Differential restriction categories II

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Given at FCMS 2010, Kananaskis

July 5, 2010

1 Introduction

Recall that the idea of a cartesian differential category is to, in a sense, "algebraicize" differential calculus, so that results can be obtained by simple algebraic manipulations, rather than using limits or geometry. Of course, another goal is to unify different ideas of "differential" that have appeared in other areas: see, for example, the differential lambda calculus of [Erhard and Regnier 2003].

We are continuing this project by adding restrictions to cartesian differential categories, so that we can talk about smooth *partial* maps, rather than merely smooth totally defined maps. In the previous talk, Jonathan described the axioms for a differential restriction category, and gave a brief description of what each axioms tells us:

Definition 1.1 *A differential restriction category is a cartesian left additive restriction category with a differential operator*

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D[f]]{} Y}$$

such that

[D.1] $D[f+g] = D[f] + D[g]$ *and* $D[0] = 0$ *(additivity of the differential operator);*

[D.2] $\langle g+h, k\rangle D[f] = \langle g, k\rangle D[f] + \langle h, k\rangle D[f]$ *and* $\langle 0, g\rangle D[f] = \overline{gf}0$ *(additivity of the derivative);*

[D.3] $D[\pi_0] = \pi_0\pi_0$*, and* $D[\pi_1] = \pi_0\pi_1$ *(derivatives of projections);*

[D.4] $D[\langle f, g\rangle] = \langle D[f], D[g]\rangle$ *(derivative of a pairing);*

[D.5] $D[fg] = \langle D[f], \pi_1 f\rangle D[g]$ *(chain rule);*

[D.6] $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle D[D[f]] = \overline{h}\langle g, k\rangle D[f]$ *(linearity of the derivative at a point);*

[D.7] $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle D[D[f]] = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle D[D[f]]$*; (symmetry of second partial derivatives);*

[D.8] $D[\overline{f}] = (1 \times \overline{f})\pi_0$ *(derivative of a restriction);*

[D.9] $\overline{D[f]} = 1 \times \overline{f}$ *(restriction of a derivative);*

(note we are using diagrammatic order of composition).

Keep in mind, of course, that there are two standard examples. The standard differential calculus example:

Example 1.2 *Natural numbers as objects, and a map $n \longrightarrow m$ is a smooth map from an open subset of $\mathcal{R}^n$ to $\mathcal{R}^m$, with the Jacobian as the differential operator.*

And algebraic geometry examples:

Example 1.3 *For any commutative ring D, natural numbers as objects, and a map $n \longrightarrow m$ consists of m rational functions of n variables, with the restriction describing the places the rational function is undefined, and the differential operator similar to the Jacobian.*

Note that while the models are analytic, or geometric, the end result is completely algebraic: proofs are given solely by manipulating the nine axioms above, rather than using limits or geometry.

As an example, here's the proof that the derivative of any constant map $1 \xrightarrow{x} X$ is 0:

$$D[x] = \langle 0, 0 \rangle D[x] = 0$$

where the first equality is since a map between terminal objects is neccesarily the 0 map, and the second is by axiom D2.

In this talk, I will briefly describe some interesting initial aspects of the theory of differential restriction categories. The first is the nature of partially additive and partially linear maps: there is some surprise in the most natural definitions of these. The second is what happens when we complete the restriction category in various ways, adding in more maps. Here, differential structure is preserved, so that one gets further extremely interesting examples of differential restriction categories, and a new perspective on an old definition.

2 Partial additive and linear maps

Our goal is to to able to describe the sense in which a map like $f(x) = 2x$, defined everywhere but $x = 5$, is linear or additive. Let us begin by recalling the definition of an additive map:

Definition 2.1 *A map f in a left additive category is **additive** if for all x, y,*

$$(x + y)f = xf + yf \text{ and } 0f = 0.$$

That is, it preserves addition and 0.

What would happen if we used this definition in a left additive *restriction* category? Well, we would be asking that the restrictions are equal, so that

$$\overline{(x+y)f} = \overline{xf+yf} = \overline{xf}\,\overline{yf}\,.$$

That is, $f(x+y)$ is defined precisely when both $f(x)$ and $f(y)$ are. Obviously, one direction of this implication is not useful. Unless the map is total, it would be strange to ask that if $f(x+y)$ is defined, then $f(x)$ and $f(y)$ must be as well. A more natural condition is thus

$$xf + yf \leq (x+y)f \text{ and } 0f = 0$$

that is, if $f(x)$ and $f(y)$ are defined, then so is $f(x+y)$, and their values are equal. In other words, in this definition, the domain is additive closed, and the map preserves the addition.

Even this, however, is too strong, a map may preserve addition while not having its domain additively closed (consider the example above!). Thus, we should only ask that when all the terms are defined, they are equal. Thus, the following is the correct definition of additive in a left additive restricion category:

Definition 2.2 *A map f in a left additive restriction category is **additive** if*

$$xf + yf \smile (x+y)f \text{ and } 0f \smile 0.$$

(Recall that two maps are compatible, written $\smile$, if $\overline{f}\,g = \overline{g}\,f$, in other words, they are equal where they are both defined). Of course, the first definition is also important, as it describes being a monoid homorphism on a submonoid; we call these maps **strongly additive**. The point, however, is that the most natural definition is the more general additive maps. Interestingly, this explicit definition of "preserving addition where it exists" is not found in classical texts.

What about linear maps? Recall the definition of linear in a cartesian differential category:

Definition 2.3 *A map f in a cartesian differential category is **linear** if $D[f] = \pi_0 f$.*

So, that, for example, $f(x) = 3x$ is linear since $D[f](x,y) = 3x$. What would happen if we used this definition in a differential restiction category? We would be asking that

$$\overline{D[f]} = \overline{\pi_0 f} \text{ that is, } 1 \times \overline{f} = \overline{f} \times 1.$$

Obviously this never occurs unless f is total. However, notice also that there is no implication bias as there was for additive maps: neither direction should imply the other. Thus, the correct definition of linear is again merely asking for compatibility.

Definition 2.4 *A map f in a differential restriction category is **linear** if $D[f] \smile \pi_0 f$.*

Then a map like $f(x) = 5x$, defined everywhere but $x = 5$, is linear. Again, this notion of "linear where defined" exists implicitly in differential calculus, but is not explicitly given by an equation as we have done above.

Notice, then, then as for cartesian differential categories, linear implies additive, but only in the weak sense. To give an example of how to prove things in a differential restriction category, here is the proof of that fact:

Proposition 2.5 *If f is linear, then f is additive.*

PROOF: For the 0 axiom:

$$
\begin{aligned}
0f &= \overline{0f}\,0f \\
&= \overline{\langle 0,0\rangle\pi_1 f}\,\langle 0,0\rangle\pi_0 f \\
&= \langle 0,0\rangle\overline{\pi_1 f}\,\pi_0 f \text{ by R4,} \\
&\leq \langle 0,0\rangle D[f] \text{ since } f \text{ linear,} \\
&= \overline{0f}\,0 \text{ by D2,} \\
&\leq 0
\end{aligned}
$$

and for the addition axiom:

$$
\begin{aligned}
\overline{(x+y)f)}(xf+yf) &= \overline{(x+y)f}(\overline{xf}xf + \overline{y}\,\overline{xf}\,\overline{x}yf) \\
&= \overline{(x+y)f}(\overline{xf}xf + \overline{\overline{y}xf}\,\overline{x}yf) \\
&= \overline{(x+y)f}(\overline{\langle x,x\rangle\pi_1 f}\langle x,x\rangle\pi_0 f + \overline{\langle y,x\rangle\pi_1 f}\langle y,x\rangle\pi_0 f) \\
&= \overline{(x+y)f}(\langle x,x\rangle\overline{\pi_1 f}\pi_0 f + \langle y,x\rangle\overline{\pi_1 f}\pi_0 f) \\
&\leq \overline{(x+y)f}(\langle x,x\rangle D[f] + \langle y,x\rangle D[f]) \text{ since } f \text{ is linear} \\
&= \overline{\langle x+y,x\rangle\pi_0 f}\langle x+y,x\rangle D[f] \text{ by D2} \\
&= \langle x+y,x\rangle\overline{\pi_0 f}D[f] \\
&= \langle x+y,x\rangle\overline{\pi_1 f}\pi_0 f \text{ since } f \text{ is linear} \\
&= \overline{x+y,x\rangle\pi_1 f}\langle x+y,x\rangle\pi_0 f \\
&= \overline{\overline{x+y}xf}\,\overline{x}(x+y)f \\
&\leq (x+y)f
\end{aligned}
$$

□

Another interesting result is the following: while one can show that partial inverses of linear maps are neccesarily linear, the same is not true for additive maps. This identifies an interesting seperation between the notions of "preserving addition" and "having a constant derivative": while the first is not strong enough to be retained by partial inverses, the second is. The point of all this is that many of these interesting subtleties, while lurking in the background of differential calculus, do not directly appear until one moves to the strictly algebraic setting of differential restriction categories.

3 Completion Processes

In the second part of the talk, I would like to describe what happens to a differential restricion category when one completes it in two different ways: first, by freely adding joins, and secondly, by freely adding relative complements. In both cases, differential structure is preserved. Particularly in the second case, this leads to another interesting example of a differential restriction category, and a new view on a classical definition.

Recall that, in general, a restriction need not have joins of compatible maps. Smooth functions defined on open subsets $\mathcal{R}^n$'s does (just take the union of the domains), but the rational functions need not, unless D is a UFD. However, there is a way to freely add joins to any restriction category.

Definition 3.1 *Given a restriction category* $\mathbb{X}$*, define* $\mathbf{Jn}(\mathbb{X})$ *to have:*

- *objects: those of* $\mathbb{X}$*;*
- *an arrow* $X \xrightarrow{A} Y$ *is a subset* $A \subseteq \mathbb{X}(X,Y)$ *such that* A *is down-closed (under the restriction order), and elements are pairwise compatible;*
- $X \xrightarrow{1_X} X$ *is given by the down-closure of the identity,* $\downarrow 1_X$*;*
- *the composite of* A *and* B *is* $\{fg : f \in A, g \in B\}$*;*
- *restriction of* A *is* $\{\overline{f} : f \in A\}$*;*
- *the join of* $(A_i)_{i \in I}$ *is given by the union of the* A_i*.*

This gives a left adjoint to the forgetful functor from join restriction categories to restriction categories.

Now, it is fairly straightforward to show that

Theorem 3.2 *If* $\mathbb{X}$ *has differential restriction structure, then so does* $\mathbf{Jn}(\mathbb{X})$*, with*

$$D[A] := \downarrow \{D[f] : f \in A\}$$

(the set $\{D[f] : f \in A\}$ need not be down-closed, so we must take its down closure). While easy to show, this is an important result, as it says we can add in the join operation to a restriction category without disturbing the existence of derivatives.

More interesting, however, is adding relative complements. Roughly, if $f' \leq f$, a *relative complement* of f' in f is a map which has the same values as f, but is defined everywhere where f' is not. A join restriction category with relative complements is a **classical restriction category**. Notice that smooth functions defined on open subsets of $\mathcal{R}^n$ is not classical, as the complement of an open subset inside another open set is not neccesarily open.

Just as for joins, though, there is a completion process which adds relative complements.

Definition 3.3 *Let* $\mathbb{X}$ *be a join restriction category. A* **classical piece** *of* $\mathbb{X}$ *is a pair of maps* $(f, f') : A \longrightarrow B$ *such that* $f' \leq f$*.*

One thinks of a classical piece as a formal relative complement $f \setminus f'$.

Definition 3.4 *Two classical pieces* $(f,f'), (g,g')$ *are* **disjoint***, written* $(f,f') \perp (g,g')$*, if* $\overline{f}\,\overline{g} = \overline{f'}\,\overline{g} \vee \overline{f}\,\overline{g'}$*. A* **raw classical piece** *consists of a finite set of classical pieces,* (f_i, f_i') *that are pairwise disjoint, and is written*

$$\bigsqcup_{i \in I} (f_i, f_i') : A \longrightarrow B.$$

Merely taking all raw classical maps is too much, however: many different joins of maps should be the same. In particular, one defines an equivalence relation on the set of raw classical maps by:

- **Breaking:** $(f, f') \equiv (f, f' \vee fe) \sqcup (ef, ef')$ for any restriction idempotent $e = \overline{e}$,
- **Collapse:** $(f, f) \equiv \emptyset$.

Definition 3.5 *A* **classical map** *is an equivalence class of raw classical maps.*

Proposition 3.6 *Given a join restriction category* $\mathbb{X}$*, there is a classical restriction category* $\mathbf{Cl}(\mathbb{X})$ *with*

- *objects those of* $\mathbb{X}$*,*
- *arrows classical maps,*
- *composition by*
$$\bigsqcup_{i\in I}(f_i, f_i') \bigsqcup_{j\in J}(g_j, g_j') := \bigsqcup_{i,j}(f_i g_j, f_i' g_j \vee f_i g_j'),$$
- *restriction by*
$$\overline{\bigsqcup_{i\in I}(f_i, f_i')} := \bigsqcup_{i\in I}(\overline{f_i}, \overline{f_i'}),$$
- *disjoint join is simply* $\sqcup$ *of classical pieces,*
- *relatative complement is*
$$(f, f') \setminus (g, g') := (f, f' \vee \overline{g}\, f) \sqcup (\overline{g'}\, f, \overline{g'}\, f').$$

This is the left adjoint to the forgetful functor from classical restriction categories to join restriction categories.

Clearly, this is a much more complicated process than the join completion. Even here, however, differential structure is preserved.

Theorem 3.7 *If* $\mathbb{X}$ *has differential restriction structure, then so does* $\mathbf{Cl}(\mathbb{X})$*, with*
$$D[(f', f)] := (D[f'], D[f])$$

Obviously, the proof requires considerably more work than the join completion, as the classical completion is defined with equivalence classes of maps. Again, the result itself is important, as we can add in a useful operation and not disturb the differential structure. However, an interesting question remains: what does this process do to smooth functions defined on open subsets of $\mathcal{R}^n$?

For example, consider $f'(x) = 2x$ defined everywhere but $x = 5$, and $f(x) = 2x$ defined everywhere. Taking a relative complement would give us a map defined *only* at $x = 5$, and has the value $2x = 10$ there. But if differential structure is retained, in what sense is this map "smooth"?

The answer, of course, is that this map is really an equivalence class of maps. In particular, imagine we have a restriction idempotent $e = \overline{e}$ (ie open subset), which includes 5. Then we have

$$(f, f') \equiv (ef, ef') \sqcup (f, f' \vee ef) = (ef, ef') \sqcup (f, f) \equiv (ef, ef')$$

So that this map is actually equivalent to any other map defined on an open subset which includes 5. This is precisely the definition of the *germ* of a function at 5. Thus, the classical completion process adds germs of functions at points.

Of course, it also allows us to take joins of germs and regular maps, so that for example we could take the join of the above map, and something like $\frac{x-1}{x-5}$, giving a total map which has "repaired" the discontinuity of the second map at 5. The fact that this restriction category is a differential restricion category, is now a remarkable fact! It tells us that there is a restriction category with regular smooth functions, with germs of functions, and with "repaired" functions, which has a differential operator that acts the same way as regular smooth maps.

In conclusion, then, the act of specifically axiomatizing what it means for a category to be a category of smooth partially defined maps leads us to new persepectives and interesting ideas on classical mathematics, and opens ups the possibility of looking for highly non-standard models of differential calculus.

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