An introduction to category theory

Math/CS Faculty Talk
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Abstraction (level 1)

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So people invented variables to express the above as

> *To solve* $x^3 + cx = d$*, find* u, v *such that* $u - v = d$ *and* $uv = (c/3)^3$*. Then* $x = \sqrt[3]{u} - \sqrt[3]{v}$*.*

(From “How to read historical mathematics”. pg. 1).

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Definition

A real vector space consists of a set of vectors V, with an operation which takes any two vectors $v, w \in V$ and produces a vector $v + w \in V$, and given any real number α produces a vector $\alpha \cdot v \in V$. (These operations then satisfy certain axioms).

Abstraction (level 2 ctd.)

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- "Symmetries" of some object: a group.
- A set with two related operations: a ring.
- Some sort of grid connected by lines: a graph.
- A set with a notion of "distance": a metric space.
- A set with a notion of "open subset": a topological space.

From level 2 to level 3...

People started noticing that there were similarities between these different mathematical structures!

- Many definitions looked similar (for example, the product of two objects).
- Many results were proven in similar ways.
- Again, people could say in vague terms why they were the same, but not precisely.

What is missing is a language *for mathematical structures themselves*.

Morphisms: the common thread

In each case, there was a notion of "morphism" of these structures.

- vector spaces have linear maps,
- groups have group homomorphisms,
- rings have ring homomorphisms,
- topological spaces have continuous maps.

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- such that $f1_X = f = 1_Y f$ and $f(gh) = (fg)h$ (composition is unital and associative).

Categories: examples

- sets and functions,
- vector spaces and linear maps,
- groups and group homomorphisms,
- rings and ring homomorphisms,
- any graph G gives a category, where the objects are the vertices of G, and a morphism from v to v' is a path from v to v' (allowing paths of length 0),
- there is a category where the objects are natural numbers, with a single morphism from n to m if n divides m.

Usefulness?

But there is a big question: what can you **do** with an arbitrary category?

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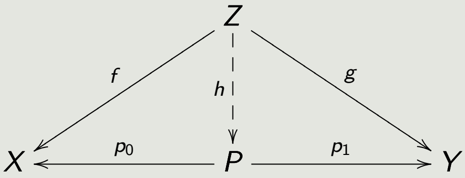
1. there are morphisms $P \xrightarrow{p_0} X$, $P \xrightarrow{p_1} Y$,
2. which are "the best possible such morphisms": for any other object Z with morphisms $Z \xrightarrow{f} X$, $Z \xrightarrow{g} Y$, there is a *unique* morphism $Z \xrightarrow{h} P$ so that $p_0 h = f$, $p_1 h = g$:



Product examples

- in sets: the product of two sets,
- in vector spaces: the product of two vector spaces,
- in groups: the product of two groups,
- in the divisibility category: the product of n and m is their gcd(!),
- products need not exist in a graph viewed as a category.

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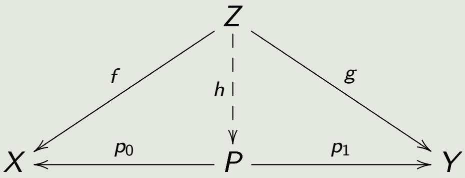

CoProducts

Definition

Let X and Y be objects in a category **C**. Say that an object Q is the **coproduct of** X **and** Y if

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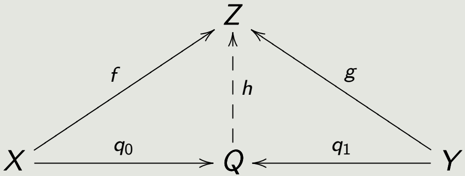
2. which are "the best possible such morphisms": for any other object Z with morphisms $Z \xleftarrow{f} X$, $Z \xleftarrow{g} Y$, there is a *unique* morphism $Z \xleftarrow{h} Q$ so that $hq_0 = f$, $hq_1 = g$:



Coproduct examples

- in sets: the disjoint union of two sets,
- in vector spaces: coproduct is the same as the product(!)
- in abelian groups: coproduct is the same as the product(!)
- in groups: coproduct is the free product,
- in the divisibility category: the coproduct of n and m is their lcm(!),
- coproducts need not exist in a graph viewed as a category.

Equalizers and coequalizers

- An important notion in many contexts is the "solution set": determining a subset on which two things are equal, eg., $\{x \in \mathbb{R} : x^2 = 2x + 3\}$
- Another important notion is "quotient sets": given an equivalence relation $\cong$ on a set X, considering the set $X/\cong$.
- These are also dual: the first is known as the equalizer, the second the coequalizer.

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- That is, any statement we can make or prove about sets, we can make or prove in any topos (assuming we don't use proof by contradiction or the axiom of choice).
- There are toposes where all functions are continuous, or smooth, or computable. A brave new world of mathematics!