functor F, so that $P(a, x) \cong \mathbf{A}(a, Fx)$. Then note that the composition constraint

$$\int^a NQ(y, a) \bullet NP(a, x) \longrightarrow N\left(\int^a Q(y, a) \otimes P(a, x)\right)$$

becomes

$$\int^a NQ(y, a) \bullet N\mathbf{A}(a, Fx) \longrightarrow N\left(\int^a Q(y, a) \otimes \mathbf{A}(a, Fx)\right).$$

Then, using the Yoneda density lemma in $\mathbf{W}$ on the left and in $\mathbf{V}$ on the right, we see that the composition constraint is really a map between identical objects:

$$NQ(y, Fx) \longrightarrow NQ(y, Fx)$$

Thus, if we can show that the composition constraint is the identity, we are done. That is, we need to show that

$$
\begin{array}{ccc}
\int^a NQ(y,a) \bullet N\mathbf{A}(a, Fx) & \dashrightarrow & N(\int^a Q(y,a) \otimes \mathbf{A}(a, Fx)) \\
\Big\downarrow{\scriptstyle m} & & \Big\uparrow{\scriptstyle N(m^{-1})} \\
NQ(y, Fx) & \xrightarrow[\ 1\ ]{} & NQ(y, Fx)
\end{array}
$$

commutes, where m is the Yoneda isomorphism. Note that in the forward direction, the Yoneda isomorphism is given by the action of the profunctor NQ. Thus, we can

expand the above as

$$NQ(y,a) \bullet N\mathbf{A}(a, Fx) \xrightarrow{\widetilde{N}} N(Q(y,a) \otimes \mathbf{A}(a, Fx))$$

with vertical arrows $(NQ)_L$, $N(Q_L)$, $N(i_a)$ and

$$NQ(y, Fx) \xrightarrow{N(m)} N(\textstyle\int^a Q(y,a) \otimes \mathbf{A}(a, Fx))$$

The left triangle commutes by definition of $(NQ)_L$, and the right triangle is N of the Yoneda isomorphism.

Thus, the composition constraint is invertible, and $N_\#$ is special. ∎

7.3.2 Autonomous

We would now like to see to what extent $N_\#$ has the structure of an autonomous lax functor. For $\mathcal{M} = \mathbf{V}\text{-}\mathbf{prof}$, $\mathcal{N} = \mathbf{W}\text{-}\mathbf{prof}$, and $F = N_\#$, we can define the κ arrow for an autonomous functor to be the identity, since $N_\#$ preserves opposites exactly.

Proposition 7.3.3. *Suppose that* $(\mathbf{V}, \otimes, I, \sigma_\mathbf{V}) \xrightarrow{N} (\mathbf{W}, \bullet, J, \sigma_\mathbf{W})$ *is a braided monoidal functor. Then the change of base* $N_\#$ *preserves opposite categories exactly: that is, for a* $\mathbf{V}$-*category* $\mathbf{X}$, $N_\#(\mathbf{X}^{op}) = (N_\#\mathbf{X})^{op}$.

Proof. Since both the change of base and opposite categories keep the same objects, both $N_*(\mathbf{X}^{\mathrm{op}})$ and $(N_*\mathbf{X})^{\mathrm{op}}$ have objects those of $\mathbf{X}$. The homs are also equal, as

$$N_*(\mathbf{X}^{\mathrm{op}})(y,x) = N((\mathbf{X}^{\mathrm{op}})(y,x)) = N(\mathbf{X}(x,y)) = (N_*\mathbf{X})(x,y) = (N_*\mathbf{X})^{\mathrm{op}}(y,x)$$

To show that the composition maps are equal, we expand out the composition in $N_*(\mathbf{X}^{\mathrm{op}})$ on the left side and in $(N_*\mathbf{X})^{\mathrm{op}}$ on the right (recalling that composition in the opposite $\mathbf{V}$-category uses the braiding followed by composition):

$$N\mathbf{X}(y,z) \bullet N\mathbf{X}(x,y) \xrightarrow{\sigma_\mathbf{W}} N\mathbf{X}(x,y) \bullet N\mathbf{X}(y,z)$$

with left vertical arrow $\widetilde{N}$ and right vertical arrow $\widetilde{N}$,

$$N(\mathbf{X}(y,z) \otimes \mathbf{X}(x,y)) \qquad N(\mathbf{X}(x,y) \otimes \mathbf{X}(y,z))$$

with left vertical arrow $N\sigma_\mathbf{V}$ and right vertical arrow Nc,

$$N(\mathbf{X}(x,y) \otimes \mathbf{X}(y,z)) \xrightarrow{Nc} N\mathbf{X}(x,z)$$

Since the last two arrows are the same, the diagram commutes by monoidal naturality of $\widetilde{N}$, and so the composition arrows are equal.

Finally, since identites remain the same in the opposite $\mathbf{V}$-category, the identity arrows in $N_*(\mathbf{X}^{\mathrm{op}})$ and $(N_*\mathbf{X})^{\mathrm{op}}$ are also equal. ∎

Moreover, one can also define the modications ϵ, ζ as isomorphisms. Indeed, for a $\mathbf{V}$-category $\mathbf{X}$, the composites they go between are

$$
\begin{array}{ccc}
N_{\#}\mathbf{X} \otimes N_{\#}\mathbf{X}^{\mathrm{op}} & \xrightarrow{\ \chi\ } & N_{\#}(\mathbf{X} \otimes \mathbf{X}^{\mathrm{op}}) \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle N_{\#}(e)} \\
J & \xrightarrow{\ \iota\ } & NI
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
J & \xrightarrow{\ n\ } & N_{\#}\mathbf{X}^{\mathrm{op}} \otimes N_{\#}\mathbf{X} \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \chi} \\
NI & \xrightarrow{\ N_{\#}n\ } & N_{\#}(\mathbf{X}^{\mathrm{op}} \otimes \mathbf{X})
\end{array}
$$

Proposition 7.3.4. *In both of the diagrams above, the left composite profunctor is isomorphic to the right composite profunctor.*

Proof. In the first diagram, the left composite has module

$$e(*, x, x') = N_{\#}\mathbf{X}(x', x) = N\mathbf{X}(x', x)$$

On the other hand, the right composite is a functor followed by a profunctor, so has module

$$(N_{\#}e)(*, \chi(x, x')) = N_{\#}e(*, (x, x')) = Ne(*, (x, x')) = N\mathbf{X}(x', x)$$

Thus the modules are equal. It is also straightforward to check that the actions are equal.

In the second diagram, the left composite has module

$$(N_{\#}n)(x', x, *) = N\mathbf{X}(x, x')$$

On the other hand, the right composite is given by

$$\int^{y' \in \mathbf{X}^{\mathrm{op}}, y \in \mathbf{X}} \chi(x', x, y', y) \otimes n(y', y, *) = \int^{y', y \in \mathbf{X}} N(\mathbf{X}^{\mathrm{op}} \otimes \mathbf{X})((x', x), (y', y)) \otimes N\mathbf{X}(y, y')$$

Applying the Yoneda density result, with

$$N(\mathbf{X}^{\mathrm{op}} \otimes \mathbf{X})^{\mathrm{op}} \xrightarrow{F} \mathbf{W}$$

given by

$$(x', x) \mapsto N\mathbf{X}(x, x')$$

gives that the above is also isomorphic to $N\mathbf{X}(x, x')$, as required. Again, it is straightforward to show that the actions are equal. ∎

To check the coherence equations, however, we need $N_{\#}$ to be monoidal.

7.3.3   Monoidal

For $N_{\#}$ to be monoidal, we would need, first of all, a profunctor

$$N\mathbf{X} \otimes N\mathbf{Y} \xrightarrow{\chi} N(\mathbf{X} \otimes \mathbf{Y}).$$

This is easy to define, as we know that there is a functor between these $\mathbf{W}$-categories: so we merely make this functor into a profunctor. However, to be a pseudonatural transformation, for profunctors P, Q, we would need an isomorphism of profunctors

$$
\begin{array}{ccc}
N\mathbf{X} \otimes N\mathbf{Y} & \xrightarrow{\ \ \chi\ \ } & N(\mathbf{X} \otimes \mathbf{Y}) \\
{\scriptstyle NP \otimes NQ}\Big\downarrow & & \Big\downarrow{\scriptstyle N(P \otimes Q)} \\
N\mathbf{X}' \otimes N\mathbf{Y}' & \xrightarrow[\ \ \chi\ \ ]{} & N(\mathbf{X}' \otimes \mathbf{Y}')
\end{array}
$$

If we take the adjoint of the bottom χ, then for objects x, y, x', y', the left composite is

$$NP(x', x) \otimes NQ(y', y)$$

while the right composite is

$$N(P(x', x) \otimes Q(y', y))$$

In general, then, we only have a comparison $\widetilde{N}$.

There are two reasons we need $N_{\#}$ to be monoidal. One is to show that it preserves monoidal objects, and the other is to express the coherence diagrams to make

$N_\#$ autonomous monoidal. However, we do not need $N_\#$ to be monoidal to preserve monoidal categories (the map monoidal objects). We already know it preserves those. This merely shows that $N_\#$ does not preserve all monoidal objects in **V-prof**. Monoidal objects in **V-prof** are the promonoidal categories of Brian Day: the above shows that N needs to be strong to preserve these. That is, $N_\#$ preserves monoidal categories, but not neccesarily promonoidal categories unless N is strong.

For the second reason, the statement and results about autonomous monoidal functors require that the functor be monoidal. However, the instances where the isomorphism is used (pgs 13-14 of [14]) only require a comparison to define the paste composite. In fact, the direction of comparison required is exactly the direction we do have as described above. That is, we do not neccesarily need $N_\#$ to be monoidal to show that it preserves autonomous monoidal categories. So the lax functor $N_\#$ does not meet the requirements to be autonomous monoidal, but appears to have enough of the neccesary structure to still preserve autonomous objects.

What is apparent is the importance of squares such as those that appear above. In these squares, we have two parallel **V**-functors (the horizontal arrows), and two parallel **V**-profunctors (the vertical arrows). To express the idea that $N_\#$ is autonomous, we do not need this square to be an isomorphism. Instead, we merely need a comparison arrow. As we shall see in the next chapter, the arrow going in the indicated direction tells us that we have a horizontal transformation. This is a type of transformation that is more than a 2-natural transformation, but less than a full pseudonatural transformation. To express this more clearly, we need to understand double categories, and specifically the double category of **V**-categories, **V**-functors, and **V**-profunctors.

Chapter 8

Change of Base as a Double Functor

In chapter 6, we saw that given a monoidal natural transformation $N \xrightarrow{\alpha} M$, we can define a module between lax functors $N_\# \xrightarrow{\alpha_\#} M_\#$. However, this a structural problem with this. Bicategories, lax functors, and modules do not form a 2-dimensional structure; one needs to add a certain type of 3-cell (a "modulation") to get a (weak) 3-dimensional structure. This leads to a disparity when one tries to define a full change of base **cmoncat** $\xrightarrow{()_\#}$ **bicat**, as there are no appropriate 3-cells for **cmoncat** to send to modulations.

Furthermore, as we saw in Chapter 7, another problem exists with using bicategories. The existing theory of structured bicategories is not adequate to express the particular nature of the change of base. Problems arise when one tries to use this theory that seem to be more related to the particular trappings of bicategory theory, rather than problems with the change of base functor itself.

All of this leads one to believe that there must be a better arrow to send α to; modules seem to be not quite the answer. In fact, if we take a step further back, one can see another problem. Given a monoidal functor $\mathbf{V} \xrightarrow{N} \mathbf{W}$, we have given two definitions of what N should be sent to: N_* and $N_\#$. This is already a clue that something has gone astray - N should be sent to a single arrow.

The way to resolve both this difficulty with N, and the problem of where to send α to, is resolved by considering double categories. In a double category, between objects, one has two types of arrows - horizontal and vertical - and cells between a pair of horizontal and vertical arrows. **V-cat** is ideally suited to be made into a double category, as it has two types of arrows between its objects: **V**-functors and

V-profunctors. (We will be dealing with a "pseudo" double category, where the vertical composition is bicategorical in nature).

Though double categories, and in particular, the double category **V-CAT** have not received as much attention as bicategories and the bicategory **V-prof**, they are even more powerful, and to define the full change of base, are absolutely neccesary. Given $\mathbf{V} \xrightarrow{N} \mathbf{W}$, we can use our previous definitions of N_* and $N_\#$ to define a single arrow: a "double lax functor" N_* between the double categories **V-CAT** and **W-CAT** which is N_* horizontally and $N_\#$ vertically. Moreover, there are nice 2-cells between double lax functors: horizontal transformations. As we shall see, one can define α as a horizontal transformation between N_* and M_*. As opposed to the definition of $\alpha_\#$ as a module, the definition of α_* as a horizontal transformation is entirely natural. Moreover, the "squares" that appeared when trying to define $N_\#$ as an autonomous lax functor are exactly the squares that are used in the definition of a horizontal transformation.

Even better, pseudo double categories, lax double functors, and horizontal transformations define a (strict) 2-category! This is a vast improvement over the situation of **bicat**, which at best could be a (weak) 3-category. We then show that assignments of **V** to the double category **V-CAT**, N to a double lax functor, and α to a horizontal transformation is 2-functorial. That is, we can define a 2-functor

$$\mathbf{cmoncat} \xrightarrow{()_*} \mathbf{doublecat}$$

This 2-functor contains, in the horizontal direction, the original 2-functor

$$\mathbf{cmoncat} \xrightarrow{()_*} \mathbf{2-cat},$$

but now also handles **V**-profunctors, and their cells between them. It is the best result one could hope for, and it is not possible without considering double categories. We conclude the chapter by showing that just as the original 2-functor $(-)_*$ was monoidal, so too is this larger version.

Though independently found here, the idea of considering change of base as taking values in double categories was first discovered by Dominic Verity in his PhD Thesis ([43]). There are some differences between that work and the present account, however. There, double categories are the most general possible; that is, they are weak both horizontally and vertically. In addition, in Verity's thesis, the emphasis is on showing that an adjoint pair of monoidal functors gets sent to an adjoint pair of double functors; here, our focus is in showing the 2-functorial nature of the change of base. Of course, the fact that this double-categorical change of base was discovered independently by two separate parties further underlines how important the idea is.

8.1 Double Categories

We begin the chapter by reviewing the definitions of double category, lax double functor, and horizontal transformation. We will also describe the elements of the double category **V-CAT**.

Succinctly, one can describe a pseudo double category as a pseudo category-object in **cat** ([21], pg. 210). Intuitively, this means that a pseudo double category has objects, two types of arrows (horizontal and vertical), and cells between a square of horizontal and vertical arrows. The horizontal arrows, together with certain cells, will form a 2-category, while the vertical arrows and other tyeps of cells will form a bicategory. We will now give an explicit definition:

Definition A (pseudo) double category $\mathcal{C}$ consists of:

- a set of objects **C**,

- for any two objects X, Y in **C**, a set of "horizontal" arrows $C_0(X, Y)$ and a set of vertical arrows $C_1(X, Y)$,

- for horizontal arrows F, G and vertical arrows P, Q, in the following configuration:

$$
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
{\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\
W & \xrightarrow[G]{} & Z
\end{array}
$$

a set of "cells",

- for horizontal arrows, composition and identities that make the objects and horizontal arrows into a category,

- for vertical arrows, composition and identities,

- for cells

$$
\begin{array}{ccccc}
X_1 & \xrightarrow{F_1} & X_2 & \xrightarrow{F_2} & X_3 \\
P\downarrow & \psi_1 & Q\downarrow & \psi_2 & \downarrow R \\
Y_1 & \xrightarrow{G_1} & Y_2 & \xrightarrow{G_2} & Y_3
\end{array}
$$

a horizontal composite

$$
\begin{array}{ccc}
X_1 & \xrightarrow{F_2 F_1} & X_3 \\
P_1\downarrow & \psi_2\psi_1 & \downarrow P_3 \\
Y_1 & \xrightarrow{G_2 G_1} & Y_3
\end{array}
$$

which is associative,

- for cells

$$
\begin{array}{ccc}
X_1 & \xrightarrow{F} & X_2 \\
P_1\downarrow & \psi_1 & \downarrow Q_1 \\
Y_1 & \xrightarrow{G} & Y_2 \\
P_2\downarrow & \psi_2 & \downarrow Q_2 \\
Z_1 & \xrightarrow{H} & Z_2
\end{array}
$$

a vertical composite

$$
\begin{array}{ccc}
X_1 & \xrightarrow{F} & X_2 \\
P_2 P_1\downarrow & \psi_2 * \psi_1 & \downarrow Q_2 Q_1 \\
Z_1 & \xrightarrow{H} & Z_3
\end{array}
$$

which is associative,

- such that the composition of cells satisfies the middle-four interchange:

$$
(\psi_4\psi_2) * (\psi_3\psi_1) = (\psi_4 * \psi_3)(\psi_2 * \psi_1),
$$

- Denote the cells whose vertical boundaries are identies as "vertically trivial". For each horizontal arrow F, we require a vertically trivial cell

$$
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
{\scriptstyle 1_X}\downarrow & {\scriptstyle 1_F} & \downarrow{\scriptstyle 1_Y} \\
X & \xrightarrow{F} & Y
\end{array}
$$

- such that the objects, horizontal arrows, and vertically trivially cells, together with the compositions and identities described above, form a 2-category $\mathbf{C_0}$.

- Denote the cells whose horizontal boundaries are identites as "horizontally trivial". For each vertical arrow P, we require a horizontally trivial cell

$$
\begin{array}{ccc}
X & \xrightarrow{1_X} & X \\
{\scriptstyle P}\downarrow & {\scriptstyle 1_P} & \downarrow{\scriptstyle P} \\
Y & \xrightarrow{1_X} & Y
\end{array}
$$

- For vertical cells $X \xrightarrow{P_1} Y \xrightarrow{P_2} Z \xrightarrow{P_3} W$, a horizontally trivial cell

$$
\begin{array}{ccc}
X & \xrightarrow{1_W} & X \\
{\scriptstyle (P_1P_2)P_3}\downarrow & {\scriptstyle \alpha_{P_1,P_2,P_3}} & \downarrow{\scriptstyle P_1(P_2P_3)} \\
W & \xrightarrow{1_W} & W
\end{array}
$$

  which is horizontally invertible,

- such that the objects, vertical arrows, and horizontally trivial cells, together with the compositions, identities, and the associator cells described above, form a bicategory $\mathbf{C_1}$ in which the units isomorphisms are strict.

There is a surprising number of examples of categories with two different types of arrows between the objects, which are related by cells.

Example 8.1.1. Sets as objects, horizontal arrows as functions, vertical arrows relations. A cell $\psi : (P, F, G, Q)$ is a containment of relations: that is, there is a cell ψ if $P(y, x)$ implies $Q(Gy, Fx)$.

Example 8.1.2. Rings as objects, horizontal arrows as ring homomorphisms, bimodules as vertical arrows. A cell $\psi : (M, F, G, N)$ is an group homomorphism $M \xrightarrow{\psi} N$ such that $\psi(rms) = f(s)\psi(m)g(s)$.

Example 8.1.3. Given a monoidal category $(\mathbf{V}, \otimes, I)$ with colimits, one can form the double category of $\mathbf{V}$-matrices: objects are sets, horizontal arrows are functions, and vertical arrows are $\mathbf{V}$-matrices. A $\mathbf{V}$-matrix M between sets X, Y is simply a function $X \times Y \xrightarrow{M(-,-)} V$. Vertical composition is a "matrix" product:

$$NM(z, x) = \sum M(z, y) \otimes M(y, x)$$

where the sum is the colimit in $\mathbf{V}$. Cells $\psi : (M, F, G, N)$ are indexed families of arrows in $\mathbf{V}$ from $M(x, y)$ to $N(fx, gy)$. This double category is useful for describing relational algebras: see Paré [38] and Clementino and Tholen [41].

Example 8.1.4. Any 2-category can be made into a double category with only identities for vertical arrows.

Example 8.1.5. Similarly, any bicategory can be made into a double category with only identites for horizontal arrows.

Example 8.1.6. A 2-category can also be made into a double category where the horizontal and vertical arrows are both the arrows of the 2-category, and the cells of the double category 2-cells between the composite arrows of the 2-category.

Example 8.1.7. The key example for our work is, for cocomplete monoidal $\mathbf{V}$, the double category $\mathbf{V}\text{-}\mathbf{CAT}$. Here the horizontal arrows are $\mathbf{V}$-functors, the vertical arrows $\mathbf{V}$-profunctors, and a cell ψ, between functors (F, G) and profunctors (P, Q) is a morphism of profunctors

$$
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\;F_*\;} & \mathbf{Y} \\
P \downarrow & \overset{\psi}{\Rightarrow} & \downarrow Q \\
\mathbf{X}' & \xleftarrow{\;G^*\;} & \mathbf{Y}'
\end{array}
$$

A cell will thus have components

$$P(y, x) \xrightarrow{\psi(y,x)} Q(Gy, Fx)$$

As we have already noted in the previous chapter, this type of cell appears frequently when one works with **V-prof**. Other examples include recent work on Cartesian Bicategories ([7], pg. 104), and work on general change of base questions ([8], pg. 101).

Since this is the double category we will be working with, let us also describe the cell compositions and identites explicity. The vertical cell identity

$$
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\ F\ } & \mathbf{Y} \\
{\scriptstyle 1_{\mathbf{X}}}\Big\downarrow & {\scriptstyle 1_F} & \Big\downarrow{\scriptstyle 1_{\mathbf{Y}}} \\
\mathbf{X} & \xrightarrow[\ F\ ]{} & \mathbf{Y}
\end{array}
$$

is the effect of F on homs: $\mathbf{X}(x', x) \xrightarrow{\ F\ } \mathbf{Y}(Fx', Fx)$.

The horizontal cell identity

$$
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\ 1_X\ } & \mathbf{X} \\
{\scriptstyle P}\Big\downarrow & {\scriptstyle 1_P} & \Big\downarrow{\scriptstyle P} \\
\mathbf{W} & \xrightarrow[\ 1_W\ ]{} & \mathbf{W}
\end{array}
$$

is given by the identity arrow $P(y, x) \xrightarrow{\ 1\ } P(y, x)$.

The vertical composition of cells

$$
\begin{array}{ccc}
\mathbf{X_1} & \xrightarrow{\ F\ } & \mathbf{X_2} \\
{\scriptstyle P_1}\Big\downarrow & {\scriptstyle \psi_1} & \Big\downarrow{\scriptstyle Q_1} \\
\mathbf{Y_1} & \xrightarrow{\ G\ } & \mathbf{Y_2} \\
{\scriptstyle P_2}\Big\downarrow & {\scriptstyle \psi_2} & \Big\downarrow{\scriptstyle Q_2} \\
\mathbf{Z_1} & \xrightarrow[\ H\ ]{} & \mathbf{Z_2}
\end{array}
$$

must have components

$$
\int^{y_1} P_2(z, y_1) \otimes P_1(y_1, x) \xrightarrow{(\psi_2\psi_1)(z,x)} \int^{y_2} Q_2(Hz, y_2) \otimes P_1(y_2, Fx)
$$

On an element y_1, it is given by the composite

$$P_2(z, y_1) \otimes P_1(y_1, x) \xrightarrow{\psi_2 \otimes \psi_1} Q_2(Hz, Gy_1) \otimes P_1(Gy_1, Fx) \xrightarrow{i_{Gy_1}} \int^{y_2} Q_2(Hz, y_2) \otimes P_1(y_2, Fx)$$

The horizontal composition of cells

$$\begin{array}{ccccc}
\mathbf{X_1} & \xrightarrow{F_1} & \mathbf{X_2} & \xrightarrow{F_2} & \mathbf{X_3} \\
P \downarrow & \psi_1 & Q & \psi_2 & \downarrow R \\
\mathbf{Y_1} & \xrightarrow{G_1} & \mathbf{Y_2} & \xrightarrow{G_2} & \mathbf{Y_3}
\end{array}$$

is given by composition

$$P(y, x) \xrightarrow{\psi_1(y, x)} Q(G_1 y, F_1 x) \xrightarrow{\psi_2(G_1 y, F_1 x)} R(G_2 G_1 y, F_2 F_1 x).$$

8.1.1 Lax Double Functors

The idea of lax functors between bicategories extends to an idea of lax double functors between (pseudo) double categories.

Definition Suppose that $\mathcal{C}$ and $\mathcal{D}$ are double categories. A lax double functor L between $\mathcal{C}$ and $\mathcal{D}$ consists of:

- a functor L_0 between the horizontal arrow categories,

- a lax functor between the vertical arrow bicategories agreeing with the above on objects; call its identity comparisons ρC and composition comparisons $\rho(p, q)$,

- a map between the cells which preserves horizontal composition.

Finally, the vertical composition is preserved up to the comparison maps:

- For a horizontal arrow $X \xrightarrow{F} Y$,

$$\begin{array}{ccccccc}
LX & \xrightarrow{LF} & LY & = & LY & \quad & LX & = & LX & \xrightarrow{LF} & LY \\
1_{LX} \downarrow & 1_{LF} & 1_{LY} \downarrow & \rho Y & \downarrow L(1_Y) & = & 1_{LX} \downarrow & \rho X & L(1_X) \downarrow & L(1_F) & \downarrow L(1_Y) \\
LX & \xrightarrow{LF} & LY & = & LY & \quad & LX & = & LX & \xrightarrow{LF} & LY
\end{array}$$

- For cells $\psi_1 : (p_1, f, g, q_1)$, $\psi_2 : (p_2, g, h, q_2)$,

$$
\begin{array}{ccc}
LX_1 \xrightarrow{LF} LX_2 = LX_2 & & LX_1 = LX_1 \xrightarrow{LF} LX_2 \\
\downarrow{LP_1} \quad L\psi_1 \quad \downarrow{LQ_1} \qquad \downarrow & & \downarrow{LP_1} \qquad \downarrow \qquad \downarrow \\
LY_1 \xrightarrow{LG} LY_2 \quad \rho \quad L(Q_2Q_1) & = & LY_1 \quad \rho \quad L(\psi_2\psi_1) \; L(Q_2Q_1) \\
\downarrow{LP_2} \quad L\psi_2 \quad \downarrow{LQ_2} \qquad \downarrow & & \downarrow{LQ_1} \qquad \downarrow \qquad \downarrow \\
LZ_1 \xrightarrow{LH} LZ_2 = LZ_2 & & LZ_1 = LZ_1 \xrightarrow{LH} LZ_2
\end{array}
$$

Example 8.1.8. Suppose we have two 2-categories, viewed as (vertically trivial) double categories. Then a lax double functor between them is a 2-functor.

Example 8.1.9. Suppose we have two bicategories, viewed as (horizontally trivial) double categories. Then a lax double functor them is a lax functor.

Example 8.1.10. As a particular case of the above, suppose we have two monoidal categories, viewed as one-object bicategories, and hence as (horizontally trivial) double categories with only one object. Then a lax double functor between them is a monoidal functor.

8.1.2 Horizontal Transformations

The key idea that is required for our change of base result is the notion of a horizontal transformation. Essentially, it is a natural transformation horizontally, while also taking vertical arrows to cells in a natural way.

Definition Suppose that we have lax double functors $\mathcal{C} \xrightarrow{L,K} \mathcal{D}$. A horizontal transformation $L \xrightarrow{\alpha} K$ consists of the following data:

- For each $X \in \mathcal{C}$, a horizontal arrow $LX \xrightarrow{\alpha X} KX$,

- For each vertical arrow $X \xrightarrow{P} Y$ in $\mathcal{C}$, a cell

$$
\begin{array}{ccc}
LX & \xrightarrow{\alpha X} & KX \\
\downarrow{LP} & \alpha P & \downarrow{KP} \\
LY & \xrightarrow{\alpha Y} & KY
\end{array}
$$

The horizontal arrows αX form a natural transformation between L_0 and K_0, and the following conditions also hold:

- For a cell $\psi : (P, F, G, Q)$,

$$
\begin{array}{ccc}
LX \xrightarrow{LF} LY \xrightarrow{\alpha Y} KY & & LX \xrightarrow{\alpha X} KX \xrightarrow{KF} KY \\
\downarrow{\scriptstyle LP} \quad L\psi \quad {\scriptstyle LQ} \quad \alpha Q \quad \downarrow{\scriptstyle KQ} & = & \downarrow{\scriptstyle LP} \quad \alpha P \quad {\scriptstyle KP} \quad K\psi \quad \downarrow{\scriptstyle KQ} \\
LW \xrightarrow{LG} LZ \xrightarrow{\alpha Z} KY & & LW \xrightarrow{\alpha W} KW \xrightarrow{KG} KZ
\end{array}
$$

- For an object X,

$$
\begin{array}{ccc}
LX \xrightarrow{\alpha X} KX = KX & & LX = LX \xrightarrow{\alpha X} KX \\
\downarrow{\scriptstyle 1_{LX}} \quad 1_{\alpha X} \quad {\scriptstyle 1_{MX}} \; \rho KX \; {\scriptstyle K(1_X)} & = & \downarrow{\scriptstyle 1_{LX}} \quad \rho X \; {\scriptstyle L(1_X)} \quad \alpha 1_X \quad \downarrow{\scriptstyle K(1_X)} \\
LX \xrightarrow{\alpha X} KX = KX & & LX = LX \xrightarrow{\alpha X} KX
\end{array}
$$

- For vertical morphisms $X \xrightarrow{P} Y \xrightarrow{Q} Z$,

$$
\begin{array}{ccc}
LX \xrightarrow{\alpha X} KX = KX & & LX = LX \xrightarrow{\alpha X} KX \\
\downarrow{\scriptstyle LP} \quad \alpha P \quad {\scriptstyle MP} & & \downarrow{\scriptstyle LP} \\
LY \xrightarrow{\alpha Y} KY \quad \rho \quad {\scriptstyle K(QP)} & = & LY \quad \rho \quad \alpha(QP) \; {\scriptstyle L(QP)} \\
\downarrow{\scriptstyle LQ} \quad \alpha Q \quad {\scriptstyle MQ} & & \downarrow{\scriptstyle LQ} \\
LZ \xrightarrow{\alpha Z} KZ = KZ & & LZ = LZ \xrightarrow{\alpha Z} KZ
\end{array}
$$

Example 8.1.11. A horizontal transformation between 2-categories and 2-functors, viewed as double categories and double functors, is a 2-natural transformation.

Example 8.1.12. A horizontal transformation between monoidal categories and monoidal functors, viewed as double categories and double functors, is a monoidal natural transformation.

Example 8.1.13. A horizontal transformation between bicategories and lax functors, viewed as double categories and double functors, is a lax natural transformation which forces the lax functors to be equal on objects. As Grandis and Paré have observed [22], these special transformations appear in the work of Carboni and Rosebrugh on lax monads of bicategories [9].

Example 8.1.14. In the work of Susan Niefield and others (see, for example, [6]), a category $\mathrm{Lax}(B^{\mathrm{op}}, \mathrm{Span})$ is described. The objects are lax functors from the opposite of a bicategory B to the bicategory of sets and spans. While not described as such, the morphisms in $\mathrm{Lax}(B^{\mathrm{op}}, \mathrm{Span})$ are horizontal transformations between lax double functors from B^{op}, considered as a double category, to the double category of sets, functions, and spans.

The previous two examples show instances where horizontal transformations appeared implicitly before the concept itself was defined, showing how fundamental these morphisms are.

8.1.3 The 2-Category Dblcat

Our last background work is to describe the compositions of lax double functors and horizontal transformations. A somewhat surprising aspect of these structures (given that we are dealing with pseudo objects and lax arrows) is that pseudo double categories, lax double functors, and horizontal transformations form a (strict) 2-category. We will not prove this here, but the result can be found in Grandis and Paré ([22], pg. 207).

An interesting aspect of that work, however, is that this 2-category actually lives inside a larger double category. The double category consists of pseudo double categories, lax double functors (horizontal), co-lax double functors (vertical), and cells between them. There are several surprising aspects of this double category: first of all, that there are viable cells between lax and co-lax double functors, and secondly, that this double category is itself strict. (The horizontal arrows and cells of this double category give rise to the 2-category **dblcat** we consider here.)

When restricted to double categories which are monoidal categories (that is, double categories with one object and only identity horizontal arrows), this gives a strict double category of monoidal categories, monoidal functors, co-monoidal functors, and cells between them. This double category itself may be interesting to investigate as a different domain for change of base questions. For now, however, we will restrict

ourselves to the usual 2-cat **cmoncat** and the 2-category **dblcat**.

The composition of lax double functors is straightforward, once one recalls that lax functors between bicategories themselves compose strictly (see Benabou [3] and Section 6.3.1).
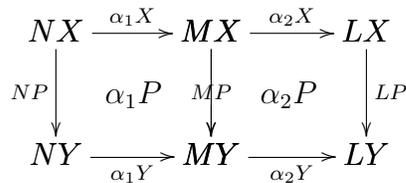
Definition Given double categories and lax double functors $\mathcal{C} \xrightarrow{L} \mathcal{D} \xrightarrow{K} \mathcal{E}$, the composite of K and L is given by composing the functors, composing the lax functors, and composing the action on cells.

The horizontal and vertical composition of horizontal transformations both use the horizontal composition in the codomain double category.

Definition Suppose that we have lax double functors and horizontal transformations

$$
\begin{array}{c}
N \\
\Downarrow \alpha_1 \\
\mathcal{C} \xrightarrow{\quad M \quad} \mathcal{D} \\
\Downarrow \alpha_2 \\
L
\end{array}
$$
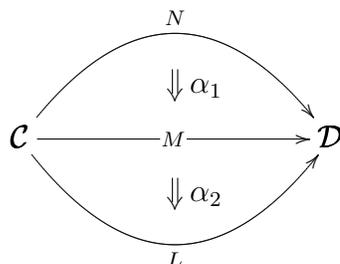
Then $(\alpha_2 \alpha_1)X$ is given by $\alpha_2 X \circ \alpha_1 X$, and for $X \xrightarrow{P} Y$, $(\alpha_2 \alpha_1)P$ is given by the horizontal composite

$$
\begin{array}{ccc}
NX \xrightarrow{\alpha_1 X} MX \xrightarrow{\alpha_2 X} LX \\
NP \downarrow \quad \alpha_1 P \quad MP \downarrow \quad \alpha_2 P \quad \downarrow LP \\
NY \xrightarrow[\alpha_1 Y]{} MY \xrightarrow[\alpha_2 Y]{} LY
\end{array}
$$

Definition Suppose that we have double functors and horizontal transformations

$$
\begin{array}{ccc}
& N_1 & & M_1 \\
\mathcal{C} & \Downarrow \alpha_1 & \mathcal{D} & \Downarrow \alpha_2 & \mathcal{E} \\
& N_2 & & M_2
\end{array}
$$
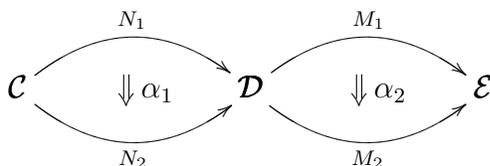
Then $(\alpha_2 * \alpha_1)X$ is given by $M_2(\alpha_1 X) \circ \alpha_2(N_1 X)$, for $X \xrightarrow{P} Y$, $(\alpha_2 * \alpha_1)P$ is given by the horizontal composite

$$
\begin{array}{ccccc}
N_2 N_1 X & \xrightarrow{\alpha_2(N_1 X)} & M_2 N_1 X & \xrightarrow{M_2(\alpha_1 X)} & M_2 M_1 X \\
{\scriptstyle N_2 N_1 P}\downarrow & {\scriptstyle \alpha_2(N_1 P)} & \downarrow{\scriptstyle M_2 N_1 P} & {\scriptstyle M_2(\alpha_1 P)} & \downarrow{\scriptstyle M_2 M_1 P} \\
N_2 N_1 Y & \xrightarrow[\alpha_2(N_1 Y)]{} & M_2 N_1 Y & \xrightarrow[M_2(\alpha_1 Y)]{} & M_2 M_1 Y
\end{array}
$$

8.2 Change of Base as a 2-functor to dblcat

Now that we have defined the 2-category **dblcat**, we can describe the change of base

$$
\textbf{cmoncat} \xrightarrow{()_*} \textbf{dblcat}
$$

We begin by describing the action of this functor on arrows.

8.2.1 N_* as a Lax Double Functor

We have already defined N_* on the horizontal (**V**-functors) and vertical (**V**-profunctors) so that N_* is a 2-functor horizontally and a lax functor vertically. Since a cell $\psi : (P, F, G, Q)$ is a **V**-form $P \xrightarrow{\psi} G^* Q F_*$, we define $N_*(\psi)(y, x)$ by $N(\psi(y, x))$; this is a **W**-form by Proposition 6.2.2. We now show that with these assignments, N_* defines a lax double functor.

Proposition 8.2.1. *Suppose that* $(\mathbf{V}, \otimes, I) \xrightarrow{N} (\mathbf{W}, \bullet, J)$ *is a monoidal functor. Then, with assignments as described above,* $\mathbf{V}\text{-}\mathbf{CAT} \xrightarrow{N_*} \mathbf{W}\text{-}\mathbf{CAT}$ *is a lax double functor.*

Proof. We have already shown that N_* is a 2-functor horizontally (Theorem 4.2.4) and a lax functor vertically (Theorem 6.2.3). We still need to show the two additional axioms for a lax double functor. The first one requires that for $\mathbf{X} \xrightarrow{F} \mathbf{Y} \in \mathbf{V}\text{-}\mathbf{cat}$,

$$
\begin{array}{ccccc}
N_* X \xrightarrow{N_* F} N_* Y \xRightarrow{\hspace{1em}} N_* Y & & & N_* X \xRightarrow{\hspace{1em}} N_* X \xrightarrow{N_* F} N_* Y \\
{\scriptstyle 1_{N_* X}}\downarrow \quad {\scriptstyle 1_{N_* F}} \quad {\scriptstyle 1_{N_* Y}}\downarrow \quad {\scriptstyle \rho Y} \quad {\scriptstyle N_*(1_Y)}\downarrow & = & & {\scriptstyle 1_{N_* X}}\downarrow \quad {\scriptstyle \rho X} \quad {\scriptstyle N_*(1_X)}\downarrow \; {\scriptstyle N_*(1_F)} \; \Big|{\scriptstyle N_*(1_Y)} \\
N_* X \xrightarrow[N_* F]{} N_* Y \xRightarrow{\hspace{1em}} N_* Y & & & N_* X \xRightarrow{\hspace{1em}} N_* X \xrightarrow[N_* F]{} N_* Y
\end{array}
$$

Recall that $N_\#$ is normal, and so the identity comparisons ρX and ρY are identity **V**-forms. Thus the above reduces to checking that $1_{N_*F} = N_*(1_F)$. The components of 1_{N_*F} are the effect of N_*F on homs, so that $1_{N_*F}(y, x) = NF(y, x)$. On the other hand, the components of $N_*(1_F)$ are N of the components of the effect of F on homs, so that $N_*(1_F)(y, x)$ is also $NF(y, x)$.

The second axiom requires that for cells $\psi_1 : (p_1, f, g, q_1)$, $\psi_2 : (p_2, g, h, q_2)$,

$$
\begin{array}{ccc}
N_*X_1 \xrightarrow{N_*F} N_*X_2 = N_*X_2 & & N_*X_1 = N_*X_1 \xrightarrow{N_*F} N_*X_2 \\
\Big\downarrow{\scriptstyle N_*P_1} \quad N_*\psi_1 \; {\scriptstyle N_*Q_1}\Big\downarrow \quad \Big\downarrow & & \Big\downarrow{\scriptstyle N_*P_1} \\
N_*Y_1 \xrightarrow{-N_*G} N_*Y_2 \quad \rho \;\; {\scriptstyle N_*(Q_2Q_1)} & = & N_*Y_1 \quad \rho \quad N_*(\psi_2\psi_1) \; {\scriptstyle N_*(Q_2Q_1)} \\
\Big\downarrow{\scriptstyle N_*P_2} \quad N_*\psi_2 \; {\scriptstyle N_*Q_2}\Big\downarrow \quad \Big\downarrow & & \Big\downarrow{\scriptstyle N_*Q_1} \\
N_*Z_1 \xrightarrow[N_*H]{} N_*Z_2 = N_*Z_2 & & N_*Z_1 = N_*Z_1 \xrightarrow[N_*H]{} N_*Z_2
\end{array}
$$

At objects $x \in \mathbf{X}, y \in \mathbf{Y}, z \in \mathbf{Z}$, this reduces to

$$
\begin{array}{ccc}
\int^{y_1} NP_2(z, y_1) \cdot NP_1(y, x) & \xrightarrow{\;\;\rho\;\;} & \int^{y_1} NQ_2(z, y_1) \cdot NP_1(y_1, x) \\
{\scriptstyle \int N\psi_1 \cdot N\psi_2}\Big\downarrow & & \Big\downarrow{\scriptstyle N(\int \psi_2 \cdot \psi_1)} \\
N \int^{y_2} P_2(Hz, y_2) \cdot P_1(y_2, Fx) & \xrightarrow[\;\;\rho\;\;]{} & N \int^{y_2} Q_2(Hz, y_2) \cdot P_1(y_2, Fx)
\end{array}
$$

If we pre-compose with an injection at y_1, then we can expand the above diagram to

$$NP_2(z,y_1) \cdot NP_1(y_1,x) \xrightarrow{\widetilde{N}} N(P_2(z,y_1) \cdot P_1(y_1,x)) \xrightarrow{N(\psi_2 \cdot \psi_1)} N(Q_2(Hz,Gy_1) \cdot Q_1(Gy_1,Fx))$$

(with the internal labels: i_{y_1}, $N(i_{y_1})$, $N\psi_2 \cdot N\psi_1$, $\int^{y_1} NP_2(z,y_1) \cdot NP_1(y,x) \xrightarrow{\rho} N\int^{y_1} P_2(z,y_1) \cdot P_1(y,x)$, $N(i_{Gy_1})$, $NQ_2(Hz,Gy_1) \cdot NQ_1(Gy_1,Fx)$, $\int N\psi_1 \cdot N\psi_2$, $N(\int \psi_2 \cdot \psi_1)$, i_{Gy_1}, $\widetilde{N}$, $\int^{y_2} NQ_2(Hz,y_2) \cdot NP_1(y_2,Fx) \xrightarrow{\rho} N\int^{y_2} Q_2(Hz,y_2) \cdot P_1(y_2,Fx)$, $N(Q_2(Hz,Gy_1) \cdot Q_1(Gy_1,Fx))$, $N(i_{Gy_1})$)

(where, to save space, $\cdot$ represents the tensor product in both $\mathbf{V}$ and $\mathbf{W}$). The internal regions all commute by definition of the maps out of the co-ends. The outer diagram commutes by naturality of $\widetilde{N}$. Thus N_* is a lax double functor.

∎

8.2.2 α_* as a Horizontal Natural Transformation

Suppose now that we have monoidal functors $\mathbf{V} \xrightarrow{N,M} \mathbf{W}$ and a monoidal natural transformation $N \xrightarrow{\alpha} M$. We wish to show that we can define a horizontal transformation $N_* \xrightarrow{\alpha_*} M_*$. For an object $\mathbf{X} \in \mathbf{V\text{-}CAT}$, define $\alpha_* X$ as it was defined in the $\mathbf{V\text{-}cat}$ case (see Proposition 4.3.1). The next proposition shows how we can define α_* on a profunctor. In contrast to the definition of $\alpha_\#$ as a module, the definition of α_* as a horizontal transformation is entirely straightforward: in effect, it is nothing more than α itself.

Proposition 8.2.2. *Suppose that* $(\mathbf{V}, \otimes, I) \xrightarrow{N,M} (\mathbf{W}, \bullet, J)$ *are monoidal functors, and* $N \xrightarrow{\alpha} M$ *is a monoidal natural transformation between them. Given a profunctor*

$\mathbf{X} \xrightarrow{\ P\ } \mathbf{Y} \in \mathbf{V\text{-}prof}$, *we can define a cell in* $\mathbf{W\text{-}CAT}$

$$
\begin{array}{ccc}
N_*X & \xrightarrow{\ \alpha_*X\ } & M_*X \\
{\scriptstyle N_*P}\downarrow & \alpha_*P & \downarrow{\scriptstyle N_*Q} \\
N_*Y & \xrightarrow[\ \alpha_*Y\ ]{} & M_*Y
\end{array}
$$

with component at $x \in \mathbf{X}, y \in \mathbf{Y}$ *given by*

$$
NP(y,x) \xrightarrow{\ \alpha P(y,x)\ } MP(y,x)
$$

Proof. We need to show that α_*P is a $\mathbf{V}$-form, so we need to show that it satisfies compatibility with left and right actions of the associated profunctors. For the right action, we need the following diagram to commute:

$$
\begin{array}{ccc}
NP(y,x) \bullet N\mathbf{X}(x,x') & \xrightarrow{\ \alpha\bullet1\ } & MP(y,x) \bullet N\mathbf{X}(x,x') \\
& {\scriptstyle \alpha\bullet\alpha} \searrow & \downarrow{\scriptstyle 1\bullet\alpha} \\
{\scriptstyle \widetilde{N}}\downarrow & & MP(y,x) \bullet M\mathbf{X}(x,x') \\
& & \downarrow{\scriptstyle \widetilde{M}} \\
N(P(y,x) \otimes \mathbf{X}(x,x')) & {\dashrightarrow}{\scriptstyle \alpha}{\dashrightarrow} & M(P(y,x) \otimes \mathbf{X}(x,x')) \\
{\scriptstyle N(P_R)}\downarrow & & \downarrow{\scriptstyle M(P_R)} \\
NP(y,x') & \xrightarrow[\ \alpha\ ]{} & MP(y,x')
\end{array}
$$

The top right triangle is bifunctoriality of $\bullet$, the bottom square is naturality of α, and the middle region is monoidal naturality of α. The left action axiom is similar, and so α_*P is a $\mathbf{W}$-form. ∎

Now that we have defined α_* on objects and vertical arrows, we need to show that it defines a horizontal transformation.

Proposition 8.2.3. *Suppose that* $(\mathbf{V}, \otimes, I) \xrightarrow{\ N,M\ } (\mathbf{W}, \bullet, J)$ *are monoidal functors, and* $N \xrightarrow{\ \alpha\ } M$ *is a monoidal natural transformation between them. With actions on objects and vertical arrows as above,* α_* *is a horizontal transformation between the lax double functors* N_*, M_*.

Proof. We know from Proposition 4.3.1 that $\alpha_* X$ defines a natural transformation. It remains to show the other three axioms. For the first, we need to show that for a cell $\psi : (P, F, G, Q)$,

$$
\begin{array}{ccccc}
N_*\mathbf{X} & \xrightarrow{N_*F} & N_*\mathbf{Y} & \xrightarrow{\alpha_*\mathbf{Y}} & M_*\mathbf{Y} \\
\downarrow{\scriptstyle N_*P} & N_*\psi \quad N_*Q & \downarrow \quad \alpha_*Q & & \downarrow{\scriptstyle M_*Q} \\
N_*\mathbf{W} & \xrightarrow{N_*G} & N_*\mathbf{Z} & \xrightarrow{\alpha_*\mathbf{Z}} & M_*\mathbf{Y}
\end{array}
\quad = \quad
\begin{array}{ccccc}
N_*\mathbf{X} & \xrightarrow{\alpha_*\mathbf{X}} & M_*\mathbf{X} & \xrightarrow{M_*F} & M_*\mathbf{Y} \\
\downarrow{\scriptstyle N_*P} & \alpha_*P \quad M_*P & \downarrow \quad M_*\psi & & \downarrow{\scriptstyle M_*Q} \\
N_*\mathbf{W} & \xrightarrow{\alpha_*W} & \boldsymbol{M_*W} & \xrightarrow{M_*G} & M_*\mathbf{Z}
\end{array}
$$

The horizontal composition of cells in **W-CAT** is simply composition, so this reduces to showing that the following diagram commutes, for $x \in \mathbf{X}, w \in \mathbf{W}$,

$$
\begin{array}{ccc}
NP(w, x) & \xrightarrow{N\psi} & NQ(Gw, Fx) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
MQ(w, x) & \xrightarrow{M\psi} & MW(Gw, Fx)
\end{array}
$$

This commutes by naturality of α.

For the second axiom, we need to show that for a **V**-category **X**,

$$
\begin{array}{ccccc}
N_*\mathbf{X} & \xrightarrow{\alpha_*\mathbf{X}} & M_*\mathbf{X} & = & M_*\mathbf{X} \\
\downarrow{\scriptstyle 1_{N_*\mathbf{X}}} & 1_{\alpha_*\mathbf{X}} \quad 1_{M\mathbf{X}} & \downarrow \; \rho M_*\mathbf{X} & M_*(1_\mathbf{X}) & \downarrow \\
N_*\mathbf{X} & \xrightarrow{\alpha_*\mathbf{X}} & M_*\mathbf{X} & = & M_*\mathbf{X}
\end{array}
\quad = \quad
\begin{array}{ccccc}
N_*\mathbf{X} & = & N_*\mathbf{X} & \xrightarrow{\alpha_*\mathbf{X}} & M_*\mathbf{X} \\
\downarrow{\scriptstyle 1_{N_*\mathbf{X}}} & \rho\mathbf{X} \quad N_*(1_\mathbf{X}) & \alpha_*1_\mathbf{X} & & \downarrow{\scriptstyle M_*(1_\mathbf{X})} \\
N_*\mathbf{X} & = & N_*\mathbf{X} & \xrightarrow{\alpha_*\mathbf{X}} & M_*\mathbf{X}
\end{array}
$$

Since both ρ's are identities, we only need to show that $1_{\alpha_*\mathbf{X}} = \alpha_* 1_\mathbf{X}$. At $x', x \in \mathbf{X}$, $1_{\alpha_*\mathbf{X}}$ is the effect on homs of α_*X, which is simply $NX(x', x) \xrightarrow{\alpha\mathbf{X}(x', x)} MX(x', x)$. On the other hand, $\alpha_* 1_\mathbf{X}$ is α applied to the identity of $\mathbf{X}(x', x)$, so it is also $\alpha\mathbf{X}(x', x)$.

For the final axiom, we need to show that given $X \xrightarrow{P} Y \xrightarrow{Q} Z \in \mathbf{V\text{-}prof}$,

$$
\begin{array}{cccc}
N_*\mathbf{X} & \xrightarrow{\alpha_*\mathbf{X}} & M_*\mathbf{X} & = M_*\mathbf{X} \\
\downarrow{\scriptstyle N_*P} & \alpha_*P \quad MP & & \downarrow \\
N_*\mathbf{Y} & \xrightarrow{\alpha_*\mathbf{Y}} & M_*\mathbf{Y} & \rho \quad M_*(QP) \\
\downarrow{\scriptstyle N_*Q} & \alpha_*Q \quad MQ & & \downarrow \\
N_*\mathbf{Z} & \xrightarrow{\alpha_*\mathbf{Z}} & M_*\mathbf{Z} & = M_*\mathbf{Z}
\end{array}
\quad = \quad
\begin{array}{cccc}
N_*\mathbf{X} & = N_*\mathbf{X} & \xrightarrow{\alpha_*\mathbf{X}} & M_*\mathbf{X} \\
\downarrow{\scriptstyle N_*P} & & & \downarrow \\
N_*\mathbf{Y} & \rho & \alpha_*(QP) & \downarrow{\scriptstyle N_*(QP)} \\
\downarrow{\scriptstyle N_*Q} & & & \downarrow \\
N_*\mathbf{Z} & = N_*\mathbf{Z} & \xrightarrow{\alpha_*\mathbf{Z}} & M_*\mathbf{Z}
\end{array}
$$

At objects $z \in \mathbf{Z}, x \in \mathbf{X}$, this reduces to checking that the following commutes:

$$\int^y NQ(z,y) \bullet NP(y,x) \xrightarrow{\int \alpha_* Q \bullet \alpha_* P} \int^y MQ(z,y) \bullet MP(y,x)$$

$$\downarrow \rho \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \rho$$

$$N(\int^y Q(z,y) \otimes P(y,x)) \xrightarrow{\alpha_*(QP)} M(\int^y Q(z,y) \otimes P(y,x))$$

If we pre-compose with an injection to $y \in \mathbf{Y}$, this reduces to checking that the following commutes:

$$NQ(z,y) \bullet NP(y,x) \xrightarrow{\alpha \bullet \alpha} MQ(z,y) \bullet MP(y,x)$$

$$\downarrow \widetilde{N} \qquad\qquad\qquad\qquad\qquad \downarrow \widetilde{M}$$

$$N(Q(z,y) \otimes P(y,x)) \xrightarrow{\alpha} M(Q(z,y) \otimes P(y,x))$$

This last diagram is exactly monoidal naturality of α. Thus all axioms are satisfied, and α_* is a horizontal transformation.

∎

8.2.3 The Full Change of Base $(-)_*$

Finally, we show that with the components defined in the previous sections, $(-)_*$ is a 2-functor between (Cocomplete Monoidal Categories, Monoidal Functors, Monoidal Natural Transformations) and (Double Categories, Lax Double Functors, Horizontal Transformations).

Theorem 8.2.4. *With components defined on objects as in Proposition 4.2.1, on arrows as in Proposition 8.2.1, and on 2-cells as in Proposition 8.2.3,*

$$\mathbf{cmoncat} \xrightarrow{(-)_*} \mathbf{dblcat}$$

defines a 2-functor.

Proof. In Theorem 4.3.2, we saw that $(-)_*$ defines a 2-functor from **moncat** to **2-cat**. In Theorem 6.2.3, we saw that $(-)_\#$ defines a functor between (Cocomplete Monoidal Categories, Monoidal Functors) and (Bicategories, Lax Functors). All that remains

to show is that $(-)_*$ preserves horizontal and vertical composition and identities of horizontal transformations. Suppose that we have monoidal natural transformations

We need to show that $(\alpha_2)_* \circ (\alpha_1)_* = (\alpha_2 \circ \alpha_1)*$. Since the original $(-)_*$ is a 2-functor, these horizontal transformations have identical action on objects. To show that they have equal action on a profunctor $\mathbf{X} \xrightarrow{P} \mathbf{Y}$, recall that vertical composition of horizontal transformations is merely horizontal composition. Moreover, horizontal composition of cells in $\mathbf{W\text{-}CAT}$ is simply composition in $\mathbf{W}$. Thus,

$$
\begin{aligned}
(\alpha_2 \circ \alpha_1) * (P)(y,x) &= (\alpha_2 P)(y,x) \circ (\alpha_1 P)(y,x) \\
&= \alpha_2 P(y,x) \circ \alpha_1 P(y,x) \\
&= (\alpha_2)_*(P)(y,x) \circ (\alpha_1)_*(P)(y,x) \\
&= (\alpha_2)_* \circ (\alpha_1)_*(P)(y,x)
\end{aligned}
$$

so that they are equal. In the same way, vertical identities are preserved.

Preservation of horizontal composition is similar, since the horizontal composition of natural transformations and horizontal transformations is done in the same way. Indeed, suppose that we have monoidal natural transformations

Recall that at an object $X \in \mathbf{V}$, $(\alpha_2 \star \alpha_1)X$ has components

$$
N_2 N_1 X \xrightarrow{\alpha_2(N_1 X)} M_2 N_1 X \xrightarrow{M_2 \alpha_1 X} M_2 M_1 X
$$

The horizontal composition of horizontal transformations is defined in exactly the same way on both objects and cells, using the horizontal composition of $\mathbf{W\text{-}CAT}$,

which, again, is simply composition. So

$$
\begin{aligned}
((\alpha_2)_* \star (\alpha_1)_*)(P)(y,x) &= (\alpha_2)_*(N_1 P)(y,x) \circ M_2(\alpha_1)_* P(y,x) \\
&= \alpha_2(N_1 P(y,x)) \circ M_2(\alpha_1(P(y,x))) \\
&= (\alpha_2 \star \alpha_1)P(y,x) \\
&= (\alpha_2 \star \alpha_1)_*(P)(y,x)
\end{aligned}
$$

Thus $(-)_*$ does preserve horizontal composition, and so is a 2-functor.

$\blacksquare$

8.3 $(-)_*$ is Monoidal

In Chapter 5, one of the first steps in showing that N_* preserved monoidal categories was to show that $(-)_*$ was monoidal. In fact, this had additional value in that it gave a direct proof that **V-cat** had a monoidal structure. Here, we give this first step in moving our work to double categories, by showing that the double category version of $(-)_*$ is monoidal.

Theorem 8.3.1. *The 2-functor* **cmoncat** $\xrightarrow{\;(-)_*\;}$ **dblcat** *is monoidal.*

Proof. To show that $(-)_*$ is monoidal, define the identity comparison maps as in Proposition 5.4.2. To define the tensor comparison maps, we need arrows

$$
\mathbf{V\text{-}CAT} \times \mathbf{W\text{-}CAT} \xrightarrow{\;\psi\;} (\mathbf{V} \times \mathbf{W})\text{-}\mathbf{CAT}
$$

As these are arrows in **dblcat**, these should be lax double functors. From Proposition 5.4.2, we know how to define this comparison on objects and horizontal arrows. Defining it on profunctors is similar. Suppose we have profunctors $\mathbf{X_1} \xrightarrow{P} \mathbf{X_2} \in \mathbf{V\text{-}prof}$, $\mathbf{Y_1} \xrightarrow{Q} \mathbf{Y_2} \in \mathbf{W\text{-}prof}$. Then we can define

$$
\psi(P \times Q)((x_2, y_2), (x_1, y_1)) := (P(x_2, x_1), Q(y_2, y_1))
$$

We need to show that this assignment carries with it comparisons that make T into a lax functor. So, suppose now that we have profunctors

$$
\mathbf{X_1} \xrightarrow{P_1} \mathbf{X_2} \xrightarrow{P_2} \mathbf{X_3}
$$

and

$$\mathbf{Y_1} \xrightarrow{Q_1} \mathbf{Y_2} \xrightarrow{Q_2} \mathbf{Y_3}$$

We then need to be able to compare

$$\psi(P_1, Q_1) \circ \psi(P_2, Q_2) \text{ and } \psi((P_1, Q_1) \circ (P_2, Q_2))$$

Evaluated at objects $(x_1, y_1), (x_3, y_3)$, the first expression reduces to

$$\left(\int^{x_2, y_2} P_2(x_3, x_2) \otimes P_1(x_2, x_1), \int^{x_2, y_2} Q_2(y_3, y_3) \otimes Q_1(y_2, y_1) \right)$$

while the second reduces to

$$\left(\int^{x_2} P_2(x_3, x_2) \otimes P_1(x_2, x_1), \int^{y_2} Q_2(y_3, y_3) \otimes Q_1(y_2, y_1) \right)$$

To get an arrow out of the first co-end, we simply define it on each component of each part of the product as an injection. Checking that all axioms are satisfied is tedious but straightforward.

Suppose now that we have cells

with $\alpha_1 \in \mathbf{V\text{-}CAT}$ and $\alpha_2 \in \mathbf{W\text{-}CAT}$. Since ψ is defined on functors and profunctors as product, we define

$$(P_1(w_1, x_1), P_2(w_2, x_2)) \xrightarrow{\psi(\alpha_1, \alpha_2)} (Q_1(G_1 w_1, F_1 x_1), Q_2(G_2 w_2, F_2 w_2))$$

as $\alpha_1 \times \alpha_2$. That ψ preserves composition of cells follows from the fact that $\times$ is a functor.

Finally, the fact that ψ is associative and unital also follows directly from the fact that $\times$ is associative and unital. ∎

This shows that if **V** is a pseudomonoid in **cmoncat** (in particular, if **V** is braided), then **V-CAT** will be a pseudomonoid in **dblcat**. In other words, **V-CAT** is a monoidal double category.

The results of this chapter suggest that the best structure that **V**-categories form is not a bicategory, but a double category. Certainly, much work can be done in the bicategory **V-prof**. However, if we take the idea that the arrows and 2-cells between monoidal categories are just as important as the monoidal categories themselves, then we must accept the fact that double categories are needed. The consequences of this are discussed in the final chapter.

Chapter 9

Conclusion

In this final chapter, we discuss the various possibilities for future work that are a result of this thesis. There are a number of different areas that could be investigated; some from the initial discussions of normed vector spaces, others from the results about change of base, and a few simply from the methodology and ideas behind some of the proofs.

9.1   Structured Double Categories

We believe that the most important result of this thesis is the (re)discovery of the fact that change of base for enriched categories, when including profunctors, is best viewed as a double functor between double categories. To re-iterate: without viewing **V-cat** as a double category, one cannot have the full 2-categorical change of base and include **V**-profunctors. Moreover, as we have seen, the squares in the double category **V-CAT** appear in numerous situations, showing again the importance of understanding the double category **V-CAT**.

This should lead to new ideas in the area of structured higher categories. Much of the recent work of the Australian school on structured bicategories (monoidal bicategories, cartesian bicategories, autonomous bicategories, etc.) has been focused on bicategories, since this was seen as the most natural structure for **V**-categories and their profunctors. However, the change of base for **V**-categories indicates that **V**-categories and their profunctors should be viewed together with **V**-functors, so that the result is a double category. Thus, the structured bicategory notions should be re-worked into structured double category notions: cartesian double categories, monoidal double categories, autonomous double categories, etc. As we have seen, this would also permit a greater understanding of the roles of the special squares that

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appear so often in the definitions of structured bicategories. As nearly all examples of structured bicategories are of the enriched category variety, we would not be losing any important examples by asking that we work with double categoies instead of bicategories. We would also hope that by moving our work to structured double categories, we will be able to show that N_* preserves autonomous objects, which we could not do when using structured bicategories.

Moreover, the very fact that **V-cat** should be viewed as a double category is itself interesting for higher category theory. Much of higher category theory has been concerned with defining "weak n-categories", where weak 2-categories are bicategories. The reason for this was because of the central status of the prototypical bicategory **V-prof**. However, if the proper weakening step is not 2-category to bicategory, but instead 2-category to weak double category, then perhaps the entire project of n-category theory needs to be rethought. That is, instead of trying to define weak n-category, perhaps one should be trying to define weak n-double categories. Indeed, the very fact that weak double categories, double functors, and horizontal transformations form a 2-category, while bicategories, lax functors, and lax natural transformations do not is reason enough to think that higher-dimensional double categories have a greater potential than higher-dimensional bicategories.

9.2 Two versions of Normed Space

As we have seen, there are two versions of what normed space should correspond to: one views them as monoidal functors from compact categories, the other as **V**-compact closed categories. Much of the thesis was taken up in trying to understand how one could transfer between these two ideas. Ultimately, this led to viewing **V**-categories as objects of a double category, and in the end, the desired transfer of structure was not achieved. This still leaves a very large project: determine under what conditions these two structures are the same.

Even if they are not the same, however, this still leaves the rather large area of

applying the ideas of normed vector spaces to category theory, using one or the other view of normed vector spaces. It would be interesting to see how many of the major theorems of analysis (such as the Hahn-Banach theorem) have corresponding versions in category theory.

9.3 Cauchy-Completeness

One benefit of moving from the bicategory **V-prof** to the double category **V-CAT** is that it eliminated the need to consider Cauchy complete **V**-categories. The Cauchy-completeness requirement was neccesary to be able to access the **V**-functors as the maps in **V-prof**. However, in the double category **V-CAT**, the **V**-functors exist as the horizontal arrows, and so we do not need to require that our categories be Cauchy-complete.

This leads to an interesting thought: by doing analysis in the double category of metric spaces, can we eliminate the need to require that metric spaces be Cauchy complete? The essential element of this project that must be determined is the nature of how Cauchy-completeness is used. If it is used, as it is for categories, merely to access a certain type of arrow, then potentially it could be eliminated. While Cauchy-completeness is not a very large restriction on a metric space, it is still an obtrusive technical condition. For example, surprisingly often, a paper in analysis will make some construction, then be forced to take the Cauchy-completion to get a space that is amenable to the standard theorems of analysis. What one really wants, of course, is to work with the original space. If the requirement of Cauchy-completeness was found to be unncessary by using the double category of metric spaces, it would simplify a rather annoying technical restriction in the work of analysts.

9.4 Meta-Theorem for Monoidal Functors

In Chapter 4, we introduced the idea of applying a monoidal functor monoidally. The meta-theorem for monoidal functors would say something along the lines of the

following: if D is a commutative diagram, and F is applied monoidally to the arrows of D, then the resulting diagram is still commutative. All of the early propositions of Chapter 4 are of this type. Unfortunately, a general statement seems difficult to formulate. We would like to be able to say that the new diagram FD should have F applied monoidally where it is appropriate, but also change instances of the associativity or unit isomorphisms as appropriate (see, for example, the stament of Lemma 4.1.3). If a general statement could be formulated, the resulting theorem would be probably easy to prove. As in most areas of category theory, it is only in formulating the actual statement that poses any difficulty. Such a theorem would be both interesting technically and useful theoretically.

9.5   Normed Modules

One of the more interesting discoveries of Chapter 3 was that the idea of normed module is the same as the usual notion of normed vector space, assuming (as analysts do) that the vector space is over $\mathbb{R}$ or $\mathbb{C}$. That is, the sub-scalar invariance of normed modules implies the exact scalar invariance of normed vector spaces. This tells us that the sub-scalar invariance notion for modules is the correct one. This, in turn, leads to the interesting possibility that there may be normed modules that exist in nature that have not yet been discovered as normed modules. That is, they have been found, but were discarded for lacking the scalar invariance of normed vector spaces. As with earlier, one could also investigate how many results from normed vector spaces carry over to the more general normed modules. Were interesting examples of normed modules to be found, they would surely enrich the study of analysis.

The ideas from this thesis could be applied in a number of different areas. Hopefully some of the ideas presented here will allow further interesting connections between the areas of functional analysis and category theory.

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