Tangent Structure

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Abstract

The manifolds built of a differential restriction category provide an abstract setting for differential geometry. However, these categories of manifolds are, in general, not differential restriction categories. The differential structure manifests itself, instead, through a tangent bundle functor with associated structural maps. The purpose of this paper is to give a precise axiomatization of this "tangent structure" and to begin to explore its consequences.

Tangent structure is of independent interest: it subsumes differential structure and is stable under a wide variety of constructions. While many of the properties of tangent structure are well-known to differential geometers, there do appear to be some basic structural aspects which have not been widely noted.

Contents

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1 Introduction

This paper is a continuation of the categorical exploration of differential structure, as described in [Blute et. al. 2006], [Blute et. al. 2008] and [Cockett et. al. 2011], into the realm of differential geometry and smooth manifolds. The reader is expected to be familiar with the ideas of those papers. In particular, this paper provides an axiomatization of differential structure at the level of smooth manifolds: we call this "tangent structure", and it is based on the properties of the tangent bundle functor.

Beginning with [Blute et. al. 2006] and continuing with [Blute et. al. 2008], the authors sought to algebraically axiomatize the fundamental properties of differentiation. In doing so, the authors bridged the gap between the differential λ-calculus [Erhard and Regnier 2003] which deals with computational resources, and the standard notion of calculus, which deals with smooth functions as covered in any basic undergraduate course. Continuing in [Cockett et. al. 2011], the authors expanded the axiomatization to deal with partial maps. Partial maps are fundamental in both of the above areas: in computation, programs that need not terminate are fundamental, and in calculus, one often works with smooth maps only defined on some open subset.

In that paper, the authors showed that differential structure is stable when one completes the category by formally adding partial maps with special properties. What is not typically true, however, is that differential restriction structure is stable when one adds more objects to the category. The fundamental example of this is when one builds manifolds out of the original category: even if the original category has differential structure, the resulting manifold category need not.

To see why this is, recall that the basic structure of a differential category is an operation which takes a map $f : X \longrightarrow Y$, and produces a map

$$D[f] : X \times X \longrightarrow Y.$$

In the standard example, this takes a smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and gives the map

$$J[f] : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

which, given a point (a, b), takes the Jacobian of f, evaluates it at b, then applies the resulting matrix to the vector a.

If we look at the category of smooth real manifolds, this structure does not exist. There is a differentiation operation, but it does not produce a map of the correct type: given a smooth map $f : M \longrightarrow N$ between smooth manifolds, differentiation produces a smooth map $T(f) : TM \longrightarrow TN$, where T is the tangent bundle functor. Thus, we should not expect the manifold completion of a differential restriction category to again be a differential restriction category. Instead, we expect that the resulting category have a tangent bundle functor T which enjoys certain properties.

To determine what these properties should be, we look at each of the differential axioms, and determine what properties they should give. Some are straightforward: for example, the differential axioms

$$\langle a + b, c\rangle D[f] = \langle a, c\rangle D[f] + \langle b, c\rangle D[f] \text{ and } \langle 0, c\rangle D[f] = 0$$

show that T is an additive bundle. Others, however, are less obvious. For example, the axiom which tells us that differentiation is linear (labelled D6 in [Blute et. al. 2008]) produces a "vertical lift" natural transformation from T to T^2.

The properties that the axioms of differentiation imply for a tangent bundle functor provides our axiomatization of "tangent structure" which, by itself, can stand alone. This is clearly of significant interest as it captures fundemantal properties which underlie differential geometry. However, this axiomatization of the tangent bundle functor appears, as far as we can tell, nowhere in the literature!

The closest we can find is the book [Kólăr et. al. 1993]. It describes each of the natural transformations which appear in our definition, however, it never packages this structure together, nor is the explicit connection with the properties of differentiation made. Of course, when one does package the structure together in this manner, other structural properties emerge. For example, having tangent structure implies the tangent bundle functor is a monad: this basic fact we can find nowhere in the differential geometry literature[1]. Both the Kleisli and the Eilenberg-Moore categories of this monad appear to be of some interest.

An overview of the paper is as follows. In the first section, we give our abstract definition of tangent structure, work out some simple properties, and discuss examples. In section 3, we recall the notion of a differential restriction category, and show that every differential restriction structure is equivalent to a "trivial" tangent structure. In section 4, we discuss Grandis' manifold construction, and show that the manifold completion of a category with tangent structure itself has tangent structure. Finally, in section 5, we generalize some results and definitions of differential geometry to our setting and include the discussion of T as a monad.

2 Tangent Structure

We begin by defining the central notion of the paper, tangent structure; in later sections we will explicitly give the connection to differential (restriction) categories.

2.1 Additive bundles

The main goal of the paper is to abstractly formalize the category of smooth manifolds with its tangent bundle functor. We begin by defining the notion of an additive bundle over an object.

Definition 2.1 *If A is an object in a restriction category $\mathbb{X}$ then an* ***additive bundle over*** *A consists of the following:*

- *A total map $X \xrightarrow{p} A$ such that the restriction pullback of n copies of $X \xrightarrow{p} A$ exists; denote these by X_n, with structure maps $p_i : X_n \longrightarrow X$;*
- *Total maps $+ : X_2 \longrightarrow X$ and $0 : M \longrightarrow X$, with $+p = p_1p = p_2p$ and $0p = 1$ such that:*

[1] The fact that there is such a monad for a cartesian differential category was independently discovered in [Manzyuk 2012]; there, the author also proves the monad has a commutative strength and a distribution over itself.

– *addition is associative, commutative, and unital; that is, each of the following diagrams commute:*

$$\begin{tikzcd} X_3 \arrow[r, "{\langle\langle p_1,p_2\rangle+,p_3\rangle}"] \arrow[d, "{\langle p_1,\langle p_2,p_3\rangle+\rangle}"'] & X_2 \arrow[d, "+"] \\ X_2 \arrow[r, "+"'] & X \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[dr, "+"] \arrow[d, "{\langle p_2,p_1\rangle}"'] & \\ X_2 \arrow[r, "+"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, "1"] \arrow[d, "{\langle p0,1\rangle}"'] & \\ X_2 \arrow[r, "+"'] & X \end{tikzcd}$$

When restricted to the total maps of $\mathbb{X}$, this is the same as asking for a commutative monoid in the slice category $\text{Total}(\mathbb{X})/A$. There appears to be no notion of slice category for restriction categories which corresponds to the requirement that the pullbacks be restriction pullbacks, while giving the notion of morphism that we wish:

Definition 2.2 *Suppose that $p : X \longrightarrow A$ and $q : Y \longrightarrow B$ are additive bundles. An **additive bundle morphism** consists of a pair of maps $f : X \longrightarrow Y$, $g : A \longrightarrow B$ so that the following diagrams commute:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "{\langle p_1 f,p_2 f\rangle}"] \arrow[d, "+"'] & Y_2 \arrow[d, "+"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "0"'] & B \arrow[d, "0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The first diagram says that the pair is a map in the arrow category; the second that the map preserves addition, and the last that it preserves zeroes. Note that even though these maps may be partial, we still ask that the diagrams commute on the nose (rather than with an inequality).

Proposition 2.3 *If $\mathbb{X}$ is a restriction category, then with the obvious composition, restriction, and identities, additive bundles in $\mathbb{X}$ and their morphisms form a restriction category. If $\mathbb{X}$ has joins, so does this category.*

PROOF: For composites, we define $(f_1, g_1) \circ (f_2, g_2) := (f_1 f_2, g_1 g_2)$. For such a map, the first and third diagrams for an additive bundle morphism obviously commute, while the second diagram commutes since

$$\begin{aligned} & \langle p_1 f_1 f_2, p_2 f_1 f_2\rangle + \\ = \;& \langle p_1 f_1, p_2 f_2 \langle p_1 f_2, p_2 f_2\rangle + \;\; \text{(by the universal property of } \langle , \rangle) \\ = \;& \langle p_1 f_1, p_2 f_2\rangle + f_2 \;\; \text{(since } f_2 \text{ is an additive bundle morphism)} \\ = \;& f_1 f_2 + \;\; \text{(since } f_1 \text{ is an additive bundle morphism)} \end{aligned}$$

as required.

For restrictions, we define $\overline{(f, g)} := (\overline{f}\,, \overline{g}\,)$. This satisfies the first diagram:

$$p\overline{g} \; = \overline{pg}\, p = \overline{fq}\, p = \overline{f}\, p$$

since q is total. For the second diagram,

$$\begin{aligned} +\overline{f} \; &= \overline{+f} \; + \\ &= \; \overline{\langle p_1\overline{f}\,, p_2\overline{f}\,\rangle} \; + \\ &= \; \overline{p_1\overline{f}}\;\overline{p_2\overline{f}} \; + \;\; \text{(since + is total)} \\ &= \; \overline{\overline{p_1 f}\, p_1, \overline{p_2 f}\, p_2} \; + \\ &= \; \langle p_1\overline{f}\,, p_2\overline{f}\,\rangle + \end{aligned}$$

as required. For the last diagram,

$$0\overline{f} = \overline{0f}\,0 = \overline{\overline{g}\,0}\,0 = \overline{g}\,0$$

since 0 is total.

If $\mathbb{X}$ has joins, we define $\bigvee_i (f_i, g_i) := (\bigvee_i f_i, \bigvee_i g_i)$. The first and third diagrams commute since joins preserve composition. The second diagram requires slightly more care. We first show that for any map (f, g) from $p : X \longrightarrow A$ to $q : Y \longrightarrow B$, $\overline{p_1 f} = \overline{p_2 f}$:

$$\begin{aligned}
\overline{p_1 f} &= \overline{p_1 f p} \text{ (since } p \text{ is total)} \\
&= \overline{p_1 p g} \text{ (by the first diagram for } (f, g)) \\
&= \overline{p_2 p g} \\
&= \overline{p_2 f p} \text{ (by the first diagram for } ((f, g)) \\
&= \overline{p_2 f} \text{ (since } p \text{ is total)}
\end{aligned}$$

Now, we need to show that the pair $(\bigvee_i f_i, \bigvee_j f_j)$ satisfies the second diagram to be an additive bundle morphism. Consider:

$$\begin{aligned}
\langle p_1 \bigvee_i f_i, p_2 \bigvee_j f_j \rangle + &= \bigvee_{i,j} \langle p_1 f_i, p_2 f_j \rangle + \\
&= \bigvee_{i,j} \langle p_1 f_i, \overline{p_1 f_i}\, p_2 f_j \rangle + \\
&= \bigvee_{i,j} \langle p_1 f_i, \overline{p_2 f_i}\, p_2 f_j \rangle + \text{ (by the result above)} \\
&= \bigvee_{i,j} \langle p_1 f_i, \overline{p_2 f_j}\, p_2 f_i \rangle + \text{ (since } f_i \smile f_j) \\
&= \bigvee_{i,j} \overline{p_2 f_j}\, \langle p_1 f_i, p_2 f_i \rangle + \\
&= \bigvee_i \langle p_1 f_i, p_2 f_i \rangle + \\
&= \bigvee_i + f_i = + \bigvee_i f_i
\end{aligned}$$

as required. $\square$

2.2 Definition of tangent structure

With additive bundles defined, we can turn to our definition of tangent structure.

Definition 2.4 *A* ***tangent structure*** *for a cartesian restriction category* $\mathbb{X}$ *consists of:*

- *(**tangent functor**) a restriction-preserving functor* $\mathbb{X} \xrightarrow{T} \mathbb{X}$*;*
- *(**tangent bundle is additive**) for each* $M \in \mathbb{X}$*,* TM *has the structure of an additive bundle over* M*; so we have maps* $p_M : TM \longrightarrow M$*, pullbacks* $T_n(M)$*, and maps* $+_M : T_2 M \longrightarrow TM$*,* $0_M : M \longrightarrow TM$*; we also ask that for each* $f : M \longrightarrow N$*, the pair* (Tf, f) *is an additive bundle morphism;*

- ***(preservation of limits)*** *for each $n, k \in \mathbb{N}$, T^n preserves products and the pullbacks of any k copies of $p_M : TM \longrightarrow M$;*

- ***(vertical lift)*** *there is a total natural transformation $T \xrightarrow{\ell} T^2$ such that for each M, the pair $(\ell_M, 0_M)$ is an additive bundle morphism from $(p : TM \longrightarrow M)$ to $(Tp : T^2M \longrightarrow TM)$;*

- ***(canonical flip)*** *there is a total natural transformation $T^2 \xrightarrow{c} T^2$ such that for each M, the pair $(c_M, 1)$ is an additive bundle morphism from $(Tp : T^2M \longrightarrow TM)$ to $(p_T : T^2M \longrightarrow TM)$.*

For more about the vertical lift and its relationship to the vertical lift in differential geometry, see the discussion before Proposition 5.3. The fact that $Tp : T^2M \longrightarrow TM$ is an additive bundle follows from the requirement that T preserves the pullbacks defining the T_n's.

Proposition 2.5 *If $(\mathbb{X}, T)$ is tangent structure, then:*

(i) $\overline{Tf} = \overline{pf}$.

(ii) *If $\mathbb{X}$ has joins, then T preserves them.*

(iii) *Each T_n is a functor, with $T_n(f) := \langle p_i f \rangle_{i \leq n}$.*

(iv) *The maps p_M, $+_M$, and 0_M are natural.*

(v) *The additive bundle objects $(T(p), T(+), T(0))$ and $(p_T, +_T, 0_T)$ are isomorphic.*

(vi) $cp_T = T(p)$, *and* $cT(p) = p_T$.

(vii) $\ell T(p) = p0 = \ell p_T$.

(viii) *For any point $x : 1 \longrightarrow M$, $T(x) = x0_M$.*

PROOF:

(i) since p is total, $\overline{T(f)} = \overline{T(f)p} = \overline{pf}$ by naturality of p.

(ii) Consider:

$$\begin{aligned}
\bigvee_{i\in I} T(f_i) &= \bigvee_{i\in I} T\left(\overline{f_i}\bigvee_{j\in I} f_j\right) \quad (\text{since } f_i \leq \bigvee j \in I)\\
&= \bigvee_{i\in I} \overline{T(f_i)}\,T\left(\bigvee_{j\in I} f_j\right) \quad (\text{since } T \text{ is a restriction functor})\\
&= \bigvee_{i\in I} \overline{pf_i}\,T\left(\bigvee_{j\in I} f_j\right) \quad (\text{by (i)})\\
&= \overline{\bigvee_{i\in I} pf_i}\,T\left(\bigvee_{j\in I} f_j\right)\\
&= \overline{p\bigvee_{i\in I} f_i}\,T\left(\bigvee_{j\in I} f_j\right)\\
&= \overline{T\left(\bigvee_{i\in I} f_i\right)}\,T\left(\bigvee_{j\in I} f_j\right) \quad (\text{by (i)})\\
&= T\left(\bigvee_{i\in I} f_i\right)
\end{aligned}$$

as required.

(iii) See proposition 4.5.

(iv) One of the axioms asks that for each $f : M \longrightarrow N$, (Tf, f) be an additive bundle morphism. That is, the following diagrams commute:

$$\begin{tikzcd}
TM \arrow[r, "Tf"] \arrow[d, "p_M"'] & TN \arrow[d, "p_N"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
T_2M \arrow[r, "\langle p_1Tf, p_2Tf\rangle"] \arrow[d, "+_M"'] & T_2N \arrow[d, "+_N"] \\
TM \arrow[r, "Tf"'] & TN
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "0_M"'] & N \arrow[d, "0_N"] \\
TM \arrow[r, "Tf"'] & TN
\end{tikzcd}$$

in other words, p, $+$, and 0 are natural. (Note then that asking that these maps be natural is actually equivalent to (Tf, f) being an additive bundle morphism).

(v) Since $(c,1)(c,1) = (1,1)$, $(c,1)$ is an additive bundle isomorphism between these objects.

(vi) Since $(c,1)$ is an additive bundle morphism from $(Tp : T^2M \longrightarrow TM)$ to $(p_T : T^2M \longrightarrow TM)$, the first diagram for additive bundle morphisms says $cp_T = T(p)$. But then since $c^2 = 1$, we also have $cT(p) = p_T$.

(vii) $\ell T(p) = p0$ since $(\ell, 0)$ is an additive bundle morphism from $(p : TM \longrightarrow M)$ to $(Tp : T^2M \longrightarrow TM)$. But by the previous result, $T(p) = cp_T$, so $\ell cp_T = p0$, and hence $\ell p_T = p0$ since $\ell c = \ell$.

(viii) By naturality of 0, $x0_M = 0_1 T(x) = T(x)$.

$\square$

Note that even though T has a natural transformation $\ell : T \longrightarrow T^2$, and a natural transformation $p : T \longrightarrow I$, (T, p, ℓ) is not a comonad: part (vii) of the previous result explicitly tells us that neither counit axiom holds. Surprisingly, however, we shall see in section 5.2 that there is a multiplication $\mu : T^2 \longrightarrow T$ which makes $(T, 0, \mu)$ into a monad.

Just as for differential categories, tangent categories allow for partial differentiation. Let s be the canonical map from $T(A \times B) \longrightarrow TA \times TB$; since T is cartesian, this map has an inverse, and we can define:

Definition 2.6 *Suppose $(\mathbb{X}, T)$ is tangent structure, and f is a map from a product: $f : A \times B \longrightarrow C$. We define the partial derivative of f for A by*

$$T_A(f) := TA \times B \xrightarrow{1 \times 0} TA \times TB \xrightarrow{s^{-1}} T(A \times B) \xrightarrow{Tf} TC$$

and f for B by

$$T_B(f) := A \times TB \xrightarrow{0 \times 1} TA \times TB \xrightarrow{s^{-1}} T(A \times B) \xrightarrow{Tf} TC.$$

Just as for cartesian differential categories, we can recover Tf from its partial derivatives:

Proposition 2.7 *If f is as above, then*

$$Tf = \langle s(1 \times p)T_A f, s(p \times 1)T_B f \rangle +$$

Proof: First, note the pairing map into the pullback is well-defined, as the two maps are equal when post-composed by p:

$$\begin{aligned} & s^{-1}(1 \times p)T_A f p \\ = \;& s^{-1}(1 \times p)(1 \times 0)sT(f)p \\ = \;& s^{-1}(1 \times p0)spf \text{ (by naturality of p)} \\ = \;& s^{-1}(1 \times p0)(p \times p)f \\ = \;& s^{-1}(p \times p)f \text{ (since 0p = 1)} \end{aligned}$$

and similarly when the other maps is post-composed by p.

To show that Tf can be recovered as described, consider:

$$\begin{aligned} & \langle s^{-1}(1 \times p)T_A f, s^{-1}(p \times 1)T_B f \rangle + \\ = \;& \langle s^{-1}(1 \times p0)Tf, s^{-1}(p0 \times 1)Tf \rangle + \text{ (as above)} \\ = \;& \langle s^{-1}(1 \times p0), s^{-1}(p0 \times 1) \rangle \langle p_1 Tf, p_2 Tf \rangle + \\ = \;& \langle s^{-1}(1 \times p0), s^{-1}(p0 \times 1) \rangle + T(f) \text{ (naturality of +)} \\ = \;& T(f) \text{ (since addition is unital)} \end{aligned}$$

as required. $\square$

We shall see more consequences of the axioms later; for now we discuss examples of tangent structure.

2.3 Examples

The category of smooth finite-dimensional manifolds and smooth maps, equipped with the tangent bundle, is the canonical example. This can be verified directly, but we will give a proof via differential restriction categories in section 3.

The results of section 3 will also show that the category of convenient manifolds and smooth maps, equipped with the "kinematic tangent bundle" [Kriegl and Michor 1997], is tangent structure. On the other hand, the "operational tangent bundle" does not give tangent structure, as it does not preserve products (for more on these tangent bundles, see section 4.3).

An important source of examples comes from synthetic differential geometry:

Proposition 2.8 *The infinitesesimally linear objects ([Kock 2006] pg. 20 definition 6.3) in a model of synthetic differential geometry have tangent structure, with $T^M := M^D$.*

PROOF: That M is infinitesesimally linear says precisely that the pullback of p over itself is given by $M^{D(2)}$, so we define $T_2M = M^{D(2)}$, and similarly $T_nM = M^{D(n)}$. Then the various bits of structure are given by applying the functor $M^{(-)}$ to particular maps between these infinitesimal objects:

- addition is described on pages 24-25 of [Kock 2006]: it is induced by the diagonalization map $\Delta : D \longrightarrow D_2$, while the 0 map is induced by the unique map $D \longrightarrow 1$;
- the map $D^2 \longrightarrow D$ which induces vertical lift is given by mapping (x_1, x_2) to $(x_1 \cdot x_2)$;
- the canonical flip $c : T^2M \longrightarrow T^2M$ is described in exercise 7.1 on page 27: it is induced by the "twist" map $t : D^2 \longrightarrow D^2$.

The coherence axioms are then straightforward to check. For example, the fact that c is an additive bundle morphism follows since the diagram

$$\begin{tikzcd}
D \times D \arrow[r, "\Delta\times 1"] \arrow[d, "t"'] & D(2) \times D \arrow[d, "t"] \\
D \times D \arrow[r, "1\times\Delta"'] & D \times D(2)
\end{tikzcd}$$

commutes. $\square$

Note we only need that the objects be *infinitesimally* linear, not microlinear, as we only need that the T_n's are the pullbacks of the T's (microlinearity asks about more general diagrams involving infinitesimal objects).

One important non-example of tangent structure is the pair $(\mathbb{X}, T_n)$, when $(\mathbb{X}, T)$ is tangent structure. These have an obvious projection $p_1p : T_n \longrightarrow I$. It is clear that the pullback of m copies of T_n along this projection equals T_{nm}, and there is an addition map

$$T_{2n} \xrightarrow{\langle\langle p_i, p_{i+n}\rangle +\rangle_{i\leq n}} T_n$$

which satisfies the required coherences. One can also define a canonical flip. For $n \geq 2$, however, there is no vertical lift: the obvious extension of the vertical lift for T gives a map from T_{n^2} to $(T_n)^2$. The structure of these functors, and other “Weil functors”, such as T^2, are clearly of great interest but beyond the scope of this paper.

For now, we turn to the promised connection between differential restriction categories and tangent structure.

3 Differential restriction categories and tangent structure

In addition to giving a formal axiomatization of the tangent bundle, one of the purposes of tangent structure is to resolve the following problem. Differential categories were invented to describe categories with a differentation operation. But there was an obvious problem with the formalization: the category of smooth manifolds and smooth maps between them was not an example. More generally, if we begin with a differential restriction category with joins ([Cockett et. al. 2011]), and form its manifold completion ([Grandis 1989]), the result is not a differential restriction category. What one does get, however, is tangent structure.

One can show this directly, but the proof is quite cumbersome, as manifolds are difficult to work with directly. Instead, we break the result into two parts. First, we show that any differential restriction category itself has tangent structure. In fact, having “trivial” tangent structure is equivalent to differential restriction structure. These results are the main focus of this section. In the following section, we show that the manifold completion of any restriction category with tangent structure again has tangent structure. Combining these two results shows that the manifold completion of a differential restriction category has tangent structure, as desired, and in particular, gives a proof that the category of finite dimensional smooth manifolds, or the category of convenient manifolds, has tangent structure.

3.1 Differential restriction categories

We begin by recalling the definition of a differential restriction category from [Cockett et. al. 2011]. Recall that the idea is to axiomatize the formal properties of the Jacobian. The D below is to be thought of as the Jacobian of f, evaluated at the second X, then applied in the direction of the first X.

Definition 3.1 *A* ***differential restriction category*** *is a cartesian left additive restriction category with an operation*

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D[f]]{} Y}$$

(“differentiation”) such that

[DR.1] $D[f+g] = D[f] + D[g]$ *and* $D[0] = 0$*;*

[DR.2] $\langle a+b, c\rangle D[f] = \langle a, b\rangle D[f] + \langle b, c\rangle D[f]$ *and* $\langle 0, a\rangle D[f] = \overline{af}0$*;*

[DR.3] $D[\pi_0] = \pi_0\pi_0$*, and* $D[\pi_1] = \pi_0\pi_1$*;*

[DR.4] $D[\langle f, g\rangle] = \langle D[f], D[g]\rangle$*;*

[DR.5] $D[fg] = \langle D[f], \pi_1 f\rangle D[g]$;

[DR.6] $\langle\langle a, 0\rangle, \langle c, d\rangle\rangle D[D[f]] = \overline{c}\langle a, d\rangle D[f]$;

[DR.7] $\langle\langle 0, b\rangle, \langle c, d\rangle\rangle D[D[f]] = \langle\langle 0, c\rangle, \langle b, d\rangle\rangle D[D[f]]$;

[DR.8] $D[\overline{f}] = (1 \times \overline{f})\pi_0 = \overline{\pi_1 f}\,\pi_0$;

[DR.9] $\overline{D[f]} = 1 \times \overline{f} = \overline{\pi_1 f}$.

In fact, however, there is an alternative version of these axioms, which is more directly relevant for tangent structure.

Proposition 3.2 *The axioms for a differential restriction category are equivalently given by replacing* [**DR.6**] *and* [**DR.7**] *with the following axioms:*

- **[DR.6′]** $\langle\langle a, 0\rangle, \langle 0, d\rangle\rangle D[D[f]] = \langle a, d\rangle D[f]$;
- **[DR.7′]** $\langle\langle a, b\rangle, \langle c, d\rangle\rangle D[D[f]] = \langle\langle a, b\rangle, \langle c, d\rangle\rangle D[D[f]]$.

PROOF: Assume that D satisfies the usual set of axioms. Clearly, it then satisfies **[DR.6′]**, by setting $c = 0$. For **[DR.7′]**, consider:

$$\begin{array}{rl}
& \langle\langle a, b\rangle, \langle c, d\rangle\rangle D^2 f \\
= & \langle\langle a, 0\rangle + \langle 0, b\rangle, \langle c, d\rangle\rangle D^2 f \\
= & \langle\langle a, 0\rangle, \langle c, d\rangle\rangle D^2 f + \langle\langle 0, b\rangle, \langle c, d\rangle\rangle D^2 f \text{ by } \textbf{[DR.2]}, \\
= & \overline{c}\,\langle a, d\rangle D^2 f + \overline{b}\,\langle\langle 0, c\rangle, \langle b, d\rangle\rangle D^2 f \text{ by } \textbf{[DR.6]} \text{ and } \textbf{[DR.7]}, \\
= & \overline{b}\,\langle a, d\rangle D^2 f + \overline{c}\,\langle\langle 0, c\rangle, \langle b, d\rangle\rangle D^2 f \text{ by } \textbf{[DR.6]} \text{ and } \textbf{[DR.7]}, \\
= & \langle\langle a, 0\rangle, \langle b, d\rangle D^2 f + \langle\langle 0, c\rangle, \langle b, d\rangle\rangle D^2 f \text{ by } \textbf{[DR.6]} \text{ again}, \\
= & \langle\langle a, c\rangle, \langle b, d\rangle\rangle D^2 f \text{ by } \textbf{[DR.2]}.
\end{array}$$

as required.

Now assume that D satisfies the alternate set of axioms, with **[DR.6]** and **[DR.7]** replaced with **[DR.6′]** and **[DR.7′]**. Clearly, it then satisfies **[DR.7]**, by setting $a = 0$. To show that it satisfies **[DR.6]**, we begin with a short calculation:

$$\begin{array}{rcl}
\overline{\langle a, d\rangle Df} & = & \overline{\langle a, d\rangle \overline{Df}} \\
& = & \overline{\langle a, d\rangle \overline{\pi_1 f}} \\
& = & \overline{\langle a, d\rangle \pi_1 f} \\
& = & \overline{\overline{a}\, df} = \overline{a}\,\overline{df}
\end{array}$$

Then to show **[DR.6]**, consider:

$$
\begin{array}{rl}
& \langle\langle a,0\rangle,\langle b,d\rangle\rangle D^2 f \\
= & \langle\langle a,b\rangle,\langle 0,d\rangle\rangle D^2 f \text{ (by } \textbf{[DR.7}'\textbf{]}) \\
= & \langle\langle a,0\rangle,\langle 0,d\rangle\rangle D^2 f + \langle\langle 0,b\rangle,\langle 0,d\rangle\rangle D^2 f \text{ (by } \textbf{[DR.2]}) \\
= & \langle a,d\rangle Df + \langle\langle 0,0\rangle,\langle b,d\rangle\rangle D^2 f \text{ (by } \textbf{[DR.6}'\textbf{]} \text{ and } \textbf{[DR.7}'\textbf{]}) \\
= & \langle a,d\rangle Df + \overline{\langle b,d\rangle Df}\, 0 \text{ (by } \textbf{[DR.2]}) \\
= & \overline{\langle b,d\rangle Df}\, \langle a,d\rangle Df \\
= & \overline{\langle b,d\rangle Df}\, \overline{\langle a,d\rangle Df}\, \langle a,d\rangle Df \\
= & \overline{b}\, \overline{df}\, \overline{a}\, \overline{df}\, \langle a,d\rangle Df \text{ (by the calculation above)} \\
= & \overline{b}\, \overline{\langle a,d\rangle Df}\, \langle a,d\rangle Df \\
= & \overline{b}\, \langle a,d\rangle Df
\end{array}
$$

as required.

□

We recall a number of examples.

Example 3.3 *Any cartesian differential category is a differential restriction category, when equipped with the trivial restriction structure ($\overline{f} = 1$ for all f).*

The standard example is of course:

Example 3.4 *Smooth functions defined on open subsets of $\mathcal{R}^n$.*

From [Cockett et. al. 2011], we also have the following more complicated example:

Proposition 3.5 *If D is a commutative ring, then the restriction category of rational functions over D, RAT_D, is a differential restriction category.*

From [Blute et. al. 2011], we also have:

Example 3.6 *The category of convenient vector spaces and smooth maps between them is a cartesian differential category; smooth maps defined on open subsets gives a differential restriction category.*

In [Cockett and Seely 2011], the authors prove a surprising result: there is a comonad **Faà** on cartesian left additive categories whose coalgebras are cartesian differential categories. In particular, we have:

Example 3.7 *If $\mathbb{X}$ is a cartesian left additive category, **Faà**$(\mathbb{X})$ is a cartesian differential category.*

One can check that for any cartesian left additive category, defining the differential of f to be $\pi_0 f$ satisfies all axioms with the exception of **[DR.2]**. For this, we would need $(a+b)f = af + bf$ and $0f = 0$ for all a, b. Of course, this is true by definition if f is additive. Thus, if all maps in $\mathbb{X}$ are additive (as in the case of the category of commutative monoids or commutative rings), then $D[f] = \pi_0 f$ does define a differential.

Example 3.8 *If $\mathbb{X}$ is an additive cartesian category, then $D[f] = \pi_0 f$ gives $\mathbb{X}$ the structure of a cartesian differential category.*

As a helpful tool for certain calculations, we note the following result about differential restriction categories from [Cockett et. al. 2011]:

Proposition 3.9 *In a differential restriction category:*

(i) $D[\overline{f}g] = (1 \times \overline{f})D[g] = \overline{\pi_1 f}\, D[g]$;

(ii) *If* $f \leq g$ *then* $D[f] \leq D[g]$;

(iii) *If* $f \smile g$ *then* $D[f] \smile D[g]$.

3.2 Differential structure as tangent structure

We now give our first main result of this section.

Proposition 3.10 *Any differential restriction category has a "trivial" tangent structure given by:*

- $TM := M \times M$, $Tf := \langle Df, \pi_1 f \rangle$;
- $p := \pi_1$
- $T_n(M) := M \times M \ldots \times M$ *($n+1$ times);*
- $+\langle a, b, c \rangle := \langle a + b, c \rangle$, $0(a) := \langle 0, a \rangle$;
- $l(\langle a, b \rangle) := \langle \langle a, 0 \rangle, \langle 0, b \rangle \rangle$;
- $c(\langle \langle a, b \rangle, \langle c, d \rangle \rangle) := \langle \langle a, c \rangle, \langle b, d \rangle \rangle$.

PROOF: That T is a functor follows from **[DR.5]** and **[DR.8]**:

$$T(f)T(g) = \langle Df, \pi_1 f \rangle \langle Dg, \pi_1 g \rangle = \langle D(fg), \pi_1 fg \rangle = T(fg) \text{ and } T(1) = \langle D(1), \pi_1 \rangle = \langle \pi_0, \pi_1 \rangle = 1.$$

T preserves restrictions since

$$T(\overline{f}\,) = \langle D(\overline{f}\,, \pi_1 \overline{f}\,\rangle = \langle \overline{\pi_1 f}\, \pi_0, \overline{\pi_1 f}\, \pi_1 \rangle = \overline{\pi_1 f}$$

while

$$\overline{T(f)} = \overline{\langle D(f), \pi_1 f \rangle} = \overline{D(f)}\, \overline{\pi_1 f} = \overline{\pi_1 f}$$

as required.

For the additive bundle structure, it is clear that $T_n(M) := M \times M \ldots \times M$ ($n+1$ times) is the pullback of n copies of $p : TM \longrightarrow M$. It is also clear that $+p = p_1 p = p_2 p$ and $0p = 1$, and the additive is associative, commutative and unital since addition of maps in a left additive category is associative, commutative and unital.

As noted in the notes after the definition of tangent structure, asking that each $T(f) = \langle Df, \pi_1 f \rangle$ is an additive bundle morphism is equivalent to asking that each of p, $+$, and 0 be natural.

That π_1 is natural is immediate:

$$T(f)\pi_1 = \langle Df, \pi_1 f\rangle \pi_1 = \overline{\pi_1 f}\, \pi_1 f = \pi_1 f.$$

$+$ is natural by [**DR.2**]:

$$\begin{array}{rl}
& \langle a, \langle b, c\rangle\rangle (T_2 f)(+_Y) \\
= & \langle \langle a, c\rangle Df, \langle \langle b, c\rangle Df, \pi_1\pi_1 c\rangle\rangle (+_x) \\
= & \langle \langle a, c\rangle Df + \langle b, c\rangle Df, \pi_1\pi_1 c\rangle \\
= & \langle \langle a+b, c\rangle Df, \pi_1\pi_1 c\rangle \text{ by } [\textbf{DR.2}], \\
= & \langle a+b, \pi_1\pi_1 c\rangle (Tf) \\
= & \langle a, \langle b, c\rangle\rangle (+_X)(Tf)
\end{array}$$

as required. The naturality of $0 : X \xrightarrow{\langle 0,1\rangle} X \times X$ similarly follows by the other part of [**DR.2**].

Obviously, T preserves products and the pullbacks defining the T_n's. That it preserves pairings follows from [**DR.3**]:

$$\begin{array}{rcl}
T(\langle f, g\rangle) & = & \langle D(\langle f, g\rangle), \pi_1\langle f, g\rangle\rangle \\
& = & \langle \langle Df, Dg\rangle), \langle \pi_1 f, \pi_1 g\rangle\rangle \text{ by } [\textbf{DR.3}], \\
& = & s\langle \langle Df, \pi_1 f\rangle, \langle Dg, \pi_1 g\rangle \\
& = & s\langle Tf, Tg\rangle
\end{array}$$

(where s is the map that switches the the two interior terms). Similarly, preservation of the projections follows from [**DR.4**].

The rest of the proof will involve calculations on maps whose domains are either T^2 or T_2. To make these calculations easier to follow, we will show that they are true when composing with an arbitrary map into T^2 or T_2, so that rather than dealing with projections of projections, we are dealing with maps in the product spaces. In addition, we will often implicitly use [**DR**.**8**] when we project out of a pairing.

The vertical lift is natural by [**DR.6**] and [**DR.2**]:

$$\begin{array}{rl}
& \langle a, c\rangle (\ell_x)(T^2 f) \\
= & \langle \langle a, 0\rangle, \langle 0, c\rangle\rangle (T^2 f) \\
= & \langle \langle a, 0\rangle, \langle 0, c\rangle\rangle \langle \langle D^2 f, \langle \pi_0\pi_1, \pi_1\pi_1\rangle Df\rangle, \pi_1\langle Df, \pi_1 f\rangle\rangle \\
= & \langle \langle \langle a, c\rangle Df, 0\rangle, \langle 0, cf\rangle\rangle \text{ by } [\textbf{DR.6}] \text{ in the first variable, and } [\textbf{DR.2}] \text{ in the second and third}, \\
= & \langle \langle a, c\rangle Df, cf\rangle \ell_Y \\
= & \langle a, c\rangle \langle Df, \pi_1 f\rangle \ell_Y \\
= & \langle a, c\rangle T(f)(\ell_Y)
\end{array}$$

as required.

The canonical flip is natural by [**DR.7′**]:

$$
\begin{array}{rl}
& \langle\langle a,b\rangle\langle c,d\rangle\rangle(T^2f)(c_Y) \\
= & \langle\langle a,b\rangle\langle c,d\rangle\rangle\langle\langle D^2f,\langle\pi_0\pi_1,\pi_1\pi_1\rangle Df\rangle,\pi_1\langle Df,\pi_1 f\rangle\rangle(c_Y) \\
= & \langle\langle\langle\langle a,c\rangle\langle b,d\rangle\rangle D^2f,\langle b,d\rangle Df\rangle\rangle,\langle\langle c,d\rangle Df,df\rangle\rangle(c_Y) \text{ by } [\mathbf{DR.7'}] \\
= & \langle\langle\langle\langle a,c\rangle\langle b,d\rangle\rangle D^2f,\langle c,d\rangle Df\rangle,\langle\langle b,d\rangle Df\rangle,df\rangle\rangle \\
= & \langle\langle a,c\rangle,\langle b,d\rangle\rangle T^2f \\
= & \langle\langle a,b\rangle\langle c,d\rangle\rangle(c_X)(T^2f)
\end{array}
$$

as required.

To show that these maps are additive bundle morphisms, we need to determine $T(+)$ and $T(0)$. By [**DR.1**] and [**DR.3**], $T(+_X) = T(\langle\pi_0+\pi_1\pi_0,\pi_1\pi_1\rangle)$ is given by

$$\langle\langle\pi_0\pi_0+\pi_0\pi_1\pi_0,\pi_0\pi_1\pi_1\rangle,\langle\pi_1\pi_0+\pi_1\pi_0\pi_1,\pi_1\pi_1\pi_1\rangle\rangle$$

while

$$T(0) = T(\langle 0,1\rangle) = \langle\langle D0,D1\rangle,\pi_1\rangle = \langle\langle 0,1\rangle,\pi_1\rangle$$

Now, the map $\langle p_1c,p_2c\rangle$ sends

$$\langle\langle a,\langle b,c\rangle\rangle,\langle d,\langle e,f\rangle\rangle\rangle \mapsto \langle\langle a,d\rangle,\langle\langle b,e\rangle,\langle c,f\rangle\rangle\rangle$$

We can then show that c preserves addition:

$$
\begin{array}{rl}
& \langle\langle a,\langle b,c\rangle\rangle,\langle d,\langle e,f\rangle\rangle\rangle\langle p_1c,p_2c\rangle(+_{TX}) \\
= & \langle\langle a,d\rangle,\langle\langle b,e\rangle,\langle c,f\rangle\rangle\rangle(+_{TX}) \\
= & \langle\langle a+b,d+e\rangle,\langle c,f\rangle\rangle \\
= & \langle\langle a+b,c\rangle,\langle d+e,f\rangle\rangle(c_X) \\
= & \langle\langle a,\langle b,c\rangle\rangle,\langle d,\langle e,f\rangle\rangle\rangle T(+_X)(c_X)
\end{array}
$$

as required. Preservation of 0 similarly uses the equation $T(0) = \langle\langle 0,1\rangle,\pi_1\rangle$, and the calculations to show l preserves addition are similar. ☐

3.3 Diagonal tangent structure as differential structure

In fact, as the following shows, tangent structure of this form is equivalent to differential restriction structure.

Definition 3.11 *A **diagonal (trivial) tangent structure** on a cartesian left additive restriction category is tangent structure for which $TM = M \times M$, and all natural transformations are given as in Proposition 3.10.*

Theorem 3.12 *Any left additive cartesian restriction category $\mathbb{X}$ with diagonal tangent structure has a differential, given by setting $Df := T(f)\pi_0$ (and so is a differential restriction category).*

PROOF: By naturality of p, we can determine that:

$$T(f) = \langle T(f)\pi_0, T(f)\pi_1\rangle = \langle Df, \pi_1 f\rangle$$

Similarly,

$$T_2(f) = \langle\langle \pi_0, \pi_1\pi_0\rangle D(f), \langle \pi_1 D(f), \pi_1\pi_1 f\rangle$$

and

$$D^2(f) = T(D(f))\pi_0 = T(T(f)\pi_0)\pi_0 = T^2(f)T(\pi_0)\pi_0 = T^2(f)\langle \pi_0\pi_0, \pi_1\pi_0\rangle\pi_0 = T^2(f)\pi_0\pi_0.$$

We begin with [**DR**.**5**]:

$$\begin{array}{rcl} D(fg) & = & T(fg)\pi_0 \\ & = & T(f)T(g)\pi_0 \\ & = & \langle T(f)\pi_0, T(f)\pi_1\rangle D(g) \\ & = & \langle Df, pi_1 f\rangle D(g) \text{ by naturality of } p = pi_1. \end{array}$$

For [**DR**.**8**]:

$$D(\overline{f}\,) = T(\overline{f}\,)\pi_0 = \overline{pi_1 f}\,\pi_0$$

and [**DR**.**9**]:

$$\overline{D(f)} = \overline{T(f)\pi_0} = \overline{T(f)} = \overline{\pi_1 f}$$

Since the functor T is cartesian, with isomorphism $s : T(M \times N) \longrightarrow TM \times TN$ given by

$$s(m_1, n_1, m_2, n_2) = (m_1, m_2, n_1, n_2),$$

we have $T(\langle f, g\rangle) = \langle Tf, Tg\rangle s$, $T(\pi_0) = s\pi_0 = \langle \pi_0\pi_0, \pi_1\pi_0$, and $T(\pi_1) = \langle \pi_1\pi_0, \pi_1\pi_1\rangle$. Then we get [**DR**.**4**]:

$$\begin{array}{rcl} D(\langle f, g\rangle) & = & T(\langle f, g\rangle)\pi_0 \\ & = & \langle Tf, Tg\rangle s\pi_0 \\ & = & \langle Tf, Tg\rangle\langle \pi_0\pi_0, \pi_1\pi_0\rangle \\ & = & \langle T(f)\pi_0, T(g)\pi_0\rangle \\ & = & \langle Df, Dg\rangle \end{array}$$

and for [**DR**.**3**]:

$$D(\pi_0) = T(\pi_0)\pi_0 = \langle \pi_0\pi_0, \pi_1\pi_0\rangle\pi_0 = \pi_0\pi_0,$$
$$D(\pi_0) = T(\pi_1)\pi_0 = \langle \pi_0\pi_1, \pi_1\pi_1\rangle\pi_0 = \pi_0\pi_1,$$
$$D(1) = T(1)\pi_0 = \pi_0$$

For [**DR**.**2**], we have

$$\begin{array}{rcl} \langle a, \langle b, c\rangle\rangle T_2 f+ & = & \langle a, \langle b, c\rangle\rangle(+)T(f) \\ \langle\langle a, c\rangle Df, \langle\langle b, c\rangle Df, cf\rangle\rangle + & = & \langle a+b, c\rangle T(f) \\ \langle\langle a, c\rangle Df + \langle b, c\rangle Df, cf\rangle & = & \langle a+b, c\rangle T(f) \\ \langle a, c\rangle Df + \langle b, c\rangle Df & = & \langle a+b, c\rangle D(f) \text{ appling } \pi_0 \text{ to both sides.} \end{array}$$

and the 0 axiom is similar. For [**DR**.**6**], let a, b, c be maps $Q \xrightarrow{M}$, and $f : M \longrightarrow N$. Then by extension, we know that

$$\begin{aligned}
\langle a, \langle b, c\rangle\rangle l T^2 f &= \langle a, \langle b, c\rangle\rangle T_2(f) l \\
\langle\langle a, 0\rangle, \langle b, c\rangle\rangle T^2 f &= \langle\langle a, c\rangle T f \pi_0, \langle\langle b, c\rangle T f \pi_0, c f\rangle\rangle l \\
\langle\langle a, 0\rangle, \langle b, c\rangle\rangle T^2 f \pi_0 \pi_0 &= \langle\langle a, c\rangle T f \pi_0, \langle\langle b, c\rangle T f \pi_0, c f\rangle\rangle l \pi_0 \pi_0 \\
\langle\langle a, 0\rangle, \langle b, c\rangle\rangle D^2 f &= \langle\langle a, c\rangle T f \pi_0, \langle\langle b, c\rangle T f \pi_0, c f\rangle\rangle \pi_0 \\
\langle\langle a, 0\rangle, \langle b, c\rangle\rangle D^2 f &= \overline{b}\, \langle a, c\rangle D(f)
\end{aligned}$$

as required.
For [**DR**.**7**], by naturality of c, we have

$$\begin{aligned}
\langle\langle a, b\rangle\langle c, d\rangle\rangle c T^2 f &= \langle\langle a, b\rangle\langle c, d\rangle\rangle T^2 f c \\
\langle\langle a, c\rangle\langle b, d\rangle\rangle T^2 f &= \langle\langle a, b\rangle\langle c, d\rangle\rangle T^2 f c \\
\langle\langle a, c\rangle\langle b, d\rangle\rangle D^2 f &= \langle\langle a, b\rangle\langle c, d\rangle\rangle T^2 f \pi_0 \pi_0 \text{ (applying } \pi_0 \text{ to both sides)} \\
\langle\langle a, c\rangle\langle b, d\rangle\rangle D^2 f &= \langle\langle a, b\rangle\langle c, d\rangle\rangle D^2 f
\end{aligned}$$

as required.

For [**DR**.**1**], for $D(0) = 0$, from the preservation of 0, we know that

$$\begin{aligned}
T(0) &= 0c \\
T(\langle 0, 1\rangle) &= \langle\langle 0, 0\rangle, \langle 1, 1\rangle\rangle c \\
\langle T0, T1\rangle s &= \langle\langle 0, 0\rangle, \langle 1, 1\rangle\rangle c \\
\langle T0, T1\rangle &= \langle\langle 0, 0\rangle, \langle 1, 1\rangle\rangle \text{ since } c = s \text{ and they are invertible,} \\
T(0)\pi_0 &= 0 \text{ (applying } \pi_0\pi_0 \text{ to both sides),} \\
D(0) &= 0.
\end{aligned}$$

and $Df + Dg = D(f + g)$ similarly follows from the preservation of addition. □

4 Manifolds

In the previous section, we showed that any differential restriction category has tangent structure. Our goal now is to show that the manifold completion of a category with tangent structure has tangent structure.

4.1 Definition of the manifold completion

We begin by briefly recalling the notion of the manifold completion of a join restriction category, first described in [Grandis 1989].

Definition 4.1 *Let $\mathbb{X}$ be a join restriction category. An* **atlas** *in $\mathbb{X}$ consists of a family of objects $(X_i)_{i \in I}$ of $\mathbb{X}$, together with, for each $i, j \in I$, a map $\phi_{ij} : X_i \longrightarrow X_j$ such that for each $i, j, k \in I$,*

[Atl. 1] $\phi_{ii}\phi_{ij} = \phi_{i,j}$ *(partial charts);*

[Atl. 2] $\phi_{ij}\phi_{jk} \leq \phi_{ik}$ *(cocycle condition);*

[Atl. 3] ϕ_{ij} *is the partial inverse of* ϕ_{ji} *(partial inverse).*

Definition 4.2 *Suppose* (X_i, ϕ_{ij}) *and* (Y_k, ψ_{kh}) *are atlases in* $\mathbb{X}$. *An* **atlas map** $A : (X_i, \phi_{ij}) \longrightarrow (Y_k, \psi_{kh})$ *is a family of maps*

$$X_i \xrightarrow{A_{ik}} Y_k$$

such that

[AtlM. 1] $\phi_{ii}A_{ik} = A_{ik}$*;*

[AtlM. 2] $\phi_{ij}A_{jk} \leq A_{ik}$*,*

[AtlM. 3] $A_{i,k}\psi_{kh} = \overline{A_{ik}}\, A_{i,h}$.

Morphisms of atlases are composed by matrix composition. Given atlas maps

$$U \xrightarrow{A} V \xrightarrow{B} W$$

we define $(AB)_{im} = \bigvee_h A_{ih}B_{hm}$. The identity map for an atlas is the atlas itself. There is a restriction given by

$$\overline{A}_{ij} = \left(\bigvee_h \overline{A_{ih}} \right) \phi_{ij}.$$

Theorem 4.3 (Grandis) *If* $\mathbb{X}$ *is a join restriction category, then* $\mathbf{Mf}(\mathbb{X})$*, with objects atlases, morphisms atlas maps, and composition, identities, and restriction as described above, is a join restriction category.*

The following is easily checked:

Proposition 4.4 **Mf** *is an endofunctor on join restriction categories and join preserving restriction functors, where*

$$\mathbf{Mf}(F)(U_i, \phi_{ij}) := (F(U_i), F(\phi_{ij})),$$

and

$$\mathbf{Mf}(F)(A_{ik}) = (F(A_{ik})).$$

Moreover, if $F \xrightarrow{\alpha} G$ *is natural, then we get a natural transformation from* $\mathbf{Mf}(F)$ *to* $\mathbf{Mf}(G)$ *by*

$$(F(U_i), F(\phi_{ij})) \xrightarrow{F(\phi_{ij})\alpha_j = \alpha_i G(\phi_{ij})} (G(U_i), G(\phi_{ij}))$$

so that **Mf** *is a 2-functor. If* α *is total, then* $\mathbf{Mf}(\alpha)$ *is as well.*

Thus, since we have a 2-functor, applying **Mf** to $T : \mathbb{X} \longrightarrow \mathbb{X}$ almost immediately gives tangent structure. The only thing to check is the preservation of the pullbacks, which we do in the next section.

4.2 Functorial Restriction Pullbacks

The pullbacks of n copies of $p : TM \longrightarrow M$ are a crucial part of tangent structure. Here, we briefly record some useful results regarding these restriction pullbacks.

Proposition 4.5 *Suppose that we have functors $F, G, H : \mathbb{X} \longrightarrow \mathbb{Y}$ between restriction categories, natural transformations $\alpha : F \longrightarrow H, \beta : G \longrightarrow H$, and for each $X \in \mathbb{X}$, there is an object $PX \in \mathbb{Y}$ and maps $l_X : PX \longrightarrow FX, r_X : PX \longrightarrow GX$ so that*

$$\begin{tikzcd} PX \arrow[r, "r_X"] \arrow[d, "l_X"'] & GX \arrow[d, "\beta_X"] \\ FX \arrow[r, "\alpha_X"'] & HX \end{tikzcd}$$

is a restriction pullback. Then:

- *P is a functor, with $P(f) := [l_X F(f), r_X G(f)]$;*
- *if F and G preserve restrictions or joins, then so does P;*
- *if both α and β are total, then l and r are natural.*

Proof:

- First, we need to check $P(f)$ is well-defined; that is, we need $l_X F(f)\alpha_Y \smile r_X G(f)\beta_Y$. In fact, they are equal:

  $$l_X F(f)\alpha_Y = l_X \alpha_X f = r_x \beta_X f = r_X G(f)\beta_Y.$$

  Then P is clearly functorial, as

  $$\begin{array}{rcl} P(f)P(g) & = & [l_X F(f), r_X G(f)][l_Y F(g), r_Y G(g)] \\ & = & [l_X F(f)F(g), r_X G(f)G(f)] \\ & = & [l_X F(fg), r_X G(fg)] \\ & = & P(fg) \end{array}$$

  and

  $$P(1) = [l_X F(1), r_X F(1)] = [l_X, r_X] = 1.$$

- If F and G preserve restrictions, then

  $$\begin{array}{rcl} P(\overline{f}\,) & = & [l_X F(\overline{f}\,), r_X G(\overline{f}\,)] \\ & = & [l_X \overline{F(f)}\,, r_X \overline{G(f)}\,] \\ & = & \overline{l_X F(f)}\,\overline{r_X G(f)}\,[l_X, r_X] \\ & = & \overline{l_X F(f)}\,\overline{r_X G(f)} \\ & = & \overline{[l_X F(f), r_X G(f)]} \\ & = & \overline{P(f)} \end{array}$$

  so P does as well. If F and G preserves joins, then so does P, as pullback maps preserve joins.

- If l_X and r_X are total, we first show that $\overline{r_X F(f)} = \overline{l_X G(f)}$:

$$\begin{array}{rcll}
\overline{r_X G(f)} & = & \overline{r_X G(f)\beta_Y} & \text{since } \beta \text{ is total,} \\
& = & \overline{r_X \beta_X H(f)} & \text{by naturality of } \beta, \\
& = & \overline{l_X \alpha_X H(f)} & \\
& = & \overline{r_X F(f)\alpha_Y} & \text{by naturality of } \alpha, \\
& = & \overline{r_X F(f)} & \text{since } \alpha \text{ is total.}
\end{array}$$

Then l is natural since

$$P(f)l_Y = [l_X F(f), r_X G(f)]l_Y = \overline{r_X G(f)}\, l_X F(f) = l_X F(f),$$

and similarly for r.

$\square$

Proposition 4.6 *Suppose we have all the conditions of the previous proposition, and $\mathbb{X}$ has joins. Then for any object $M = (U_i, \phi_{ij})$ in $\mathbf{Mf}(\mathbb{X})$, the pullback also exists.*

Proof: The diagram commutes since **Mf** is a functor. Thus, it suffices to show the universal property. Suppose we have

$$\begin{tikzcd}
(V_m,\psi_{mn}) \arrow[drr, "B"] \arrow[ddr, "A"'] & & \\
& (PU_i,P\phi_{ij}) \arrow[r, "\mathbf{Mf}(r)"'] \arrow[d, "\mathbf{Mf}(l)"] & (GU_i,G\phi_{ij}) \arrow[d, "\mathbf{Mf}(\alpha)"'] \\
& (FU_i,F\phi_{ij}) \arrow[r, "\mathbf{Mf}(\alpha)"] & HU_i, H\phi_{ij}
\end{tikzcd}$$

so that $A\mathbf{Mf}(\alpha) \smile B\mathbf{Mf}(\beta)$. Now, compatability implies pointwise compatibility, so we have

$$A\mathbf{Mf}(\alpha)_{mk} \smile B\mathbf{Mf}(\beta)_{mk}$$

for each m and k. By the lemma about $\mathbf{Mf}(\alpha)$, this gives

$$A_{mk}\alpha_k \smile B_{mk}\alpha_k.$$

Then by the universal property of the pullback in $\mathbb{Y}$, we know there exists a map $V_m \xrightarrow{[A_{mk},B_{mk}]} PU_k$. We claim these maps together form a manifold map. For ATM2,

$$\psi_{mn}[A_{nk}, B_{nk}] = [\psi_{mn}A_{nk}, \psi_{mn}B_{nk}] \leq [A_{mk}, B_{mk}],$$

and ATM1 is similar. For ATM3,

$$\begin{array}{rcll}
[A_{mk}, B_{mk}]P\phi_{kj} & = & [A_{mk}, B_{mk}][l_k F(\phi_{kj}, r_k G(\phi_{kh})] & \text{by definition of } P, \\
& = & [A_{mk}F(\phi_{kj}), B_{mk}G(\phi_{kj})] & \\
& = & [\overline{A_{mk}}\, A_{mj}, \overline{B_{mk}}\, B_{mj}] & \text{by ATM3 for A and B,} \\
& = & \overline{A_{mk}}\, \overline{B_{mk}}\, [A_{mj}, B_{mj}] & \\
& = & \overline{[A_{mk}, B_{mk}]}\, [A_{mj}, B_{mj}] &
\end{array}$$

so it is a manifold map.

For its restriction, recall that compatibility of $A\mathbf{Mf}(\alpha)$ and $B\mathbf{Mf}(\beta)$ also implies that we have

$$\bigvee_i \overline{(A\mathbf{Mf}(\alpha))_{mi}}\,\overline{(B\mathbf{Mf}(\beta))_{mj}} = \bigvee_i \overline{(B\mathbf{Mf}(\beta))_{mi}}\,\overline{(A\mathbf{Mf}(\alpha))_{mj}}$$

that is,

$$\bigvee_i \overline{A_{mi}\alpha_i}\,\overline{B_{mj}\alpha_j} = \bigvee_i \overline{B_{mi}\beta_i}\,\overline{A_{mj}\beta_j}\,,$$

but since α and β are total, this reduces to

$$\bigvee_i A_{mi}B_{mj} = \bigvee_i B_{mi}A_{mj}.$$

Now, we want to show that $\overline{[A,B]}_{mn} = (\overline{A}\,\overline{B}\,)_{mn}$. Indeed, consider

$$\begin{aligned}
(\overline{A}\,\overline{B}\,)_{mn} &= \bigvee_i \overline{A_{mi}} \bigvee_j \overline{B_{mj}}\,\psi_{mn} \text{ by definition of manifold map restriction,}\\
&= \bigvee_i \overline{A_{mi}} \bigvee_j \overline{B_{mj}}\,\overline{A_{mi}}\,\psi_{mn}\\
&= \bigvee_i \overline{A_{mi}} \bigvee_j \overline{A_{mj}}\,\overline{B_{mi}}\,\psi_{mn} \text{ by the above calculation,}\\
&= \bigvee_i \overline{A_{mi}}\,\overline{B_{mi}}\,\psi_{mn}\\
&= \bigvee_i \overline{[A_{mi},B_{mi}]}\,\psi_{mn}\\
&= \overline{[A,B]}_{mn}
\end{aligned}$$

as required.

Finally, suppose that we have some manifold map $(V_m,\psi_{mn}) \xrightarrow{C} (PU_i, P\phi_{ij})$ such that

$$C\mathbf{Mf}(l) \le A \text{ and } C\mathbf{Mf}(r) \le B.$$

This gives, for each m and i,

$$C_{mi}l_i \le A_{mi} \text{ and } C_{mi}r_i \le B_{mi},$$

so that by the universal property of the pullback in $\mathbb{Y}$, we have

$$C_{mi} \le [A_{mi}, B_{mi}]$$

so that $C \le [A,B]$, as required.

□

Thus, we have:

Corollary 4.7 *The manifold completion of join restriction category with tangent structure has tangent structure.*

In particular:

Corollary 4.8 *If $\mathbb{X}$ is a join differential restriction structure, then* $\mathbf{Mf}(\mathbb{X})$ *has tangent structure.*

We note that if $\mathbb{X}$ does not have joins but has differential structure then one can always join complete $\mathbb{X}$ without destroying the differential structure ([Cockett et. al. 2011], section 5).

4.3 Comparison with other tangent bundle functor definitions

Before moving on to further tangent structure theory, we briefly compare the tangent bundle we get through the above process with the usual definitions of the tangent bundle of a smooth manifold.

Let M be a smooth manifold. Perhaps the most "geometric" standard definition of the tangent bundle is the following:

Definition 4.9 *(Kinematic tangent bundle) If V is a vector space, a* ***kinematic tangent vector*** *is an equivalence class of smooth curves $f : \mathbb{R} \longrightarrow V$ with $f \sim g$ if $f(0) = g(0)$ and $f'(0) = g'(0)$. The set of all kinematic tangent vectors forms the* ***kinematic tangent bundle*** *KM. Given a smooth map $f : X \longrightarrow Y$, one defines $Kf : KX \longrightarrow KY$ by $Kf(c) := cf$.*

The idea is that a tangent vector at a is an infitesimally small curve through a[2].

However, this geometric definition is equivalent to the "local product" definition of the tangent structure of a differential restriction category.

Proposition 4.10 *In the category of smooth maps between cartesian spaces, $K = T$.*

Proof: Given a kinematic tangent vector $f : \mathbb{R} \longrightarrow X$, we define a pair of elements of X by $(D(f)(1,0), f(0))$. Given a pair of elements (x, a) of M, we define a kinematic tangent vector f by $f(r) := f(0) + r \cdot x$. It is clear that these two definitions are well-defined inverses of one another.

For the action on maps, the local product definition gives us that the result of applying $T(f)$ to c is

$$\langle c'(0), c(0) \rangle \langle Df, \pi_1 f \rangle$$

while the kinematic definition gives us

$$\langle (cf)'(0), f(c(0)) \rangle$$

(where $g'(x) = Df(1, x)$). These two definitions then agree by the chain rule. $\Box$

Since these definitions agree on the base category, they also agree on the categories of manifolds, and hence we have:

Corollary 4.11 *In the category of smooth manifolds, $K = T$.*

The second standard definition of the tangent bundle is the "operational" tangent bundle. For a smooth manifold M, let $C^\infty(M)$ denote the vector field of smooth maps from M to $\mathbb{R}$.

[2] Synthetic differential geometry formalizes this by including infinitesimal spaces D, and defining a tangent vector to be a map $f : D \longrightarrow X$.

Definition 4.12 *Let x be a point in a smooth manifold M. An* ***operational tangent vector at** x* *is a linear map $\alpha : C^\infty(M) \longrightarrow \mathbb{R}$ which satisfies*

$$\alpha(fg) = \alpha(f) \cdot g(x) + \alpha(g) \cdot f(x).$$

(These are known as linear derivations*). The set of all operational tangent vectors over all points of M forms the* ***operational tangent bundle*** *DM. Given a smooth map $f : M \longrightarrow N$, one defines $D(f)(\alpha)$ as $C^\infty(f)\alpha$.*

This "functional analysis" version of the tangent bundle is popular because it is typically easier to manipulate. Unfortunately, for the generality we are working, it is the wrong definition.

First, this definition does not work without a base object $\mathbb{R}$, and so is impossible to define in a general differential restriction category. However, even when this object exists, as, for example, in the category of smooth maps between convenient vector spaces, this definition is not equivalent to the kinematic definition. There is an obvious map from KM to DM: given a curve c, one can define

$$\alpha_c(f) := cf'(0)$$

which is easily checked to be a linear derivation. However, in general this map is only invertible under special circumstances: for more information, see 28.7 of [Kriegl and Michor 1997].

5 T as a monad

In this final section, we show that T has the structure of a monad. The Kleisli category of this monad is related to vector fields, however, so before showing the monad structure of T, we give a short discussion of vector fields for tangent structure.

5.1 Vector fields

The notion of a vector field is of central importance in differential geometry, and generalizes easily to our setting.

Definition 5.1 *If $(\mathbb{X}, T)$ is tangent structure, and M an object of T, a section of $p : TM \longrightarrow M$ of p is a* ***vector field****. That is, a vector field is a map $v : M \longrightarrow TM$ such that $vp = 1_M$.*

The additive structure of vector fields also works in our setting.

Proposition 5.2 *If $(\mathbb{X}, T)$ is tangent structure, and M an object of T, then the set of vector fields $\chi(M)$ has the structure of a commutative monoid. Moreover, if we have a map $f : M \longrightarrow N$, then $T(f)$ preserves these operations; that is, $(v + w)T(f) = vT(f) + wT(f)$ and $0T(f) = 0$.*

Proof: Given two vector fields $v, w : M \longrightarrow TM$, we have $vp = wp = 1$, so we get a map $\langle v, w\rangle : T_2M \longrightarrow TM$. We then define $v + w := \langle v, w\rangle +$, and the 0 vector field as the 0 map $M \longrightarrow TM$. By the axioms for a tangent structure, this is a commutative monoid.

For the second claim, we have:

$$
\begin{array}{rl}
& \langle v, w\rangle +_M T(f) \\
= & \langle v, w\rangle T_2(f) +_N \ \text{(by naturality of } +) \\
= & \langle v, w\rangle\langle p_1T(f), p_2T(f)\rangle +_N \ \text{(by definition of } T_2) \\
= & \langle vT(f), wT(f)\rangle +_N \\
= & vT(f) + wT(f)
\end{array}
$$

as required; the 0 result follows similarly. □

As we shall see in the next section, the addition of vector fields is in fact a specific case of composition in a Kleisli category.

Before we get to that, however, we shall also describe a particular non-zero vector field that is defined on each object of the form TM (note that each object has a canonical zero vector field $0 : M \longrightarrow TM$, since $0p = 1$). To describe this other canonical vector field, we need to revisit our notion of vertical lift. Recall that the vertical lift of our axioms is a natural transformation $\ell : T \longrightarrow T^2$. In the differential geometry literature, however, the standard vertical lift is a natural transformation $v : T_2 \longrightarrow T^2$. Here, we show that the standard vertical lift can be recovered from the weak version of our axioms. One then uses the standard vertical lift to define the canonical vector field on TM.

Proposition 5.3 *If $(\mathbb{X}, T)$ is tangent structure, then there is a natural transformation*

$$v : T_2 \longrightarrow T^2$$

with $vp_T = p_2$, and $vT(p) = p_1p0$.

PROOF: The map is given by

$$v := \langle p_1\ell, p_2 0_T c\rangle +_T c$$

First the first claim, consider

$$
\begin{array}{rcl}
vp_T & = & \langle p_1\ell, p_2 0_T c\rangle +_T cp_T \\
& = & \langle p_1\ell, p_2 0_T c\rangle +_T T(p) \ \text{(by proposition 2.5)} \\
& = & \langle p_1\ell, p_2 0_T c\rangle T_2(p) + \ \text{(naturality of } +) \\
& = & \langle p_1\ell, p_2 0_T c\rangle\langle p_1T(p), p_2T(p)\rangle + \\
& = & \langle p_1\ell T(p), p_2 0_T cT(p)\rangle + \\
& = & \langle p_1p0, p_2 0_T p_T\rangle + \ \text{(by proposition 2.5} \\
& = & \langle p_2p0, p_2\rangle + \ \text{(since } 0p = 1 \text{ and } p_1p0 = p_2p0) \\
& = & p_2\langle p0, 1\rangle + \\
& = & p_2 \ \text{(unit of the addition)}
\end{array}
$$

as required.

For the second claim, consider

$$\begin{array}{rcl} vT(p) & = & \langle p_1\ell, p_2 0_T c\rangle +_T cT(p) \\ & = & \langle p_1\ell, p_2 0_T c\rangle +_T p_T \text{ (by proposition 2.5)} \\ & = & \langle p_1\ell, p_2 0_T c\rangle (p_1 p)_T \text{ (since } +p = p_1 p) \\ & = & p_1 \ell p_T \\ & = & p_1 p 0 \text{ (by proposition 2.5)} \end{array}$$

as required. □

We can then define the so-called "Liouville vector field" on an object of the form TM.

Proposition 5.4 *If M is an object of a tangent category $(\mathbb{X}, T)$, then there is a vector field on TM given by*

$$TM \xrightarrow{\langle 1,1\rangle v} T^2 M$$

PROOF: This is a vector field since $\langle 1,1\rangle v p_T = \langle 1,1\rangle p_2 = 1$, as required. □

5.2 Monad structure of T

As we saw in the previous section, a category with tangent structure allows for an additive structure on vector fields. But this is part of a much larger structure. As we shall see in this section, if $(\mathbb{X}, T)$ is tangent structure, then T is in fact a monad, and the Kleisli category $\mathbb{X}_T$ contains vector fields and their addition - but is much more general.

We begin with an important map which "forgets" a double tangent vector in T^2:

Lemma 5.5 *The map*

$$T^2(M) \xrightarrow{u_M := [Tp, p_T]} T_2 M$$

is a natural transformation from T^2 to T_2.

PROOF: Let $M \xrightarrow{f} N$ be an arbitrary map, and consider

$$\begin{array}{rcl} (u_M)(T_2 f) & = & [Tp, p_T][p_1 T(f), p_2 T(f)] \text{ by definition of } T_2(f), \\ & = & [T(p)T(f), p_T T(f)] \\ & = & [T(pf), p_T T(f)] \end{array}$$

while

$$\begin{array}{rcl} (T^2 f)(u_N) & = & T^2(f)[Tp, p_T] \\ & = & [T^2(f)T(p), T^2(f)p_T] \\ & = & [T(T(f)p), p_T T(f)] \text{ by naturality of } p, \\ & = & [T(pf), p_T T(f)] \text{ by naturality of } p, \end{array}$$

so that the two are equal, as required. □

In the context of synthetic differential geometry, a right inverse for u can be considered as notion of affine connection [Kock and Reyes 1979], though we do not pursue that idea here.

We can now give the monad structure of T.

Proposition 5.6 *If $\mathbb{X}$ is a cartesian restriction category with tangent structure T, then T is a monad, with unit $M \xrightarrow{0} TM$, and multiplication μ given by the composite*

$$T^2M \xrightarrow{u} T_2M \xrightarrow{+} TM.$$

Proof: By definition, the 0 and + are natural, and we have shown u is natural above. Thus, we only need to check the unit and associativity axioms. For one unit axiom, consider

$$\begin{aligned}
& T(0)u+ \\
= \;& T(0)[T(p),p]+ \\
= \;& [T(0)T(p),T(0)p]+ \\
= \;& [T(0p),p0]+ \text{ by functoriality of } T \text{ and naturality of } p, \\
= \;& [T(1),p0]+ \text{ by coherence for } p, \\
= \;& [1,p0]+ \\
= \;& 1 \text{ by the unit axiom for the addition of tangent vectors.}
\end{aligned}$$

For the other unit axiom, consider

$$\begin{aligned}
& 0[T(p),p]+ \\
= \;& [0T(p),0p]+ \\
= \;& [0cp,1]+ \text{ by coherence of } c \text{ and } p, \\
= \;& [T(0)p,1]+ \text{ by preservation of } 0, \\
= \;& [p0,1]+ \text{ by naturality of } p, \\
= \;& 1 \text{ by the unit axiom for the addition of tangent vectors.}
\end{aligned}$$

For the associativity, we need to show the commutativity of the outside of the following diagram:

$$\begin{tikzcd}
T^3M \arrow[r, "T(u)"] \arrow[dd, "u_T"'] \arrow[ddr, dashed, "u_3"'] & TT_2M \arrow[rr, "T(+)"] \arrow[d, dashed, "w"] & & T^2M \arrow[dd, "u"] \\
& T_2TM \arrow[r, dashed, "+_T"'] & T^2M \arrow[ur, dashed, "c"] & \\
T_2TM \arrow[d, "+_T"'] & T_3M \arrow[rr, dashed, "+_r"] \arrow[d, dashed, "+_l"'] & & T_2M \arrow[d, "+"] \\
T^2M \arrow[r, "u"] & T_2M \arrow[rr, "+"] & & TM
\end{tikzcd}$$

where we have added the dashed arrows, and

$$u_3 = [T(p_T)T(p), p_{T^2}T(p), p_{T^2}p_T].$$

The bottom right diagram commutes by the associativity of tangent vector addition, and the top right diagram by preservation of addition. Thus, we only need to show the commutativity of the left region and the middle region.

To show the left region commutes, we begin by expanding the right composite of that region:

$$\begin{aligned}(u_3)(+_l) &= [T(p_T)T(p), p_{T^2}T(p), p_{T^2}p_T][[p_1, p_2]+, p_3] \\ &= [[T(p_T)T(p), p_{T^2}T(p)]+, p_{T^2}p_T]\end{aligned}$$

Now, since these are two maps into T_2, to show they are equal, it suffices to show they are equal when post-composed by p_1 and p_2. Indeed, if we consider

$$\begin{aligned}(u_T)(+_T)(u)p_1 &= [T(p_T), p_{T^2}](+_T)[T(p), p_T]p_1 \\ &= [T(p_T), p_{T^2}](+_T)T(p) \\ &= [T(p_T), p_{T^2}]T_2(p)(+) \text{ by naturality of } +, \\ &= [T(p_T), p_{T^2}][p_1T(p), p_2T(p)](+) \text{ by definition of } T_2(f), \\ &= [T(p_T)T(p), p_{T^2}T(p)](+) \\ &= (u_3)(+_l)p_1 \text{ (by above)}\end{aligned}$$

and

$$\begin{aligned}(u_T)(+_T)(u)p_2 &= [T(p_T), p_{T^2}](+_T)[T(p), p_T]p_2 \\ &= [T(p_T), p_{T^2}](+_T)p_T \\ &= [T(p_T), p_{T^2}](p_2p_T) \text{ by coherence of } +, \\ &= p_{T^2}p_T \\ &= (u_3)(+_l)p_2 \text{ (by above)}\end{aligned}$$

as required. Thus the left region commutes.

For the middle region, we first calculate

$$\begin{aligned}T(u)w &= T(u)[T(p_1)c, T(p_2)c] \\ &= [T(up_1)c, T(up_2)c] \\ &= [T([T(p), p_T]p_1)c, T([T(p), p_T]p_2)c] \\ &= [T^2(p)c, T(p_T)c]\end{aligned}$$

So that the top composite is

$$[T^2(p)c, T(p_T)c](+_T)(c)[T(p), p_T]$$

while the middle composite is

$$\begin{aligned}(u_3)(+_r) &= [T(p_T)T(p), p_{T^2}T(p), p_{T^2}p_T][p_1, [p_2, p_3]+] \\ &= [[T(p_T)T(p), [p_{T^2}T(p), p_{T^2}p_T]+]\end{aligned}$$

Again, since these are two maps into T_2, to show they are equal, it suffices to show they are equal when post-composed by p_1 and p_2. Indeed, if we consider

$$\begin{array}{rl} & [T^2(p)c, T(p_T)c](+_T)(c)[T(p), p_T]p_1 \\ = & [T^2(p)c, T(p_T)c](+_T)(c)T(p) \\ = & [T^2(p)c, T(p_T)c](+_T)p_T \text{ by coherence of } c, \\ = & [T^2(p)c, T(p_T)c](p_2)(p_T) \text{ by coherence of } +_T, \\ = & T(p_T)cp_T \\ = & T(p_T)T(p) \text{ by coherence of } c, \\ = & (u_3)(+_r)p_1 \text{ (by above)} \end{array}$$

and

$$\begin{array}{rl} & [T^2(p)c, T(p_T)c](+_T)(c)[T(p), p_T]p_2 \\ = & [T^2(p)c, T(p_T)c](+_T)(c)p_T \\ = & [T^2(p)c, T(p_T)c](+_T)T(p) \text{ by coherence of } c, \\ = & [T^2(p)c, T(p_T)c]T_2(p)(+) \text{ by naturality of } +, \\ = & [T^2(p)c, T(p_T)c][p_1T(p), p_2T(p)](+) \text{ by definition of } T_2(f), \\ = & [T^2(p)cT(p), T(p_T)cT(p)](+) \\ = & [T^2(p)p_T, T(p_T)p_T](+) \text{ by coherence of } c, \\ = & [p_{T^2}T(p), p_{T^2}p_T](+) \text{ by naturality of } p \text{ (twice)}, \\ = & (u_3)(+_r)p_2 \end{array}$$

as requred. Thus, the diagram commutes, and T is a monad. $\Box$

We also record how μ interacts with l and c:

Lemma 5.7 *If $(\mathbb{X}, T)$ is tangent structure, then $c\mu = \mu$, and $l\mu = p0$.*

PROOF: For the first claim:

$$\begin{array}{rcl} c\mu & = & c\langle T(p), p_T\rangle + \\ & = & \langle cT(p), cp_T\rangle + \\ & = & \langle p_T, T(p)\rangle + \text{ (by Proposition 2.5)} \\ & = & \langle T(p), p_T\rangle + \text{ (by commutativity of } +) \\ & = & \mu \end{array}$$

For the second claim,

$$l\mu = l\langle T(p), p_T\rangle + = \langle p0, p0\rangle + = p0$$

by the unitality of $+$. $\Box$

It is somewhat surprising that the fact that T is a monad (also discovered independently for cartesian differential categories in [Manzyuk 2012]) has been overlooked in differential geometry. This is especially surprising considering the next result: the Kleisi category of this monad is a generalization of the addition of vector fields.

Proposition 5.8 *If $v, w : M \longrightarrow TM$ are vector fields on M, then the Kleisli composite of v and w is given by $v + w$.*

PROOF: For arbitrary maps in the Kleisi category $v : A \longrightarrow TB$, $w : B \longrightarrow TC$, vw is given by the formula

$$vT(w)\langle T(p), p_T\rangle + = \langle vT(wp), vpw\rangle + .$$

But if v and w are vector fields, then $wp = vp = 1$, so the above simplifies to $\langle v, w\rangle +$, as required. $\Box$

This shows that the Kleisli category $\mathbb{X}_T$ may be of some importance: the maps are generalized vector fields, while the composition of these vector fields is a generalization of vector field addition. We were not able to find a reference to these generalized vector fields or their composition in the literature.

While the Kleisli category appears as a generalization of vector fields, the Eilenberg-Moore algebras are harder to understand. An algebra $\alpha : TM \longrightarrow M$ describes a way to associate each tangent vector on the space to another point on the space, in a way that is compatible with addition. In particular, sending the tangent vector to its base point is an algebra:

Proposition 5.9 *If $(\mathbb{X}, T)$ is tangent structure, then for any object M, the projection map $p_M : TM \longrightarrow M$ is an algebra for the monad T.*

PROOF: We begin by showing that p is a morphism of monads from $(T, 0, \mu)$ to the identity monad. For this, we need to show the diagrams

$$\begin{tikzcd} T^2 \arrow[r, "\mu"] \arrow[d, "T(p)"'] & T \arrow[d, "p"] \\ T \arrow[r, "p"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "0"'] \arrow[dr, "1"] & \\ T \arrow[r, "p"'] & I \end{tikzcd}$$

commute. For the first diagram, we have

$$\mu p = \langle T(p), p_T\rangle + p = \langle T(p), p_T\rangle p_1 p = T(p)p$$

by the properties of additive bundles. The second diagram, $0p = 1$, is also an axiom of additive bundles.

If $\alpha : T \longrightarrow S$ is a morphism of monads, then α sends S-algebras to T-algebras via pre-composition. Since any object M is canonically an algebra for the identity functor, and p is a morphism of monads to the identity functor, any M is thus a T-algebra, with structure map $p : TM \longrightarrow M$. $\Box$

Of course, any object of the form TM also has an associated free algebra $\mu : T^2M \longrightarrow TM$. For example, on $\mathbb{R}^2 = T(\mathbb{R})$, the free algebra structure sends

$$\langle a, b, c, d\rangle \mapsto \langle b + c, d\rangle.$$

Again, it would be interesting to know if this algebra, or other algebras of this monad, occur in the literature on differential geometry.

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