Towards cotangent categories

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Overview

Tangent categories are a *minimal* categorical setting for differential geometry.

- Tangent categories span a wide variety of examples from differential geometry and algebraic geometry to abstract homotopy theory (functor calculus).
- Many structures can be defined in a tangent category, including vector bundles, differential forms, and connections.
- In this talk the focus is on how to define and work with the *cotangent* bundle in tangent categories.

Plan:

1. Review of tangent categories and differential bundles.
2. Define *linear dualization* in a tangent category and how this gives a cotangent bundle.
3. Can we axiomatically define a *cotangent category*?

Tangent category definition

Tangent categories abstract the structure of the tangent bundle functor on the category of smooth manifolds.

Definition (Rosický 1984, modified Cockett/Cruttwell 2014)

A **tangent category** consists of a category $\mathbb{X}$ with:

- an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of $p_M : TM \longrightarrow M$ along itself exists (and is preserved by each T^m), call this pullback $T_n M$;
- for each $M \in \mathbb{X}$, $p_M : TM \longrightarrow M$ has the structure of a commutative monoid in the slice category $\mathbb{X}/M$, in particular there are natural transformations $+ : T_2 \longrightarrow T$, $0 : 1_{\mathbb{X}} \longrightarrow T$;

Tangent category definition (continued)

Definition

- (canonical flip) there is a natural transformation $c : T^2 \longrightarrow T^2$ which preserves additive bundle structure and satisfies $c^2 = 1$;
- (vertical lift) there is a natural transformation $\ell : T \longrightarrow T^2$ which preserves additive bundle structure and satisfies $\ell c = \ell$;
- various other coherence equations for ℓ and c;
- (tangent spaces have trivial tangent bundle) the following is a pullback:

$$\begin{array}{ccc} T_2 M & \xrightarrow{\nu} & T^2 M \\ {\scriptstyle \pi_0 p_M}\downarrow & & \downarrow{\scriptstyle T(p_M)} \\ M & \xrightarrow[0_M]{} & TM \end{array}$$

where $\nu = \langle \pi_0 0_M, \pi_1 \ell \rangle T(+)$.

Examples

- Smooth manifolds with their tangent bundle.
- Convenient manifolds (a certain type of infinite-dimensional manifold) with their *kinematic* tangent bundle.
- The infinitesimally linear objects in a model of synthetic differential geometry (SDG)
- The category of C^∞-rings.
- Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.
- (MacAdam) The category of all small categories with finite limits is a tangent category, where

$$T(\mathbb{X}) = \text{ Beck modules in } \mathbb{X} \text{ (Abelian group objects in } \mathbb{X})$$

- Abelian functor calculus gives a tangent category, and Goodwillie functor calculus gives an (infinity) tangent category.
- The vector fields in any tangent category form a new tangent category (as do many other constructions).

Differential bundles

The analog of vector bundles in a tangent category are:

Definition (Cockett/Cruttwell 2015)

A **differential bundle** in a tangent category consists of an additive bundle $(q : E \longrightarrow M, \sigma, \zeta)$ and a map $\lambda : E \longrightarrow TE$ such that the following is a pullback:

$$\begin{array}{ccc} E_2 & \xrightarrow{\nu} & TE \\ {\scriptstyle \pi_0 q}\downarrow & & \downarrow{\scriptstyle T(q)} \\ M & \xrightarrow[0_M]{} & TM \end{array}$$

where E_2 is the pullback of q along itself, and $\nu = \langle \pi_0 0_E, \pi_1 \lambda \rangle T(\sigma)$.

A differential bundle E over 1 has

$$TE \cong E \times E.$$

(MacAdam) In the tangent category of smooth manifolds, differential bundles = vector bundles (with their local triviality condition!)

Differential bundles continued

Definition

A **linear morphism of differential bundles** from $(q : E \longrightarrow M, \sigma, \zeta, \lambda)$ to $(q' : E' \longrightarrow M', \sigma', \zeta', \lambda')$ consists of a morphism in the arrow category (f, f') which preserve the λ's:

$$\begin{array}{ccc} E & \xrightarrow{f'} & E' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ M & \xrightarrow[f]{} & M' \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{f'} & E' \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda'} \\ TE & \xrightarrow[T(f')]{} & TE' \end{array}$$

call the associated category $\text{Lin}(\mathbb{X})$.

- In the tangent category of smooth manifolds, morphisms of differential bundles = linear bundle morphisms.
- With some additional pullback assumptions, $\text{Lin}(\mathbb{X})$ is a fibration over the base category $\mathbb{X}$.
- T can be seen as a functor $\mathbb{X} \longrightarrow \text{Lin}(\mathbb{X})$.

Cotangent bundle?

Recall that the cotangent bundle T^*M is defined by taking the *dual* of each of the tangent spaces of M.

- The cotangent bundle *is not* an endofunctor on the category of smooth manifolds.
- It does give an endofunctor when restricted to etale maps...but not general smooth maps.
- It does give a functor to a different category, though: the *dual* of the fibration of differential/vector bundles.

The dual fibration

Any fibration $F : \mathbb{A} \longrightarrow \mathbb{B}$ has an associated **dual** fibration given by taking the opposite category in each fibre. For example:

- Consider the simple fibration over a Cartesian category which has objects pairs (A, A') with maps $(f, f') : (A, A') \longrightarrow (B, B')$ such that

$$f : A \longrightarrow B \text{ and } f' : A \times A' \longrightarrow B'$$

- Its dual fibration has objects pairs (A, A') with maps $(f, f^*) : (A, A') \longrightarrow (B, B')$ such that

$$f : A \longrightarrow B \text{ and } f^* : A \times B' \longrightarrow A'.$$

- This is also known as the category of *lenses*, and has appeared in many places (database theory, functional programming, dialectica categories, machine learning): its morphisms have a very useful *bidirectional* nature.

More dual fibrations

- Recall that the arrow category/codomain fibration of a category $\mathbb{X}$ with pullbacks has objects maps $q : E \longrightarrow M$ and morphisms pairs (f, f') which give commuting squares

$$\begin{array}{ccc} E & \xrightarrow{f'} & E' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

- Its dual fibration again has objects maps $q : E \longrightarrow M$ but now a morphism is a pair (f, f^*) with

$$f : M \longrightarrow M' \text{ and } f^* : E' \times_{M'} M \longrightarrow E$$

  (This is sometimes called the category of **dependent** lenses: notice it again has a bidirectional nature!)
- The dual of the fibration $\text{Lin}(\mathbb{X})$, $\text{Lin}^*(\mathbb{X})$ is the same as above, except now the objects are differential bundles, and f^* must be linear.

Linear dualization

Definition

If $\mathbb{X}$ is a tangent category, a **linear dualization on** $\mathbb{X}$ consists of a fibration functor (ie., it preserves Cartesian arrows)

$$\begin{array}{ccc} \mathrm{Lin}(\mathbb{X}) & \xrightarrow{D} & \mathrm{Lin}^*(\mathbb{X}) \\ & \searrow \quad \swarrow & \\ & \mathbb{X} & \end{array}$$

which is compatible with the tangent structure, that is

$$\begin{array}{ccc} \mathrm{Lin}(\mathbb{X}) & \xrightarrow{D} & \mathrm{Lin}^*(\mathbb{X}) \\ {\scriptstyle \bar{T}}\downarrow & & \downarrow{\scriptstyle \tilde{T}} \\ \mathrm{Lin}(\mathbb{X}) & \xrightarrow{D} & \mathrm{Lin}^*(\mathbb{X}) \end{array}$$

commutes (where $\bar{T}$ and $\tilde{T}$ are functors induced by T).

Cotangent bundle functor

In any tangent category $\mathbb{X}$ with a linear dualization D, we get an associated functor T^* by composing T with D:

$$\mathbb{X} \xrightarrow{T} \mathsf{Lin}(\mathbb{X}) \xrightarrow{D} \mathsf{Lin}^*(\mathbb{X}).$$

This assigns to each M its "cotangent bundle" $q : T^*M \longrightarrow M$, and assigns to a map $f : M \longrightarrow N$ its "pullback" (as it is called in differential geometry)

$$T^*(f) : T^*N \times_N M \longrightarrow T^*M$$

(in other terminology, it assigns to each f a dependent lens).

Pullback of covector fields

For the next few slides, we'll assume $\mathbb{X}$ is a tangent category with a linear dualization.

Definition

Define a **covector field on** M in $\mathbb{X}$ to be a map $\omega : M \longrightarrow T^*(M)$ which is a section of $q : T^*(M) \longrightarrow M$.

It is a standard result that covector fields can be "pulled back", and this also holds in our setting:

Lemma

If $f : M \longrightarrow N$ is a map and $\omega : N \longrightarrow T^*(N)$ is a covector field on N, then

$$M \xrightarrow{\langle f\omega, 1\rangle} T^*(N) \times_N M \xrightarrow{T^*(f)} T^*(M)$$

is a covector field on M.

Cotangent bundle functor on etale maps

We also get an induced endofunctor on the base category when restricted to etale maps maps:

- In any tangent category, define a **etale** map to be a map $f : M \to N$ such that

$$\begin{array}{ccc} TM & \xrightarrow{T(f)} & TN \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

  is a pullback (this agrees with the usual notion in smooth manifolds).
- That is, $(f, T(f))$ is a Cartesian arrow in the fibration $\text{Lin}(\mathbb{X})$, so D of it is a Cartesian arrow in $\text{Lin}^*(\mathbb{X})$
- But Cartesian arrows in a dual fibration correspond to Cartesian arrows in the original fibration, so applying the above to an etale map gives a morphism of $\text{Lin}(\mathbb{X})$, and taking the top component of this map gives an endofunctor

$$T^* : \text{etale}(\mathbb{X}) \longrightarrow \text{etale}(\mathbb{X}).$$

Some other structure on the cotangent bundle

There are some other results one gets in this abstract setup:

- The canonical flip $c : T^2 \to T^2$ gives linear isomorphisms

$$\begin{array}{ccc} T^2M & \xrightarrow{c_M} & T^2M \\ {\scriptstyle T(p_m)}\downarrow & & \downarrow{\scriptstyle p_{TM}} \\ TM & \xrightarrow[1_{TM}]{} & TM \end{array}$$

  Applying D to this gives isomorphisms

$$T(T^*M) \cong T^*(TM)$$

- Can build the Liouville-Hamilton vector field on the cotangent bundle: a vector field on T^*M, ie., a map

$$T^*(M) \to T(T^*(M))$$

- Can build the canonical covector field on the cotangent bundle, ie., a map

$$T^*(M) \to T^*(T^*(M))$$

Cotangent categories?

Can we directly axiomatize a "category equipped with a cotangent bundle"?

- Why? In many places the cotangent bundle is seen as a more natural structure than the tangent bundle, so it might be nice to axiomatize it directly.
- We could define a cotangent category as a category $\mathbb{X}$ equipped with a functor

$$T^* : \mathbb{X} \longrightarrow \mathsf{AdBun}^*(\mathbb{X})$$

  where $\mathsf{AdBun}(\mathbb{X})$ is the category of additive bundles in $\mathbb{X}$.
- Some of the other structure is less clear, though: for example, as we saw on the previous slide, the canonical flip gives an isomorphism

$$T(T^*M) \cong T^*(TM)$$

  not an isomorphism $T^*(T^*(M)) \cong T^*(T^*(M))$.
- Also not clear how to define the analog of differential bundles directly in the case cotangent structure.

Conclusions

In conclusion:

- A tangent category equipped with a *linear dualization* functor has an associated cotangent bundle functor, and the technology of the dual fibration is very helpful in defining this abstractly.
- Some standard results about the cotangent bundle can be recovered from this abstract setup (and some non-standard results immediately follow as well).
- It is not yet clear (to us) how to define a cotangent category, but we're still thinking about it.

References

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