A tale of two tangent bundles

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Throughout the 1980's and 1990's, Kriegl and Michor worked on a framework that would allow them to discuss smooth manifolds modelled on infinite dimensional vector spaces.

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A **convenient vector space** is a locally convex vector space E such that any smooth curve $\mathbb{R} \xrightarrow{c} E$ has a smooth antiderivative.

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They build a large amount of theory on "convenient manifolds": spaces that locally look like a convenient vector space with all transition maps smooth.

The tangent bundle

There are two tangent bundle definitions:

Definition

If E is a convenient vector space, a **kinematic tangent vector** is an equivalence class of smooth curves $f : \mathbb{R} \longrightarrow E$ with $f \sim g$ if $f(0) = g(0)$ and $f'(0) = g'(0)$. Locally, the **kinematic tangent bundle** of a convenient manifold M is its set of kinematic tangent vectors.

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Definition

Let x be a point in a smooth manifold M. An **operational tangent vector at** x is a linear map $\alpha : C^\infty(M) \longrightarrow \mathbb{R}$ which satisfies

$$\alpha(fg) = \alpha(f) \cdot g(x) + \alpha(g) \cdot f(x).$$

The set of all operational tangent vectors over all points of M forms the **operational tangent bundle** DM.

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- As another example, the authors are forced to consider 12 (!) different definitions of differential form, some based on the kinematic tangent bundle, some on the operational.
- We will investigate which definition is “right” via differential categories and synthetic differential geometry.

Differential categories

We'll start by looking at conveninent smooth manifolds via a definition of Blute, Cockett and Seely (2007):

Definition

A cartesian differential category consists of a cartesian left additive category which has, for each map $f : X \longrightarrow Y$, a map $D[f] : X \times X \longrightarrow Y$ satisfying seven axioms (chain rule, D preserves addition, symmetry of partial derivatives, etc.)

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- Think of D as the Jacobian, evaluted at the second X, in the direction of the first X.
- As an example of the axioms, the chain rule is given by asking that $D[fg] = \langle D[f], \pi_1 f \rangle D[g]$.

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- Cartesian spaces: objects natural numbers, a map $f : n \longrightarrow m$ is a smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

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- Examples from differential linear logic.
- Cockett and Seely (2011): cartesian differential categories are comonadic over left additive cartesian categories, so every left additive cartesian category has an associated cofree cartesian differential category (a slightly generalized version may be comonadic over cartesian categories!).
- Most relevant for us: Blute, Erhard and Tasson (2011) showed convenient vector spaces and smooth maps are a cartesian differential category.

Differential restriction categories

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- Using a different name for this structure, Grandis (1989) showed that starting with any suitably well-behaved restriction category, one can build a category of manifolds.
- But what do we get when the restriction category has compatible differential structure?

Tangent structure

If we start with a differential restriction category (Cockett, Cruttwell, Gallagher 2011) and build its category of manifolds, we get the following structure (Cruttwell and Cockett 2011):

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Definition

Tangent structure for a cartesian category $\mathbb{X}$ consists of an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$ with:

- a natural map $TX \xrightarrow{p_X} X$ which has the structure of a commutative monoid in $\mathbb{X} \setminus X$ for each X;
- T preserves products and certain pullbacks;
- two natural transformation $l : T \longrightarrow T^2$ ("vertical lift") and $c : T^2 \longrightarrow T^2$ ("canonical flip") which preserve the commutative monoid structure.

There is a sense in which the properties of this tangent bundle are equivalent to the properties of a cartesian differential category.

Tangent structure on convenient manifolds is the kinematic tangent bundle

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- The kinematic tangent bundle is exactly the one we get with our general theory.
- In essence, a kinematic tangent vector is simply a choice of two points: $c(0)$ and $c'(0)$, and this matches with differential categories, with derivative $D[f] : X \times X \longrightarrow Y$.
- Thus our general theory immediately gives us a host of results about the kinematic tangent bundle.

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Thus, for differential categories and tangent structure, the kinematic tangent bundle is the right tangent bundle; the operational tangent bundle is simply "something else".

Synthetic differential geometry

In contrast to the low-level approach of differential categories is synthetic differential geometry, which defines a “smooth topos” as a topos with a special “infinitesimal object” D.

Definition

For any object X in a smooth topos, one can define its tangent bundle as X^D.

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Definition

For any object X in a smooth topos, one can define its tangent bundle as X^D.

When restricted to the "infinitesimally linear" objects, this endofunctor is tangent structure (as defined earlier).

SDG and smooth convenient manifolds

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- In the early 1980's, Anders Kock showed that the category of convenient vector spaces fully and faithfully embeds inside a model of SDG, the "Cahiers" or "Dubuc" topos.
- It is easy to show that this embedding extends to smooth convenient manifolds as well.
- So, we can ask the question: which tangent bundle, if either, does the "synthetic" tangent bundle correspond to?

Synthetic != Operational

- Suppose one has a fully faithful embedding of smooth convenient manifolds into a smooth topos which preserves products. Then the synthetic tangent bundle cannot equal the operational tangent bundle, as the synthetic tangent bundle preserves products ($(X \times Y)^D \cong X^D \times Y^D$), while the operational one does not!

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- Since Kock's embedding preserves products, the synthetic tangent bundle cannot be the operational tangent bundle.

But could the synthetic tangent bundle be the kinematic tangent bundle?

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- This is where the operational tangent bundle comes from: as maps $C^\infty(M) \longrightarrow D = \mathrm{Spec}(\mathbb{R}[\epsilon])$.
- But this doesn't work for smooth convenient manifolds! This embedding is not full and faithful.
- Instead, the embedding directly defines an action of a Weil algebra on each smooth convenient manifold.

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- In particular, the action of the ring of dual numbers $\mathbb{R}[\epsilon]$ corresponds to exponentiation by D.
- And the action of the ring of dual numbers that Kock defines is the kinematic tangent bundle.

So, as with differential categories, the kinematic tangent bundle is the right tangent bundle.

In the text?

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- The only place they “need” the operational tangent bundle is to define the Lie bracket of kinematic vector fields: but one can do this via SDG without the operational tangent bundle.

While this appears not to be explicitly recognized in the text itself, all their results also point to the kinematic tangent bundle being the right thing.

Conclusion

By working with the theory directly, the authors find it hard to distinguish the kinematic and operational tangent bundles: only viewed through the general theory of differential categories or synthetic differential geometry is it apparent that the correct tangent bundle is the kinematic one. Knowing this would have saved them a lot of effort!

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Some further points to consider:

- This discussion also applies to non-Hausdorff paracompact smooth finite dimensional manifolds: for these as well, the kinematic tangent bundle is the correct definition.
- Since the operational definition more closely relates to constructions in algebraic geometry, it is often the preferred definition; this gives an instance where the kinematic definition is preferred.