Reconsidering Cartesian differential categories

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- the definition of Cartesian differential categories;
- a problematic non-example and a solution;
- (generalized) Cartesian differential categories as coalgebras;
- an application: de Rham cohomology.

Cartesian differential categories

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- For a smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the Jacobian is a smooth map

$$J(f) : \mathbb{R}^n \longrightarrow \mathsf{Lin}[\mathbb{R}^n, \mathbb{R}^m].$$

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$$J(f) : \mathbb{R}^n \longrightarrow \mathrm{Lin}[\mathbb{R}^n, \mathbb{R}^m].$$

- We don't want to assume any closed structure, so we uncurry, thinking of the Jacobian as a map

$$J(f) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

which is linear in the first variable.

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- a category with products;
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Call this a **Cartesian left additive category**.

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(Blute/Cockett/Seely) A **Cartesian differential category** consists of a Cartesian left additive category $\mathbb{X}$, which has for each map $f : X \longrightarrow Y$, a map $D[f] : X \times X \longrightarrow Y$, such that:

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1. $D(f+g) = D(f) + D(g), D(0) = 0$;
2. $D(\langle f, g \rangle) = \langle Df, Dg \rangle$;
3. $D(1) = \pi_0, D(\pi_0) = \pi_0\pi_0, D(\pi_1) = \pi_0\pi_1$;
4. $D(fg) = \langle Df, \pi_1 f \rangle D(g)$;
5. $\langle a+b, c \rangle D(f) = \langle a, c \rangle D(f) + \langle b, c \rangle D(f), \langle 0, c \rangle D(f) = 0$;
6. $\langle \langle a, 0 \rangle, \langle b, c \rangle \rangle D(D(f)) = \langle a, c \rangle D(f)$;
7. $\langle \langle 0, b \rangle, \langle c, d \rangle \rangle D(D(f)) = \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D(D(f))$.

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- The second last axiom says $D(f) : X \times X \longrightarrow Y$ is itself linear in its first variable.
- If the Cartesian closed structure is closed, one asks that $\mathrm{ev} : [X, Y] \times X \longrightarrow Y$ be linear in its first variable (Bucciarelli/Ehrhard/Manzonetto).

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- The category whose objects are open subsets of $\mathbb{R}^n$'s, maps smooth maps, is not an example.
- The derivative of a smooth map $U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ does not have type
$$D(f) : U \times U \longrightarrow V.$$
- Instead, it has type
$$D(f) : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^m.$$

The domain of $D(f)$ plays two roles

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We need to formalize this.

Generalized definition

Definition

A **generalized Cartesian differential category** consists of a Cartesian category $\mathbb{X}$, which has for each object X, an associated monoid $(L(X), +_X, e_X)$ (preserving products and idempotent), and for each map $f : X \longrightarrow Y$, an associated map $D(f) : L(X) \times X \longrightarrow L(Y)$, such that:

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1. for each X, $+_X$ and e_X are linear;
2. $D(\langle f, g \rangle) = \langle Df, Dg \rangle$;
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Given a Cartesian left additive category $\mathbb{X}$, define a Cartesian left additive category **Faà**($\mathbb{X}$) with:

- an object is a pair of objects (A, X);

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- an object is a pair of objects (A, X);
- a map $f : (A, X) \longrightarrow (B, Y)$ consists of a sequence of maps $(f_i)_{i \in \mathbb{N}}$, where:
  - $f_0 : X \longrightarrow Y$;
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- composition of f_0 is usual, f_1 by chain rule, higher maps by "higher-order" chain rule.

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- making it into a comonad.

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Every coalgebra with $C(X) = (X, X)$ is a Cartesian differential category, with derivative $D(f) = C(f)_1$.

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Even better:

- *Every* coalgebra is a generalized Cartesian differential category, with derivative $D(f) = C(f)_1$.
- So every Cartesian category $\mathbb{X}$ has an associated generalized Cartesian differential category $\mathbf{Fa\grave{a}}(\mathbb{X})$.

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- is the coalgebras for a more natural comonad.

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- reproduces the usual definition when applied to Cartesian spaces;
- reproduces the de Rham cohomology for convenient vector spaces and their open subsets.

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with:

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Let $\Omega_k(X)$ be the set of k-forms of an object X; they can be given the structure of an Abelian group.

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$$\sum_{i=0}^{k} (-1)^i \langle \pi_i, \pi_0, \pi_1, \ldots \widehat{\pi}_i, \pi_{i+1} \ldots \pi_k \rangle D_X(\omega)$$

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- Note: this uses all the axioms of a generalized Cartesian differential category!

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- allows for examples involving open subsets;
- more natural as the coalgebras for a certain comonad;
- while more general, it is still powerful enough to define constructions such as de Rham cohomology.