General connections in tangent categories

Geoff Cruttwell
Mount Allison University
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"Differential geometry is the study of a connection on a principal bundle." (R. Sharpe)

"I personally feel that the next person to propose a new definition of a connection should be summarily executed." (M. Spivak)

Overview

- Tangent categories provide an abstract framework for differential geometry.
- Much recent work has been done to show how to formulate various ideas from differential geometry in arbitrary tangent categories.
- One particular example is the notion of a **connection on a vector bundle**, formulated for tangent categories by Cockett and Cruttwell and (very) recently re-formulated by Lucyshyn-Wright.

Overview

- Today, I'll describe a version of connection that applies to more general types of bundles (due to Ehresmann) that can also be described in tangent categories.
- I'll also show how the general formulation relates to other formulations of the connection notion, including the particular example of connections on a principal bundle.
- The more general version of connection is (I believe) also easier to understand than connections on specific types of bundles.

Tangent category definition

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

A **tangent category** consists of a category $\mathbb{X}$ with:

- **tangent bundle functor**: an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- **projection of tangent vectors**: a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of p_M along itself exists (and is preserved by each T^m), call this pullback $T_n M$;
- **addition and zero tangent vectors**: for each $M \in \mathbb{X}$, p_M has the structure of a commutative monoid in the slice category $\mathbb{X}/M$; in particular there are natural transformations $+ : T_2 \longrightarrow T$, $0 : 1_{\mathbb{X}} \longrightarrow T$;

Tangent category definition (continued)

Definition

- **symmetry of mixed partial derivatives**: a natural transformation $c : T^2 \longrightarrow T^2$;
- **linearity of the derivative**: a natural transformation $\ell : T \longrightarrow T^2$;
- **the vertical bundle of the tangent bundle is trivial**:

$$\begin{array}{ccc} T_2(M) & \xrightarrow{\langle \pi_0 \ell, \pi_1 0_{TM} \rangle T(+)} & T^2(M) \\ {\scriptstyle \pi_0 p_M = \pi_1 p_M} \downarrow & & \downarrow {\scriptstyle T(p_M)} \\ M & \xrightarrow[0_M]{} & T(M) \end{array}$$

  is a pullback;
- various coherence equations for ℓ and c.

$\mathbb{X}$ is a **Cartesian tangent category** if $\mathbb{X}$ has products and T preserves them.

Examples

(i) Finite dimensional smooth manifolds with the usual tangent bundle.

(ii) Convenient manifolds with the kinematic tangent bundle.

(iii) Any Cartesian differential category (includes all Fermat theories by a result of MacAdam, and Abelian functor calculus by a result of Bauer et. al.).

(iv) The microlinear objects in a model of synthetic differential geometry (SDG).

(v) Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.

(vi) The category of C^∞-rings.

(vii) With additional pullback assumptions, tangent categories are closed under slicing.

Note: Building on work of Leung, Garner has shown how tangent categories are a type of enriched category.

Intuitive idea of general connections

Simply: a **general connection** on a “bundle” $q : E \longrightarrow M$ is a choice of a horizontal and vertical co-ordinate system for TE.

Bundles

(Provisional definition)

Definition

Say a map $q : E \longrightarrow M$ in a tangent category is a **bundle** if

(i) All pullbacks along q exist and are preserved by each T^n.

(ii) All pullbacks along $T(q)$ exist and are preserved by each T^n.

Example

Fibre bundles in the category of smooth manifolds.

Example

Any map between microlinear objects in a model of SDG.

Vertical bundle

Definition

If $q : E \longrightarrow M$ is a bundle, its **vertical bundle**, $V(E)$, is the following pullback:

$$\begin{array}{ccc} V(E) & \xrightarrow{\quad i \quad} & T(E) \\ \downarrow & & \downarrow T(q) \\ M & \xrightarrow[\quad 0 \quad]{} & T(M) \end{array}$$

This has a "lift" map $\ell_V : V(E) \longrightarrow T(V(E))$ inherited from ℓ_E.

Horizontal bundle

Definition

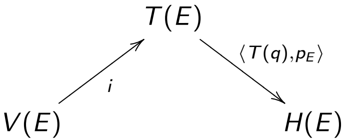
If $q : E \longrightarrow M$ is a bundle, its **horizontal bundle**, $H(E)$, is the following pullback:

$$\begin{array}{ccc} H(E) & \longrightarrow & T(M) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_M} \\ E & \xrightarrow[q]{} & M \end{array}$$

This has a “lift” map $\ell_H : H(E) \longrightarrow T(H(E))$ inherited from ℓ_M.

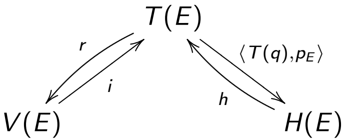
Associated maps

A bundle then has the following diagram of maps:



General connection

A **connection** on such a bundle is then required to have maps r, h:



General connection continued

satisfying various axioms:

- r a retract of i and h is a section of $\langle T(q), p_E \rangle$;
- r **stays in the same fibre**: $rip_E = p_E$;
- h **stays in the same fibre**: $hp = \pi$;
- r **is linear**: $r\ell_V = \ell T(r)$;
- h **is linear**: $h\ell = \ell_H T(h)$;
- **Orthogonality**: $hr = \pi 0$;
- **Sum decomposition**: $ri + \langle T(q), p_E \rangle h = 1_{TE}$.

Note: there should be other ways of expressing these axioms (see Lucyshyn-Wright's alternative versions of connections on differential bundles in tangent categories).

Different definitions of connection

So why do differential geometry books have so many different definitions of connection?

- In the category of smooth manifolds, it suffices to give either an h or an r.
- Ehresmann's version just gives an h.
- As we'll briefly see, for particular types of bundles, the vertical bundle can be trivialized, giving a simpler description of an r; the standard definitions are re-formulated versions of such an r.

Connections on the tangent bundle

Recall an axiom for a tangent category is that the following is a pullback:

$$
\begin{array}{ccc}
T_2(M) & \xrightarrow{\langle \pi_0 \ell, \pi_1 0_T \rangle T(+)} & T^2(M) \\
{\scriptstyle \pi_0 p = \pi_1 p} \downarrow & & \downarrow {\scriptstyle T(p)} \\
M & \xrightarrow[0]{} & T(M)
\end{array}
$$

So in this case $V(T(M)) \cong T_2(M)$, i.e., the vertical bundle of $p : TM \longrightarrow M$ is trivial.

- Thus, to give an r on the tangent bundle is to give a $k : T^2 M \longrightarrow TM$.
- For smooth manifolds, this itself can be re-formulated as giving a **covariant derivative** (an operation on vector fields of M).
- The covariant derivative itself can also be described by giving **Christoffel symbols of the second kind** or **connection coefficients**.

Connections on a vector bundle

More generally, for a vector bundle $q : E \longrightarrow M$, the vertical bundle is also trivial; that is, there is a map v making the following a pullback:

$$\begin{array}{ccc} E \times_M E & \xrightarrow{\quad v \quad} & T(E) \\ {\scriptstyle \pi_0 p = \pi_1 p}\downarrow & & \downarrow{\scriptstyle T(p)} \\ M & \xrightarrow[\quad 0 \quad]{} & T(M) \end{array}$$

- Thus, to give an r for such a bundle is to give a $k : TE \longrightarrow E$.
- Similarly, this itself can be re-formulated as giving a **Koszul derivative** (an operation on sections of the vector bundle).

Principal G-bundles

Definition

If G is a group in a tangent category, a **principal G-bundle** consists of a bundle $q : E \longrightarrow M$ and a fibre-preserving left G-action $\alpha : G \times E \longrightarrow E$ which is free and transitive, i.e. such that

$$G \times E \xrightarrow{\langle \alpha, \pi_1 \rangle} E \times_M E$$

is invertible.

Example

Principal G-bundles in the category of smooth manifolds.

Connections on Principal G-bundles

Suppose (G, e, m) is a group in a Cartesian tangent category, and let $T_e G$ denote the tangent space of G at e (the pullback of p_G along e), with inclusion $j : T_e G \longrightarrow TG$.

Theorem

If $q : E \longrightarrow M$, $\alpha : G \times E \longrightarrow E$ is a principal G-bundle in a tangent category, then

$$\begin{array}{ccc} T_e G \times E & \xrightarrow{(j \times 0) T(\alpha)} & T(E) \\ {\scriptstyle \pi_1 q} \downarrow & & \downarrow {\scriptstyle T(q)} \\ M & \xrightarrow[0]{} & T(M) \end{array}$$

is a pullback, i.e., its vertical bundle is trivial.

- Thus, to give an r in this case it suffices to give an $\omega : TE \longrightarrow T_e G$.
- In smooth manifolds, $T_e G$ is known as the **Lie algebra** of G, and so such an ω is a **Lie-algebra valued 1-form** on E.

Conclusions

- The general definition of connection can be formulated in tangent categories.
- On specific types of bundles, the general definition can be expressed in different ways which mirror the classical definitions.
- I believe the general definition is the easiest to understand.
- One can also describe notions of curvature and parallel transport for these general connections in tangent categories.

References

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