

# Differential Restriction Categories

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## Abstract

We combine two recent ideas: cartesian differential categories, and restriction categories. The result is a new structure which axiomatizes the category of smooth maps defined on open subsets of  $\mathcal{R}^n$  in a way that is completely algebraic. We also give other models for the resulting structure, discuss what it means for a partial map to be additive or linear, and show that differential restriction structure can be lifted through various completion operations.

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## 1 Introduction

In [Blute et. al. 2008], the authors proposed an alternative way to view differential calculus. The derivative was seen as an operator on maps, with many of its typical properties (such as the chain rule) axioms on this operation. The resulting categories were called cartesian differential categories, and the standard model is smooth maps between the spaces  $\mathcal{R}^n$ . One interesting aspect of this project was the algebraic feel it gave to differential calculus. The seven axioms of a cartesian differential category described all the necessary properties that the standard Jacobian has. Thus, instead of reasoning with epsilon arguments, one could reason about calculus by manipulating algebraic axioms.

Moreover, as shown in [Bucciarelli et. al. 2010], cartesian (closed) differential categories provide a semantic basis for modeling the simply typed differential lambda-calculus described in [Erhard and Regnier 2003]. This latter calculus is linked to various resource calculi which, as their name suggests, are useful in understanding the resource requirements of programs. Thus, models of computation in settings with a differential operator are of interest in the semantics of computation when resource requirements are being considered.

Fundamental to computation is the possibility of non-termination. Thus, an obvious extension of cartesian differential categories is to allow partiality of maps. Of course, this has a natural analogue in the standard model: smooth maps defined on open subsets of  $\mathcal{R}^n$  are a notion of partial smooth map which is ubiquitous in analysis.

To axiomatize these ideas, we combine cartesian differential categories with the restriction categories of [Cockett and Lack 2002]. Again, the axiomatization is completely algebraic: there are two operations (differentiation and restriction) that satisfy seven axioms for the derivative, four for

the restriction, and two for the interaction of derivation and restriction.

Our goal in this paper is not only to give the definitions and examples of these “differential restriction categories”, but also to show how natural the structure is. There are a number of points of evidence for this claim. In a differential restriction category, one can define what it means for a partial map such as

$$f(x) = \begin{cases} 2x & \text{if } x \neq 5; \\ \uparrow & \text{if } x = 5. \end{cases}$$

to be “linear”. One can give a similar description for the notion of “additive”. The differential interacts so well with the restriction that not only does it preserve the order and compatibility relations, it also preserves joins of maps, should they exist.

Moreover, differential restriction structure is surprisingly robust<sup>1</sup>. In the final two sections of the paper, we show that differential structure lifts through two completion operations on restriction categories. The first completion is the join completion, which freely add joins of compatible maps to a restriction category. We show that if differential structure is present on the original restriction category, then one can lift this differential structure to the join completion.

The second completion operation is much more drastic: it adds “classical” structure to the restriction category, allowing one to classically reason about the restriction category’s maps. Again, we show that if the original restriction category has differential structure, then this differential structure lifts to the classical setting. This is perhaps the most surprising result of the paper, as one typically thinks of differential structure as being highly non-classical. In particular, it is not obvious how differentials of functions defined at a single point should work. We show that what the classical completion is doing is adding germs of functions, so that a function defined on a point (or a closed set) is defined by how it works on any open set around that point (or closed set). It is these germs of functions on which one can define differential restriction structure.

The paper is laid out as follows. In Section 2, we review the theory of restriction categories. This includes reviewing the notions of joins of compatible maps, as well as the notion of a cartesian restriction category.

In Section 3, we define differential restriction categories. We must begin, however, by defining left additive restriction categories. Left additive categories are categories in which it is possible to add two maps, but the maps themselves need not preserve the addition (for example, the set of smooth maps between  $\mathcal{R}^n$ ). Such categories were an essential base for defining cartesian differential categories, as the axioms need to discuss what happens when maps are added. Here, we describe left additive restriction categories, in which the maps being added may only be partial. One interesting aspect of this section is the definition of additive maps (those maps which do preserve the addition), which is slightly more subtle than its total counterpart.

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<sup>1</sup>With the exception of being preserved when we take manifolds. Understanding what happens when we take manifolds of a differential restriction category will be considered in a future paper: see the concluding section of this paper for further remarks.

With the theories of cartesian restriction categories and left additive restriction categories described, we are finally able to define differential restriction categories. One surprise is that the differential automatically preserves joins. Again, as with additive maps, the definition of linear is slightly more subtle than its total counterpart. We also show that rational functions over a commutative ring forms a differential restriction category.

In Section 4, we extensively develop a family of differential restriction categories: rational functions over a commutative ring. Rational functions, having “poles”, are a natural candidate for restriction structure. We show that the natural formal derivative on these functions, together with this restriction, forms a differential restriction category. While many of the ideas of this section are implicit in algebraic geometry, focusing on differential and restriction structure makes the ideas explicit.

In the next two sections, we describe what happens when we join or classically complete the underlying restriction category of a differential restriction category, and show that the differential structure lifts in both cases. Again, this is important, as it shows how robust differential restriction structure is, as well as allowing one to differentiate in a classical setting.

Finally, in 7, we discuss further work. In particular, the next step will be to use differential restriction categories and the manifold completion process of [Grandis 1989] to define smooth manifolds.

On that note, we would like to compare our approach to other categorical theories of smooth maps. Lawvere’s synthetic differential geometry (carried out in [Dubuc 1979], [Kock 2006], and [Moerdijk and Reyes 1991]) is one such example. The notion of smooth topos is central to Lawvere’s program. A smooth topos is a topos which contains an object of “infinitesimals”. One thinks of this object as the set  $D = \{x : x^2 = 0\}$ . Smooth toposes give an extremely elegant approach to differential geometry. For example, one defines the tangent space of an object  $X$  to be the exponential  $X^D$ . This essentially makes the tangent space the space of all infinitesimal paths in  $X$ , which is precisely the intuitive notion of what the tangent space is.

The essential difference between the synthetic differential geometry approach and ours is the level of power of the relative settings. A smooth topos is, in particular, a topos, and so enjoys a great number of powerful properties. The differential restriction categories we describe here have fewer assumptions: we only ask for finite products, and assume no closed structure or subobject classifier. Thus, our approach begins at a much more basic level. While the standard model of a differential restriction category is smooth maps defined on open subsets of  $\mathcal{R}^n$ , the standard model of a smooth topos is a certain completion of smooth maps between all smooth manifolds. In contrast to the synthetic differential geometry approach, our goal is thus to see at what minimal level differential calculus can be described, and only then move to more complicated objects such as smooth manifolds.

A number of authors have described others notions of smooth space: see, for example, [Chen 1977], [Frölicher 1982], [Sikorski 1972]. All have a similar approach, and the similarity is summed up in [Stacey 2008]:

“...we know what it means for a map to be smooth between certain subsets of Euclidean space and so in general we declare a function smooth if whenever, we examine it using those subsets, it is smooth. This is a rather vague statement - what do we mean by ‘examine’? - and the various definitions can all be seen as ways of making this precise.”

Thus, in each of these approaches, the author assumes an existing knowledge of smooth maps defined on open subsets of  $\mathcal{R}^n$ . Again, our approach is more basic: we are seeking to understand the nature of these smooth maps between  $\mathcal{R}^n$ . In particular, one could define Chen spaces, or Frölicher spaces, based on a differential restriction category other than the standard model, and get new notions of generalised smooth space.

Finally, it is important to note that none of these other approaches work with *partial* maps. Our approach, in addition to starting at a more primitive level, gives us the ability to reason about the partiality of maps which is so central to differential calculus, geometry, and computation.

## 2 Restriction categories review

In this section, we begin by reviewing the theory of restriction categories. Restriction categories were first described in [Cockett and Lack 2002] as an alternative to the notion of a “partial map category”. In a partial map category, one thinks of a partial map from  $A$  to  $B$  as a span

$$\begin{array}{ccc} & A' & \\ m \swarrow & & \searrow f \\ A & & B \end{array}$$

where the arrow  $m$  is a monic. Thus,  $A'$  describes the domain of definition of the partial map. By contrast, a restriction category is a category which has to each arrow  $f : A \longrightarrow B$  a “restriction”  $\bar{f} : A \longrightarrow A$ . One thinks of this  $\bar{f}$  as giving the domain of definition: in the case of sets and partial functions, the map  $\bar{f}$  is given by

$$\bar{f}x = \begin{cases} x & \text{if } f(x) \text{ defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

There are then four axioms which axiomatize the behaviour of these restrictions (see below).

There are two advantages of restriction categories when compared to partial map categories. The first is that they are more general than partial map categories. In a partial map category, one needs to have as objects each of the possible domains of definition of the partial functions. In a restriction category, this is not the case, as the domain is expressed by the restrictions. This is important for the examples considered below. In particular, the canonical example of a differential restriction category will have objects the spaces  $\mathcal{R}^n$ , and maps the smooth maps defined on open subsets of these spaces. This is not an example of a partial map category, as the open subsets are not objects, but it is naturally a restriction category, with the same restriction as for sets and partial functions.

The second advantage is that the theory is completely algebraic. In partial map categories, one deals with equivalence classes of spans and their pullbacks. As a result, they are often difficult to work with directly. In a restriction category, one simply manipulates equations involving the restriction operator, using the four given axioms. As cartesian differential categories give a completely algebraic description of the derivatives of smooth maps, bringing these two algebraic theories together is a natural approach to capturing smooth maps which are partially defined.

## 2.1 Definition and examples

Restriction categories are axiomatized as follows. Note that throughout this paper, we are using diagrammatic order of composition, so that “ $f$ , followed by  $g$ ”, is written  $fg$ .

**Definition 2.1** *Given a category,  $\mathbb{X}$ , a **restriction structure** on  $\mathbb{X}$  gives for each,  $A \xrightarrow{f} B$ , a restriction arrow,  $A \xrightarrow{\bar{f}} A$ , that satisfies four axioms:*

[R.1]  $\bar{f}f = f$ ;

[R.2] *If  $\text{dom}(f) = \text{dom}(g)$  then  $\bar{g}\bar{f} = \bar{f}\bar{g}$ ;*

[R.3] *If  $\text{dom}(f) = \text{dom}(g)$  then  $\overline{\bar{g}f} = \bar{g}\bar{f}$ ;*

[R.4] *If  $\text{dom}(g) = \text{cod}(f)$  then  $f\bar{g} = \overline{fg}$ .*

*A category with a specified restriction structure is a **restriction category**.*

We have already seen two examples of restriction categories: sets and partial functions, and smooth functions defined on open subsets of  $\mathcal{R}^n$ . Many more examples can be found in [Cockett and Lack 2002], as well as in [Cockett and Hofstra 2008], where restriction categories are used to describe categories of partial computable maps.

A rather basic fact is that each restriction  $\bar{f}$  is idempotent. We record this together with some other basic consequences of the definition:

**Lemma 2.2** *If  $\mathbb{X}$  is a restriction category then:*

- (i)  $\bar{f}$  is idempotent;
- (ii)  $\bar{f}\bar{f}g = \bar{f}g$ ;
- (iii)  $\overline{\bar{f}g} = \bar{f}g$ ;
- (iv)  $\overline{\bar{f}} = \bar{f}$ ;
- (v)  $\overline{\bar{f}\bar{g}} = \bar{f}\bar{g}$ ;
- (vi) *If  $f$  is monic then  $\bar{f} = 1$  (and so in particular  $\bar{1} = 1$ );*
- (vii)  $\bar{f}g = g$  implies  $\bar{g} = \bar{f}\bar{g}$ .

PROOF:

- (i) By [R.3] and [R.1] we have  $\overline{f} \overline{f} = \overline{f f} = \overline{f}$ .
- (ii) By [R.3] and [R.1] we have  $\overline{f} \overline{f g} = \overline{f f g} = \overline{f g}$ .
- (iii) We use [R.4], [R.3], and (ii) to conclude  $\overline{f g} = \overline{\overline{f g} f} = \overline{f g f} = \overline{f g}$ .
- (iv) By (iii) we have  $\overline{f} = \overline{1 f} = \overline{1 \overline{f}} = \overline{\overline{f}}$ .
- (v) By [R.3] we have  $\overline{\overline{f g}} = \overline{f g} = \overline{f} \overline{g}$ ;
- (vi) Since  $\overline{f} f = 1 f$ , when  $f$  is monic we conclude that  $\overline{f} = 1$ .
- (vii) By [R.3] we have  $\overline{g} = \overline{\overline{f g}} = \overline{f} \overline{g}$ .

□

## 2.2 Partial map categories

As alluded to in the introduction to this section, an alternative way of axiomatizing categories of partial maps is via spans where one leg is a monic. We recall this notion here. These will be important, as we shall see that rational functions over a commutative rig naturally embed in a larger partial map category.

**Definition 2.3** *Let  $\mathbb{X}$  be a category, and  $\mathcal{M}$  a class of monics in  $\mathbb{X}$ .  $\mathcal{M}$  is a **stable system of monics** in case*

**SSM1** *All isomorphisms are in  $\mathcal{M}$*

**SSM2**  *$\mathcal{M}$  is closed to composition*

**SSM3** *For any  $m : B' \rightarrow B \in \mathcal{M}$ ,  $f : A \rightarrow B \in \mathbf{C}$  the following pullback, called an  $\mathcal{M}$ -pullback, exists and  $m' \in \mathcal{M}$ :*

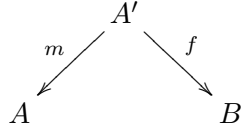
$$\begin{array}{ccc} A' & \xrightarrow{f} & B' \\ m' \downarrow & & \downarrow m \\ A & \xrightarrow{f} & B \end{array}$$

**Definition 2.4** *An  $\mathcal{M}$ -Category is a pair  $(\mathbb{X}, \mathcal{M})$  where  $\mathbb{X}$  is a category with a specified system of stable monics  $\mathcal{M}$ .*

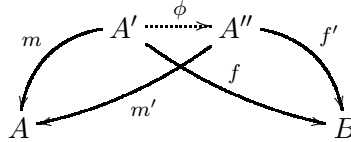
**Definition 2.5** *Let  $(\mathbb{X}, \mathcal{M})$  be an  $\mathcal{M}$ -Category. Define  $\text{Par}(\mathbb{X}, \mathcal{M})$  to be the category where*

**Obj:** *The objects of  $\mathbb{X}$*

**Arr:**  $A \xrightarrow{(m,f)} B$  are classes of spans  $(m, f)$ ,

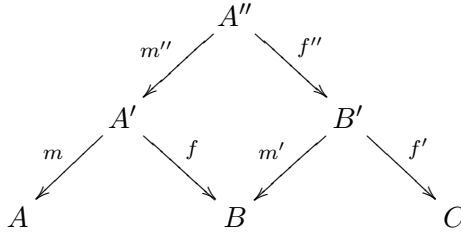


where  $m \in \mathcal{M}$ . The classes of spans are quotiented by the equivalence relation  $(m, f) \sim (m', f')$  if there is an isomorphism,  $\phi$ , such that both triangles in the following diagram commute.



**Id:**  $A \xrightarrow{(1_A, 1_A)} A$

**Comp:** By pullback; i.e. given  $A \xrightarrow{(m,f)} B, B \xrightarrow{(m',f')} C$ , the pullback



gives a composite  $A \xrightarrow{(m''m, f''f')} C$ . (Note that without the equivalence relation on the arrows, the associative law would not hold.)

Moreover, this has restriction structure: given an arrow  $(m, f)$ , we can define its restriction to be  $(m, m)$ . From [Cockett and Lack 2002], we have the following completeness result:

**Theorem 2.6** *Every restriction category is a full subcategory of a category of partial maps.*

However, it is not true that every full subcategory of a category of partial maps is a category of partial maps, so the restriction notion is more general.

### 2.3 Joins of compatible maps

An important aspect of the theory of restriction categories is the idea of the join of two compatible maps. We first describe what it means for two maps to be compatible, that is, equal where they are both defined.

**Definition 2.7** *Two parallel maps  $f, g$  in a restriction category are compatible, written  $f \smile g$ , if  $\overline{f}g = \overline{g}f$ .*



Note that compatibility is *not* transitive. Recall also the notion of when a map  $f$  is less than or equal to a map  $g$ :

**Definition 2.8**  $f \leq g$  if  $\overline{f}g = f$ .

This captures the notion of  $g$  having the same values as  $f$ , but having a smaller domain of definition. An important alternative characterization of compatibility is the following:

**Lemma 2.9** *In a restriction category,*

$$f \smile g \Leftrightarrow \overline{f}g \leq f \Leftrightarrow \overline{g}f \leq g.$$

PROOF: If  $f \smile g$ , then  $\overline{f}g = \overline{g}f \leq f$ . Conversely, if  $\overline{f}g \leq f$ , then by definition,  $\overline{\overline{f}g} = \overline{f}g$ , so  $\overline{g}f = \overline{f}g$ .  $\square$

Intuitively, the join of two compatible maps  $f$  and  $g$  will be a map which is defined everywhere  $f$  and  $g$  are, and takes their common value where both  $f$  and  $g$  are defined. Naturally, there will also be the concept of a nullary join; that is, a nowhere-defined map. An arbitrary restriction category need not have joins.

**Definition 2.10** *Let  $\mathbb{X}$  be a restriction category.  $\mathbb{X}$  is a **finite join restriction category** if each homset  $\mathbb{X}(A, B)$  has a map  $\emptyset_{AB}$  such that*

$$\mathbf{J1} \text{ for any } f : A \longrightarrow B, \emptyset_{AB} \leq f;$$

$$\mathbf{J2} \text{ for any } f : A \longrightarrow B, f\emptyset_{BC} = \emptyset_{AC};$$

*and for any two compatible maps  $f, g : A \longrightarrow B$ , there is a join  $f \vee g : A \longrightarrow B$  such that*

$$\mathbf{J3} \text{ } f \leq f \vee g;$$

$$\mathbf{J4} \text{ } g \leq f \vee g;$$

$$\mathbf{J5} \text{ If } h : A \longrightarrow B \text{ such that } f, g \leq h, \text{ then } (f \vee g) \leq h;$$

$$\mathbf{J6} \text{ For any } C \xrightarrow{s} A, s(f \vee g) = (sf) \vee (sg).$$

Obviously, sets and partial functions have all joins - simply take the union of the domains of the compatible maps. Similarly, since the union of open sets is open, smooth functions on open subsets also have joins.

Note that the definition only asks for compatibility of joins with composition on the left. In the following proposition, we show that this implies compatibility with composition on the right, as well as establish several other useful results.

**Proposition 2.11** *In any join restriction category,*

$$(i) \overline{\emptyset} = \emptyset;$$

$$(ii) \text{ For every } g, \emptyset g = \emptyset;$$

$$(iii) \overline{f \vee g} = \overline{f} \vee \overline{g};$$

(iv)  $(f \vee g)h = (fh) \vee (gh)$ .

(v) if  $\bar{g} = \emptyset$ , then  $g = \emptyset$ ;

(vi) if  $(f_i)_{i \in I}$  is a finite compatible family of arrows,

$$\overline{f_j} \left( \bigvee_{i \in I} f_i \right) = f_j$$

for any  $j \in I$ .

PROOF:

(i)  $\emptyset \leq \overline{\emptyset}$  so that  $\emptyset = \overline{\emptyset}$ .

(ii)  $\emptyset g = \overline{g\emptyset} = g\overline{\emptyset} = g\emptyset = \emptyset$ ;

(iii) It is clear that  $\overline{f \vee g} \leq \overline{f \vee \overline{g}}$ , and so  $\overline{f \vee g} = \overline{\overline{f \vee \overline{g}}}$ . We must show the reverse inequality

$$\begin{aligned} \overline{f \vee g}(\overline{f \vee \overline{g}}) &= \overline{f \vee g}(\overline{f \vee \overline{g}}) \\ &= \overline{(\overline{f \vee \overline{g}})(f \vee g)} = \overline{(\overline{f \vee \overline{g}})f \vee (\overline{f \vee \overline{g}})g} \\ &= \overline{f(\overline{f \vee \overline{g}})f \vee \overline{g}(\overline{f \vee \overline{g}})g} = \overline{ff \vee \overline{g}gf \vee \overline{g}} \end{aligned}$$

(iv) Again it is clear that  $(fh) \vee (gh) \leq (f \vee g)h$ , we must show that the reverse inequality holds. To do this we shall first establish that  $(f \vee g)\overline{h} = (f\overline{h}) \vee (g\overline{h})$  as

$$\begin{aligned} (f \vee g)\overline{h} &= \overline{(f \vee g)\overline{h}(f \vee g)} = \overline{((f \vee g)\overline{h}f \vee (f \vee g)\overline{h}g)} \\ &= \overline{((f \vee g)\overline{h}f \vee (f \vee g)\overline{h}g)} = \overline{f(f \vee g)\overline{h}f \vee \overline{g}(f \vee g)\overline{h}g} \\ &= \overline{f(f \vee g)\overline{h}f \vee \overline{g}(f \vee g)\overline{h}g} = \overline{f\overline{h}f \vee \overline{g}\overline{h}g} = \overline{f\overline{h} \vee \overline{g}\overline{h}} \end{aligned}$$

It remains to show that  $(f \vee g)h \leq (fh) \vee (gh)$  and to show this it suffices to show  $\overline{(f \vee g)h} = \overline{(fh) \vee (gh)}$  but for this we have:

$$\begin{aligned} \overline{(fh) \vee (gh)} &= \overline{f\overline{h} \vee \overline{g}\overline{h}} = \overline{f\overline{h} \vee \overline{g}\overline{h}} = \overline{f\overline{h} \vee \overline{g}\overline{h}} \\ &= \overline{(f \vee g)\overline{h}} = \overline{(f \vee g)h} \end{aligned}$$

(v) If  $\bar{g} = \emptyset$ , then  $\bar{g}g = \emptyset g$ , so  $g = \emptyset$ .

(vi) Consider:

$$\begin{aligned} &\overline{f_j} \left( \bigvee f_i \right) \\ &= \bigvee \overline{f_j} f_i \\ &= \bigvee \overline{f_i} f_j \text{ since each } f_i \text{ is compatible with } f_j \\ &= f_j \text{ since } \overline{f_i} f_j \leq f_j \text{ for each } i, \text{ and equal to } f_j \text{ for } i = j \end{aligned}$$

as required. □

## 2.4 Cartesian restriction categories

Not surprisingly, cartesian differential categories involve cartesian structure. Thus, to develop the theory which combines cartesian differential categories with restriction categories, it will be important to recall how cartesian structure interacts with restrictions. This was first described in [Cockett and Lack 2007], and we recall the basic idea here.

**Definition 2.12** *Let  $\mathbb{X}$  be a restriction category. A **restriction terminal object**  $1$  is an object in  $\mathbb{X}$  such that for any object  $A$ , there is a unique total map  $!_A : A \rightarrow 1$  which satisfies  $!_1 = id_1$ . Further, these maps  $!$  must satisfy the property that for any map  $f : A \rightarrow B$ ,  $f!_B \leq !_A$ , i.e.  $f!_B = \overline{f!_B} !_A = \overline{f!_B} !_A = \overline{f} !_A$ .*

A **restriction product** of objects  $A, B$  in  $\mathbb{X}$  is defined by total projections

$$\pi_0 : A \times B \rightarrow A \quad \pi_1 : A \times B \rightarrow B$$

satisfying the property that for any object  $C$  and maps  $f : C \rightarrow A, g : C \rightarrow B$  there is a pairing map,  $\langle f, g \rangle : C \rightarrow A \times B$  such that both triangles below exhibit lax commutativity

$$\begin{array}{ccc} & C & \\ f \swarrow & \vdots \langle f, g \rangle & \searrow g \\ A & \xrightarrow{\langle f, g \rangle} & A \times B & \xrightarrow{\pi_1} & B \\ \pi_0 \swarrow & & & & \searrow \end{array}$$

that is,

$$\langle f, g \rangle \pi_0 = \overline{\langle f, g \rangle} f \quad \text{and} \quad \langle f, g \rangle \pi_1 = \overline{\langle f, g \rangle} g.$$

In addition, we ask that  $\overline{\langle f, g \rangle} = \overline{f} \overline{g}$ .

We require lax commutativity as a pairing  $\langle f, g \rangle$  should only be defined as much as both  $f$  and  $g$  are.

**Definition 2.13** *A restriction category  $\mathbb{X}$  is a **cartesian restriction category** if  $\mathbb{X}$  has a restriction terminal object and all restriction products.*

Clearly, both sets and partial functions, and smooth functions defined on open subsets of  $\mathcal{R}^n$  are cartesian restriction categories.

The following contains a number of useful results.

**Proposition 2.14** *In any cartesian restriction category,*

- (i)  $\langle f, g \rangle \pi_0 = \overline{f} g$  and  $\langle f, g \rangle \pi_1 = \overline{g} f$ ;
- (ii) if  $e = \overline{e}$ , then  $e \langle f, g \rangle = \langle ef, g \rangle = \langle f, eg \rangle$ ;
- (iii)  $f \langle g, h \rangle = \langle fg, fh \rangle$ ;
- (iv) if  $f \leq f'$  and  $g \leq g'$ , then  $\langle f, g \rangle \leq \langle f', g' \rangle$ ;

(v) if  $f \smile f'$  and  $g \smile g'$ , then  $\langle f, g \rangle \smile \langle f', g' \rangle$ ;

(vi) if  $f$  is total, then  $(f \times g)\pi_1 = \pi_1 g$ . If  $g$  is total,  $(f \times g)\pi_0 = \pi_0 f$ .

PROOF:

(i) By the lax commutativity,  $\langle f, g \rangle \pi_0 = \overline{\langle f, g \rangle} f = \overline{f} \overline{g} f = \overline{g} f$  and similarly with  $\pi_1$ .

(ii) Note that

$$e\langle f, g \rangle \pi_0 = e\overline{g} f = \overline{e} \overline{g} f = \overline{e} \overline{g} f = \overline{e} \overline{g} f = \langle f, eg \rangle \pi_0$$

A similar result holds with  $\pi_1$ , and so by universality of pairing,  $e\langle f, g \rangle = \langle f, eg \rangle$ . By symmetry, it also equals  $\langle ef, g \rangle$ .

(iii) Note that

$$f\langle g, h \rangle \pi_0 = f\overline{h} g = \overline{f} \overline{h} g = \langle fg, fh \rangle \pi_0$$

where the second equality is by R4. A similar result holds for  $\pi_1$ , and so the result follows by universality of pairing.

(iv) Consider

$$\begin{aligned} & \overline{\langle f, g \rangle} \langle f', g' \rangle \\ &= \overline{f} \overline{g} \langle f', g' \rangle \text{ by (i)} \\ &= \langle f', \overline{f} \overline{g} g' \rangle \text{ by (ii)} \\ &= \langle f', \overline{f}, g \rangle \text{ since } g \leq g' \\ &= \langle \overline{f} f', g \rangle \text{ by (ii)} \\ &= \langle f, g \rangle \text{ since } f \leq f'. \end{aligned}$$

Thus  $\langle f, g \rangle \leq \langle f', g' \rangle$ .

(v) By Lemma 2.9, we only need to show that  $\overline{\langle f, g \rangle} \langle f', g' \rangle \leq \langle f, g \rangle$ . But, again by Lemma 2.9, we have  $\overline{f} f' \leq f$  and  $\overline{g} g' \leq g$ , so by (iv) we get  $\langle \overline{f} f', \overline{g} g' \rangle \leq \langle f, g \rangle$  and thus by (ii) and (i), we get  $\overline{\langle f, g \rangle} \langle f', g' \rangle \leq \langle f, g \rangle$ .

(vi)

$$(f \times g)\pi_1 = \langle \pi_0 f, \pi_1 g \rangle \pi_1 = \overline{\pi_0 f} \pi_1 g = \pi_1 g$$

□

If  $\mathbb{X}$  is a cartesian restriction category which also has joins, then the two structures are automatically compatible:

**Proposition 2.15** *In any cartesian restriction category with joins,*

(i)  $\langle f \vee g, h \rangle = \langle f, h \rangle \vee \langle g, h \rangle$  and  $\langle f, \emptyset \rangle = \langle \emptyset, f \rangle = \emptyset$ ;

(ii)  $(f \vee g) \times h = (f \times h) \vee (g \times h)$  and  $f \times \emptyset = \emptyset \times f = \emptyset$ .

PROOF:

(i) Since  $\overline{\langle f, \emptyset \rangle} = \overline{f} \overline{\emptyset} = \overline{f} \emptyset = \emptyset$ , by Proposition 2.11, we have  $\langle f, \emptyset \rangle = \emptyset$ . For pairing,

$$\begin{aligned}
\langle f \vee g, h \rangle &= \overline{\langle f \vee g, h \rangle} \langle f \vee g, h \rangle \\
&= \overline{f \vee g} \overline{h} \langle f \vee g, h \rangle \\
&= (\overline{f} \vee \overline{g}) \langle f \vee g, h \rangle \\
&= (\overline{f} \langle f \vee g, h \rangle) \vee (\overline{g} \langle f \vee g, h \rangle) \\
&= \langle \overline{f}(f \vee g), h \rangle \vee \langle \overline{g}(f \vee g), h \rangle \\
&= \langle f, h \rangle \vee \langle g, h \rangle
\end{aligned}$$

as required.

(ii) Using part (a),  $f \times \emptyset = \langle \pi_0 f, \pi_1 \emptyset \rangle = \langle \pi_0 f, \emptyset \rangle = \emptyset$  and

$$\begin{aligned}
(f \vee g) \times h &= \langle \pi_0(f \vee g), \pi_1 h \rangle \\
&= \langle (\pi_0 f) \vee (\pi_0 g), \pi_1 h \rangle \\
&= \langle \pi_0 f, \pi_1 h \rangle \vee \langle \pi_0 g, \pi_1 h \rangle \\
&= (f \times h) \vee (g \times h)
\end{aligned}$$

□

We shall see that this pattern continues with left additive and differential restriction categories: if the restriction category has joins, then it is automatically compatible with left additive or differential structure.

### 3 Differential restriction categories

Before we define differential restriction categories, we need to define left additive restriction categories. Left additive categories were introduced in [Blute et. al. 2008] as a precursor to differential structure. To axiomatize how the differential interacts with addition, one must define categories in which it is possible to add maps, but not have these maps necessarily preserve the addition (as is the case with smooth maps defined on real numbers). The canonical example of one of these left additive categories is the category of commutative monoids with *arbitrary* functions between them. These functions have a natural additive structure given pointwise:  $(f + g)(x) := f(x) + g(x)$ , as well as 0 maps:  $0(x) := 0$ . Moreover, while this additive structure does not interact well with postcomposition by a function, it does with precomposition:  $h(f + g) = hf + hg$ , and  $f0 = 0$ . This is essentially the definition of a left additive category.

#### 3.1 Left additive restriction categories

To define left additive *restriction* categories, we need to understand what happens when we add two partial maps, as well as the nature of the 0 maps. Intuitively, the maps in a left additive category are added pointwise. Thus, the result of adding two partial maps should only be defined where the original two maps were both defined. Moreover, the 0 maps should be defined everywhere. Thus, the most natural requirement for the interaction of additive and restriction structure is that  $\overline{f + g} = \overline{f} \overline{g}$ , and that the 0 maps be total.

**Definition 3.1**  $\mathbb{X}$  is a left additive restriction category if each  $\mathbb{X}(A, B)$  is a commutative monoid which in its interaction with the restriction satisfies  $\overline{f+g} = \overline{f}\overline{g}$  and  $\overline{0} = 1$ , and furthermore is left additive:  $f(g+h) = fg + fh$  and  $f0 = \overline{f}0$ .

It is important to note the difference between the last axiom ( $f0 = \overline{f}0$ ) and its form for left additive categories ( $f0 = 0$ ).  $f0$  need not be total, so rather than ask that this be equal to 0 (which is total), we must instead ask that  $f0 = \overline{f}0$ . This phenomenon will return when we define differential restriction categories. In general, any time an axiom is not linear, we must modify the axiom to include the restrictions of the maps that are lost.

There are two obvious examples of left additive restriction categories: commutative monoids with arbitrary partial functions between them, and the subcategory of these consisting of continuous or smooth functions defined on open subsets of  $\mathcal{R}^n$ .

Some results about left additive structure:

**Proposition 3.2** In any left additive restriction category:

- (i)  $f + g = \overline{g}f + \overline{f}g$ ;
- (ii) if  $e = \overline{e}$ , then  $e(f + g) = ef + g = f + eg$ ;
- (iii) if  $f \leq f', g \leq g'$ , then  $f + g \leq f' + g'$ ;
- (iv) if  $f \smile f', g \smile g'$ , then  $(f + g) \smile (f' + g')$ .

PROOF:

$$(i) \quad f + g = \overline{f+g}(f+g) = \overline{f}\overline{g}(f+g) = \overline{g}\overline{f}f + \overline{f}\overline{g}g = \overline{g}f + \overline{f}g$$

(ii)

$$\begin{aligned} & f + eg \\ &= \overline{eg}f + \overline{f}eg \text{ by (i)} \\ &= \overline{e}\overline{g}f + \overline{e}\overline{f}g \\ &= \overline{e}(\overline{g}f + \overline{f}g) \\ &= e(f + g) \text{ by (i)} \end{aligned}$$

(iii) Suppose  $f \leq f', g \leq g'$ . Then:

$$\begin{aligned} & \overline{f+g}(f'+g') \\ &= \overline{f}\overline{g}(f'+g') \\ &= \overline{g}\overline{f}f' + \overline{f}\overline{g}g' \\ &= \overline{g}f + \overline{f}g \text{ since } f \leq f', g \leq g' \\ &= f + g \text{ by (i)}. \end{aligned}$$

so  $(f + g) \leq (f' + g')$ .

(iv) Suppose  $f \smile f'$ ,  $g \smile g'$ . By (reference), it suffices to show that  $\overline{f+g}(f'+g') \leq f+g$ . Again by (reference) we have  $\overline{f}f' \leq f$ ,  $\overline{g}g' \leq g$ , so by (ii), we can start with

$$\begin{aligned} \overline{f}f' + \overline{g}g' &\leq f+g \\ \overline{\overline{g}g'}\overline{f}f' + \overline{\overline{f}f'}\overline{g}g' &\leq f+g \\ \overline{\overline{g}g'}\overline{f}f' + \overline{\overline{f}f'}\overline{g}g' &\leq f+g \text{ by R3} \\ \overline{f}\overline{g}(\overline{g'}f' + \overline{f'}g') &\leq f+g \text{ by left additivity} \\ \overline{f+g}(f'+g') &\leq f+g \text{ by (i)} \end{aligned}$$

□

If  $\mathbb{X}$  has joins and left additive structure, then they are automatically compatible:

**Proposition 3.3** *If  $\mathbb{X}$  is a left additive restriction category with joins, then:*

(i)  $f + \emptyset = \emptyset$ ;

(ii)  $\bigvee_i f_i + \bigvee_j g_j = \bigvee_{i,j} f_i + g_j$ .

PROOF:

(i)  $\overline{f+\emptyset} = \overline{f}\overline{\emptyset} = \overline{f}\emptyset = \emptyset$ , so by Proposition 2.11,  $f + \emptyset = \emptyset$ .

(ii) Consider:

$$\begin{aligned} \left(\bigvee_i f_i\right) + \left(\bigvee_j g_j\right) &= \overline{\left(\bigvee_i f_i\right) + \left(\bigvee_j g_j\right)} \left(\bigvee_i f_i\right) + \left(\bigvee_j g_j\right) \\ &= \left(\bigvee_i \overline{f_i}\right) \left(\bigvee_j \overline{g_j}\right) \left(\bigvee_i f_i\right) + \left(\bigvee_j g_j\right) \\ &= \left(\bigvee_{i,j} \overline{f_i} \overline{g_j}\right) \left(\bigvee_i f_i\right) + \left(\bigvee_j g_j\right) \\ &= \bigvee_{i,j} \overline{g_j} \overline{f_i} \left(\bigvee_i f_i\right) + \overline{f_i} \overline{g_j} \left(\bigvee_j g_j\right) \\ &= \bigvee_{i,j} \overline{g_j} f_i + \overline{f_i} g_j \text{ by Proposition 2.11,} \\ &= \bigvee_{i,j} f_i + g_j, \end{aligned}$$

as required.

□

### 3.2 Additive and strongly additive maps

Before we get to the definition of a differential restriction category, it will be useful to have a slight detour, and investigate the nature of the additive maps in a left additive restriction category. In a left additive category, arbitrary maps need not preserve the addition, in the sense that

$$(x+y)f = xf + yf \text{ and } 0f = 0,$$

is not taken as axiom. Those maps which do preserve the addition (in the above sense) form an important subcategory, and such maps are called additive. Similarly, it will be important to identify which maps in a left additive restriction category are additive.

Here, however, we must be a bit more careful in our definition. Suppose we took the above axioms as our definition of additive in a left additive restriction category. In particular, asking for that equality would be asking for the restrictions to be equal, so that

$$\overline{(x+y)f} = \overline{xf+yf} = \overline{xf} \overline{yf}$$

That is,  $xf$  and  $yf$  are defined exactly when  $(x+y)f$  is. Obviously, this is a problem in one direction: it would be nonsensical to ask that  $f$  be defined on  $x+y$  implies that  $f$  is defined on both  $x$  and  $y$ . The other direction seems more logical: asking that if  $f$  is defined on  $x$  and  $y$ , then it is defined on  $x+y$ . That is, in addition to being additive as a function, its domain is also additively closed.

Even this, however, is often too strong for general functions. A standard example of a smooth partial function would be something  $2x$ , defined everywhere but  $x=5$ . This map does preserve addition, wherever it is defined. But it is not additive in the sense that its domain is not additively closed. Thus, we need a weaker notion of additivity: we merely ask that  $(x+y)f$  be *compatible* with  $xf+yf$ . Of course, the stronger notion, where the domain is additively closed, is also important, and will be discussed further below.

**Definition 3.4** *Say that a map  $f$  in a left additive restriction category is **additive** if for any  $x, y$ ,*

$$(x+y)f \smile xf+yf \text{ and } 0f \smile 0$$

We shall see below that for total maps, this agrees with the usual definition. We also have the following alternate characterizations of additivity:

**Lemma 3.5** *A map  $f$  is additive if and only if for any  $x, y$ ,*

$$\overline{xf} \overline{yf} (x+y)f \leq xf+yf \text{ and } 0f \leq 0$$

or

$$(x\overline{f} + y\overline{f})f \leq xf+yf \text{ and } 0f \leq 0.$$

PROOF: Use the alternate form of compatibility (Lemma 2.9) for the first part, and then R4 for the second. □

**Proposition 3.6** *In any left additive restriction category,*

- (i) *total maps are additive if and only if  $(x+y)f = xf+yf$ ;*
- (ii) *restriction idempotents are additive;*
- (iii) *additive maps are closed under composition;*
- (iv) *if  $g \leq f$  and  $f$  is additive, then  $g$  is additive;*



(v) 0 maps are additive, and additive maps are closed under addition.

PROOF: In each case, the 0 axiom is straightforward, so we only show the addition axiom.

(i) It suffices to show that if  $f$  is total, then  $\overline{(x+y)f} = \overline{xf+yf}$ . Indeed, if  $f$  is total,

$$\overline{(x+y)f} = \overline{x+y} = \overline{xy} = \overline{xfyf} = \overline{xf+yf}.$$

(ii) Suppose  $e = \bar{e}$ . Then by R4,

$$(xe + ye)\bar{e} = \overline{\overline{xe + ye}\bar{e}} (xe + ye) \leq xe + ye$$

so that  $e$  is additive.

(iii) Suppose  $f$  and  $g$  are additive. Then

$$\begin{aligned} & \overline{xfgyfg}(x+y)fg \\ &= \overline{xfgyfgxfyf}(x+y)fg \\ &\leq \overline{xfgyfg}(xf+yf)g \text{ since } f \text{ is additive,} \\ &\leq xfg + yfg \text{ since } g \text{ is additive,} \end{aligned}$$

as required.

(iv) If  $g \leq f$ , then  $g = \bar{g}f$ , and since restriction idempotents are additive, and the composites of additive maps are additive,  $g$  is additive.

(v) For any 0 map,  $(x+y)0 = 0 = 0+0 = x0+y0$ , so it is additive. For addition, suppose  $f$  and  $g$  are additive. Then we have

$$(x+y)f \smile xf+yf \text{ and } (x+y)g \smile xg+yg.$$

Since adding preserves compatibility, this gives

$$(x+y)f + (x+y)g \smile xf+yf + xg+yg.$$

Then using *left* additivity of  $x, y$ , and  $x+y$ , we get

$$(x+y)(f+g) \smile x(f+g) + y(f+g)$$

so that  $f+g$  is additive.

□

What is not true, however, is that if  $f$  is additive and has a partial inverse  $g$ , then  $g$  is partially additive. Indeed, consider the left additive restriction category of arbitrary partial maps from  $\mathcal{Z}$  to  $\mathcal{Z}$ . In particular, consider the partial map  $f$  which is only defined on  $\{p, q, r\}$  for  $r \neq p+q$ , and maps those points to  $\{n, m, n+m\}$ . In this case,  $f$  is additive, since  $(p+q)f$  is undefined. However,  $f$ 's partially inverse  $g$ , which sends  $\{n, m, n+m\}$  to  $\{p, q, r\}$  is not additive, since  $ng+mg \neq (n+m)g$ . The problem is that  $f$ 's domain is not additively closed. This leads us to the following definition.

**Definition 3.7** Say that a map  $f$  in a left additive restriction category is **strongly additive** if for any  $x, y$ ,

$$xf + yf \leq (x + y)f \text{ and } 0f = 0.$$

An alternate description, which can be useful for some proofs, is the following:

**Lemma 3.8**  $f$  is strongly additive if and only if  $(\overline{xf} + \overline{yf})f = xf + yf$  and  $0f = 0$ .

PROOF:

$$\begin{aligned} & xf + yf \leq (x + y)f \\ \Leftrightarrow & \overline{xf + yf} (x + y)f = xf + yf \\ \Leftrightarrow & \overline{xf} \overline{yf} (x + y)f = xf + yf \\ \Leftrightarrow & (\overline{xf}x + \overline{yf}y)f = xf + yf \\ \Leftrightarrow & (\overline{xf} + \overline{yf})f = xf + yf \text{ by R4.} \end{aligned}$$

□

Intuitively, the strongly additive maps are the ones which are additive in the previous sense, but whose domains are also closed under addition and contain 0. Note then that not all restriction idempotents will be strongly additive, and a map less than or equal to a strongly additive map need not be strongly additive. Excepting this, all of the previous results about additive maps hold true for strongly additive ones, and in addition, a partial inverse of a strongly additive map is strongly additive.

**Proposition 3.9** In a left additive restriction category,

- (i) strongly additive maps are additive, and if  $f$  is total, then  $f$  is additive if and only if it is strongly additive;
- (ii) if  $f$  is strongly additive, then so is  $\overline{f}$ ;
- (iii) identities are strongly additive, and if  $f$  and  $g$  are strongly additive, then so is  $fg$ ;
- (iv) 0 maps are strongly additive, and if  $f$  and  $g$  are strongly additive, then so is  $f + g$ ;
- (v) if  $f$  is strongly additive and has a partial inverse  $g$ , then  $g$  is also strongly additive.

PROOF: In most of the following proofs, we omit the proof of the 0 axiom, as it is straightforward.

- (i) Since  $\leq$  implies  $\smile$ , strongly additive maps are additive, and by previous discussion, if  $f$  is total, the restrictions of  $xf + yf$  and  $(x + y)f$  are equal, so  $\smile$  implies  $\leq$ .
- (ii) Suppose  $f$  is strongly additive. Then using the alternate description of strongly additive,

$$\begin{aligned} & (\overline{xf} + \overline{yf})\overline{f} \\ = & \overline{\overline{xf} + \overline{yf} f (\overline{xf} + \overline{yf})} \text{ by R4,} \\ = & \overline{xf + yf} (\overline{xf} + \overline{yf}) \text{ since } f \text{ strongly additive,} \\ = & \overline{xf} \overline{yf} (\overline{xf} + \overline{yf}) \\ = & \overline{xf} + \overline{yf} \end{aligned}$$

and  $0\overline{f} = \overline{0f}0 = \overline{0}0 = 0$ , so  $\overline{f}$  is strongly additive.

(iii) Identities are total and additive, so are strongly additive. Suppose  $f$  and  $g$  are strongly additive. Then

$$\begin{aligned} & xfg + yfg \\ & \leq (xf + yf)g \text{ since } g \text{ strongly additive,} \\ & \leq (x + y)fg \text{ since } f \text{ strongly additive,} \end{aligned}$$

so  $fg$  is strongly additive.

(iv) Since any 0 is total and additive, 0's are strongly additive. Suppose  $f$  and  $g$  are strongly additive. Then

$$\begin{aligned} & x(f + g) + y(f + g) \\ & = xf + xg + yf + yg \text{ by left additivity,} \\ & \leq (x + y)f + (x + y)g \text{ since } f \text{ and } g \text{ are strongly additive,} \\ & = (x + y)(f + g) \text{ by left additivity,} \end{aligned}$$

so  $f + g$  is strongly additive.

(v) Suppose  $f$  is strongly additive and has a partial inverse  $g$ . Using the alternate form of strongly additive,

$$\begin{aligned} & (x\bar{g} + y\bar{g})g \\ & = (xgf + ygf)g \\ & = (xg\bar{f} + yg\bar{f})fg \text{ since } f \text{ is strongly additive,} \\ & = (xg\bar{f} + yg\bar{f})\bar{f} \\ & = xg\bar{f} + yg\bar{f} \text{ since } \bar{f} \text{ strongly additive,} \\ & = xg + yg \end{aligned}$$

and  $0g = 0fg = 0\bar{f} = 0$ , so  $g$  is strongly additive.

□

Finally, note that neither additive nor strongly additive maps are closed under joins. For additive, the join of the additive maps  $f : \{n, m\} \rightarrow \{p, q\}$  and  $g : \{n + m\} \rightarrow r$ , where  $p + q \neq r$ , is not additive. For strongly additive, if  $f$  is defined on multiples of 2 and  $g$  on multiples of 3, their join is not closed under addition, so is not strongly additive.

### 3.3 Cartesian left additive restriction categories

In a differential restriction category, we will need both cartesian and left additive structure. Thus, we describe here how cartesian and additive restriction structures must interact.

**Definition 3.10**  $\mathbb{X}$  is a **cartesian left additive restriction category** if the product functor preserves addition (that is  $(f + g) \times (h + k) = (f \times h) + (g \times k)$  and  $0 = 0 \times 0$ ) and the maps  $\pi_0, \pi_1$ , and  $\Delta$  are additive.

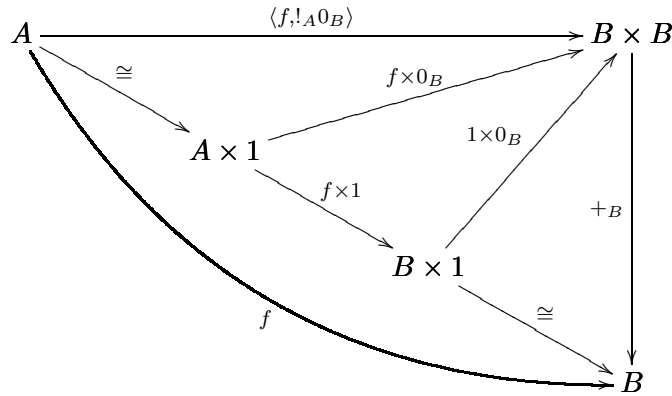
If  $\mathbb{X}$  is a cartesian restriction category then each object becomes canonically a (total) commutative monoid by  $+_X = \pi_0 + \pi_1 : X \times X \rightarrow X$  and  $0 : 1 \rightarrow X$ . Surprisingly, assuming these total commutative monoids are coherent with the cartesian structure, one can then recapture the additive structure, as the following theorem shows. Thus, in the presence of cartesian restriction structure, it suffices to give additive structure on the total maps to get a cartesian left additive restriction category.

**Theorem 3.11** *Suppose that  $\mathbb{X}$  is a cartesian restriction category, with each object  $A$  having the structure of a total commutative monoid,  $A \times A \xrightarrow{+_A} A$  and  $1 \xrightarrow{0_A} A$ , such that the projections and diagonal are additive and we have an exchange axiom:*

$$+_X \times Y = (X \times Y) \times (X \times Y) \xrightarrow{\text{ex}} (X \times X) \times (Y \times Y) \xrightarrow{+_X \times +_Y} X \times Y.$$

Then  $\mathbb{X}$  can be given the structure of a cartesian left additive category, where, for  $A \xrightarrow{f,g} B$ ,  $f + g := \langle f, g \rangle +_B$  and  $0_{AB} := !_A 0_B$ .

PROOF: That this gives a commutative monoid on each  $\mathbb{X}(A, B)$  follows directly from the commutative monoid axioms on  $B$  and the coherences of the cartesian structure. For example, to show  $f + 0 = f$ , we need to show  $\langle f, !_A 0_B \rangle = f$ . Indeed, we have



the right-most shape commutes a commutative monoid axiom for  $B$ , and the other shapes commute by coherences of the cartesian structure. The other commutative monoid axioms are similar.

For the interaction with restriction,

$$\overline{f + g} = \overline{\langle f, g \rangle +_B} = \overline{\langle f, g \rangle}_{+B} = \overline{\langle f, g \rangle} = \overline{f} \overline{g},$$

and  $\overline{0_{AB}} = \overline{!_A 0_B} = 1$  since  $!$  and  $0$  are themselves total.

For the interaction with composition,

$$f(g + h) = f \langle g, h \rangle 0_C = \langle fg, fh \rangle 0_C = fg + fh$$

and

$$f 0_{BC} = f !_B 0_C = \overline{f} !_A 0_C = \overline{f} 0_{AC}$$

as required.

The requirement that  $(f + g) \times (h + k) = (f \times h) + (g \times k)$  follows from the exchange axiom:

$$\begin{array}{ccc}
A \times C & \xrightarrow{\langle f \times h, g \times k \rangle} & (B \times D) \times (B \times D) \\
& \searrow \langle f, g \rangle \times \langle h, k \rangle & \swarrow \text{ex} \\
& & (B \times B) \times (D \times D) \\
& \searrow \langle f, g \rangle +_B \times \langle h, k \rangle +_D & \swarrow +_{B \times D} \\
& & B \times D
\end{array}$$

the right triangle is the exchange axiom, and the other two shapes commute by the cartesian coherences. Checking that  $\pi_0$ ,  $\pi_1$  and  $\delta$  are additive is similar.  $\square$

**Proposition 3.12** *In a cartesian left additive restriction category:*

(i)  $\langle f, g \rangle + \langle f', g' \rangle = \langle f + f', g + g' \rangle$  and  $\langle 0, 0 \rangle = 0$ ;

(ii) if  $f$  and  $g$  are additive, then so is  $\langle f, g \rangle$ ;

(iii) the projections are strongly additive, and if  $f$  and  $g$  are strongly additive, then so is  $\langle f, g \rangle$ ,

(iv)  $f$  is additive if and only if

$$(\pi_0 + \pi_1)f \smile \pi_0 f + \pi_1 f \text{ and } 0f \smile 0;$$

(that is, in terms of the monoid structure on objects,  $(+)(f) \smile (f \times f)(+)$  and  $0f \smile 0$ ,

(v)  $f$  is strongly additive if and only if

$$(\pi_0 + \pi_1)f \geq \pi_0 f + \pi_1 f \text{ and } 0f = 0;$$

(that is,  $(+)(f) \geq (f \times f)(+)$  and  $0f \geq 0$ ).

Note that  $f$  being strongly additive only implies that  $+$  and  $0$  are lax natural transformations.

PROOF:

(i) Since the second term is a pairing, it suffices to show they are equal when post-composed with projections. Post-composing with  $\pi_0$ , we get

$$\begin{aligned}
& (\langle f, g \rangle + \langle f', g' \rangle)\pi_0 \\
&= \langle f, g \rangle\pi_0 + \langle f', g' \rangle\pi_0 \text{ since } \pi_0 \text{ is additive,} \\
&= \overline{g}f + \overline{g'}f' \\
&= \overline{g} \overline{g'}(f + f') \\
&= \overline{g + g'}(f + f') \\
&= \langle f + f', g + g' \rangle\pi_0
\end{aligned}$$

as required. The  $0$  result is direct.

(ii) We need to show

$$(x + y)\langle f, g \rangle \smile x\langle f, g \rangle + y\langle f, g \rangle;$$

however, since the first term is a pairing, it suffices to show they are compatible when post-composed by the projections. Indeed,

$$(x + y)\langle f, g \rangle \pi_0 = (x + y)\bar{g}f \smile x\bar{g}f + y\bar{g}f$$

while since  $\pi_0$  is additive,

$$(x\langle f, g \rangle + y\langle f, g \rangle)\pi_0 = x\langle f, g \rangle\pi_0 + y\langle f, g \rangle\pi_0 = x\bar{g}f + y\bar{g}f$$

so the two are compatible, as required. Post-composing with  $\pi_1$  is similar.

(iii) Since projections are additive and total, they are strongly additive. If  $f$  and  $g$  are strongly additive,

$$\begin{aligned} & x\langle f, g \rangle + y\langle f, g \rangle \\ = & \langle xf, xg \rangle + \langle yf, yg \rangle \\ = & \langle xf + yf, xg + yg \rangle \text{ by (i)} \\ \leq & \langle (x + y)f, (x + y)g \rangle \text{ since } f \text{ and } g \text{ are strongly additive,} \\ = & (x + y)\langle f, g \rangle \end{aligned}$$

so  $\langle f, g \rangle$  is strongly additive.

(iv) If  $f$  is additive, the condition obviously holds. Conversely, if we have the condition, then  $f$  is additive, since

$$(x + y)f = \langle x, y \rangle(\pi_0 + \pi_1)f \smile \langle x, y \rangle(\pi_0f + \pi_1f) = xf + yf$$

as required.

(v) Similar to the previous proof.

□

### 3.4 Differential restriction categories

With cartesian left additive restriction categories defined, we turn to defining differential restriction categories. To do this, we begin by recalling the notion of a cartesian differential category. The idea is to axiomatize the Jacobian of smooth maps. Normally, the Jacobian of a map  $f : X \rightarrow Y$  gives, for each point of  $X$ , a linear map  $X \rightarrow Y$ . That is,  $D[f] : X \rightarrow [X, Y]$ . However, we don't want to assume that our category has closed structure. Thus, uncurrying, we get that the derivative should be of the type  $D[f] : X \times X \rightarrow Y$ . The second coordinate is simply the point at which the derivative is being taken, while the first coordinate is the direction in which this derivative is being evaluated. With this understanding, the first five axioms of a cartesian differential category should be relatively clear. Axioms 6 and 7 are slightly more tricky, but in the essence they say that the derivative is linear, and that the order of partial differentiation does not matter. For more discussion of these axioms, see [Blute et. al. 2008].

**Definition 3.13** A *cartesian differential category* is a cartesian left additive category with a differentiation operation

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow{D[f]} Y}$$

such that

[CD.1]  $D[f + g] = D[f] + D[g]$  and  $D[0] = 0$  (additivity of differentiation);

[CD.2]  $\langle g + h, k \rangle D[f] = \langle g, k \rangle D[f] + \langle h, k \rangle D[f]$  and  $\langle 0, g \rangle D[f] = 0$  (additivity of a derivative in its first variable);

[CD.3]  $D[1] = \pi_0$ ,  $D[\pi_0] = \pi_0 \pi_0$ , and  $D[\pi_1] = \pi_0 \pi_1$  (derivatives of projections);

[CD.4]  $D[\langle f, g \rangle] = \langle D[f], D[g] \rangle$  (derivatives of pairings);

[CD.5]  $D[fg] = \langle D[f], \pi_1 f \rangle D[g]$  (chain rule);

[CD.6]  $\langle \langle g, 0 \rangle, \langle h, k \rangle \rangle D[D[f]] = \langle g, k \rangle D[f]$  (linearity of the derivative);

[CD.7]  $\langle \langle 0, h \rangle, \langle g, k \rangle \rangle D[D[f]] = \langle \langle 0, g \rangle, \langle h, k \rangle \rangle D[D[f]]$  (independence of partial differentiation).

We now give the definition of a differential restriction category. Axioms 8 and 9 are the additions to the above. Axiom 8 says that the differential of a restriction is similar to the derivative of an identity, with the partiality of  $f$  now included. Axiom 9 says that the restriction of a differential is nothing more than  $1 \times \bar{f}$ : the first component, being simply the co-ordinate of the direction the derivative is taken, is always total. In addition to these new axioms, one must also modify axioms 2 and 6 to take into account the partiality when one loses maps, and remove the first part of axiom 3 ( $D[1] = \pi_0$ ), since axiom 8 makes it redundant.

**Definition 3.14** A *differential restriction category* is a cartesian left additive restriction category with a differentiation operation

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow{D[f]} Y}$$

such that

[DR.1]  $D[f + g] = D[f] + D[g]$  and  $D[0] = 0$ ;

[DR.2]  $\langle g + h, k \rangle D[f] = \langle g, k \rangle D[f] + \langle h, k \rangle D[f]$  and  $\langle 0, g \rangle D[f] = \bar{g}f0$ ;

[DR.3]  $D[\pi_0] = \pi_0 \pi_0$ , and  $D[\pi_1] = \pi_0 \pi_1$ ;

[DR.4]  $D[\langle f, g \rangle] = \langle D[f], D[g] \rangle$ ;

[DR.5]  $D[fg] = \langle D[f], \pi_1 f \rangle D[g]$ ;

[DR.6]  $\langle \langle g, 0 \rangle, \langle h, k \rangle \rangle D[D[f]] = \bar{h} \langle g, k \rangle D[f]$ ;

$$[\text{DR.7}] \quad \langle \langle 0, h \rangle, \langle g, k \rangle \rangle D[D[f]] = \langle \langle 0, g \rangle, \langle h, k \rangle \rangle D[D[f]];$$

$$[\text{DR.8}] \quad D[\bar{f}] = (1 \times \bar{f})\pi_0;$$

$$[\text{DR.9}] \quad \overline{D[f]} = 1 \times \bar{f}.$$

Of course, any cartesian differential category is a differential restriction category, when equipped with the trivial restriction structure ( $\bar{f} = 1$  for all  $f$ ). The standard example with a non-trivial restriction is smooth functions defined on open subsets of  $\mathcal{R}^n$ ; that this is a differential restriction category is readily verified. In the next section, we will present a more sophisticated example (rational functions over a commutative ring).

There is an obvious notion of differential restriction functor:

**Definition 3.15** *If  $\mathbb{X}$  and  $\mathbb{Y}$  are differential restriction categories, a **differential restriction functor**  $\mathbb{X} \xrightarrow{F} \mathbb{Y}$  is a restriction functor such that*

- $F$  preserves the addition and zeroes of the homsets;
- $F$  preserves products strictly:  $F(A \times B) = FA \times FB$ ,  $F1 = 1$ ;
- $F$  preserves the differential:  $F(D[f]) = D(F[f])$ .

The differential itself automatically preserves both the restriction ordering and the compatibility relation:

**Proposition 3.16** *In a differential restriction category:*

- (i)  $D[\bar{f}g] = (1 \times \bar{f})D[g]$ ;
- (ii) If  $f \leq g$  then  $D[f] \leq D[g]$ ;
- (iii) If  $f \smile g$  then  $D[f] \smile D[g]$ .

PROOF:

- (i) Consider:

$$\begin{aligned} & D[\bar{f}g] \\ &= \langle D\bar{f}, \pi_1\bar{f} \rangle D[g] \text{ by [D5]} \\ &= \langle (1 \times \bar{f})\pi_0, \pi_1\bar{f} \rangle D[g] \text{ by [D8]} \\ &= \langle (1 \times \bar{f})\pi_0, (1 \times \bar{f})\pi_1 \rangle D[g] \text{ by naturality} \\ &= (1 \times \bar{f})D[g] \text{ by Lemma 2.14.} \end{aligned}$$

as required.

- (ii) If  $f \leq g$ , then

$$\overline{D[f]} D[g] = (1 \times \bar{f})D[g] = D[\bar{f}g] = D[f],$$

so  $D[f] \leq D[g]$ .



(iii) If  $f \smile g$ , then

$$\overline{D[f]} D[g] = (1 \times \overline{f}) D[g] = D[\overline{f} g] = D[\overline{g} f] = (1 \times \overline{g}) D[f] = \overline{D[g]} D[f],$$

so  $D[f] \smile D[g]$ .

□

Moreover, just as for cartesian and left additive structure, if  $\mathbb{X}$  has joins and differential structure, then they are automatically compatible:

**Proposition 3.17** *In a differential restriction category with joins,*

(i)  $D[\emptyset] = \emptyset$ ,

(ii)  $D[\bigvee_i f_i] = \bigvee_i D[f_i]$ .

PROOF:

(i)  $\overline{D[\emptyset]} = 1 \times \overline{\emptyset} = \emptyset$ , so by Lemma 2.11,  $D[\emptyset] = \emptyset$ .

(ii) Consider:

$$\begin{aligned} & \bigvee_{i \in I} D[f_i] \\ &= \bigvee_{i \in I} D \left[ \overline{f_i} \bigvee_{j \in I} f_j \right] \text{ by Lemma 2.11} \\ &= \bigvee_{i \in I} (1 \times \overline{f_i}) D \left[ \bigvee_{j \in I} f_j \right] \text{ by Lemma 3.16} \\ &= \left( 1 \times \bigvee_{i \in I} \overline{f_i} \right) D \left[ \bigvee_{j \in I} f_j \right] \\ &= \left( 1 \times \overline{\bigvee_{i \in I} f_i} \right) D \left[ \bigvee_{j \in I} f_j \right] \\ &= \overline{D \left[ \bigvee_{i \in I} f_i \right]} D \left[ \bigvee_{j \in I} f_j \right] \text{ by D9} \\ &= D \left[ \bigvee_{i \in I} f_i \right] \end{aligned}$$

as required.

□

### 3.5 Linear maps

Just as we had to modify the definition of additive maps for left additive restriction categories, so too do we have to modify linear maps when dealing with differential restriction categories. Recall that in a cartesian differential category, a map is linear if  $D[f] = \pi_0 f$ . If we asked for this in a differential restriction category, we would have

$$\overline{\pi_0 f} = \overline{D[f]} = 1 \times \overline{f} = \overline{\pi_1 f},$$

which is never true unless  $f$  is total. In contrast to the additive situation, however, there is no obvious preference for one side to be more defined than the other. Thus, a map will be linear when  $D[f]$  and  $\pi_0 f$  are compatible.

**Definition 3.18** *A map  $f$  in a differential restriction category is **linear** if*

$$D[f] \smile \pi_0 f$$

We shall see below that for total  $f$ , this agrees with the usual definition. We also have the following alternate characterizations of linearity:

**Lemma 3.19** *In a differential restriction category,*

$$\begin{aligned} & f \text{ is linear} \\ \Leftrightarrow & \overline{\pi_1 f} \pi_0 f \leq D[f] \\ \Leftrightarrow & \overline{\pi_0 f} D[f] \leq \pi_0 f \end{aligned}$$

PROOF: Use the alternate form of compatibility (Lemma 2.9). □

Linear maps then have a number of important properties. Note one surprise: while additive maps were not closed under partial inverses, linear maps are.

**Proposition 3.20** *In a differential restriction category:*

- (i) *if  $f$  is total,  $f$  is linear if and only if  $D[f] = \pi_0 f$ ;*
- (ii) *if  $f$  is linear, then  $f$  is additive;*
- (iii) *restriction idempotents are linear;*
- (iv) *if  $f$  and  $g$  are linear, so is  $fg$ ;*
- (v) *if  $g \leq f$  and  $f$  is linear, then  $g$  is linear;*
- (vi) *0 maps are linear, and if  $f$  and  $g$  are linear, so is  $f + g$ ;*
- (vii) *projections are linear, and if  $f$  and  $g$  are linear, so is  $\langle f, g \rangle$ ;*
- (viii)  *$\langle 1, 0 \rangle D[f]$  is linear for any  $f$ ;*
- (ix) *if  $f$  is linear and has a partial inverse  $g$ , then  $g$  is also linear.*

PROOF:

(i) It suffices to show that if  $f$  is total,  $\overline{D[f]} = \overline{\pi_0 f}$ . Indeed, if  $f$  is total,

$$\overline{D[f]} = 1 \times \overline{f} = \overline{f} \times 1 = \overline{\pi_0 f}.$$

(ii) For the 0 axiom:

$$\begin{aligned} 0f &= \overline{0f}0f \\ &= \overline{\langle 0, 0 \rangle \pi_1 f} \langle 0, 0 \rangle \pi_0 f \\ &= \langle 0, 0 \rangle \overline{\pi_1 f} \pi_0 f \text{ by R4,} \\ &\leq \langle 0, 0 \rangle D[f] \text{ since } f \text{ linear,} \\ &= \overline{0f}0 \text{ by D2,} \\ &\leq 0 \end{aligned}$$

and for the addition axiom:

$$\begin{aligned} \overline{(x+y)f}(xf+yf) &= \overline{(x+y)f}(\overline{xf}xf + \overline{yf}\overline{xy}yf) \\ &= \overline{(x+y)f}(\overline{xf}xf + \overline{yf}\overline{xy}yf) \\ &= \overline{(x+y)f}(\langle x, x \rangle \overline{\pi_1 f} \langle x, x \rangle \pi_0 f + \langle y, x \rangle \overline{\pi_1 f} \langle y, x \rangle \pi_0 f) \\ &= \overline{(x+y)f}(\langle x, x \rangle \overline{\pi_1 f} \pi_0 f + \langle y, x \rangle \overline{\pi_1 f} \pi_0 f) \\ &\leq \overline{(x+y)f}(\langle x, x \rangle D[f] + \langle y, x \rangle D[f]) \text{ since } f \text{ is linear} \\ &= \overline{\langle x+y, x \rangle \pi_0 f} \langle x+y, x \rangle D[f] \text{ by D2} \\ &= \langle x+y, x \rangle \overline{\pi_0 f} D[f] \\ &= \langle x+y, x \rangle \overline{\pi_1 f} \pi_0 f \text{ since } f \text{ is linear} \\ &= \overline{x+y, x} \pi_1 f \langle x+y, x \rangle \pi_0 f \\ &= \overline{x+yx}\overline{f}\overline{x}(x+y)f \\ &\leq (x+y)f \end{aligned}$$

as required.

(iii) Suppose  $e = \overline{e}$ . Then consider

$$\begin{aligned} &\overline{\pi_1 e} \pi_0 \overline{e} \\ &= \overline{\pi_1 e} \overline{\pi_0 e} \pi_0 \\ &\leq \overline{\pi_1 e} \pi_0 \\ &= \langle \pi_0 e, \pi_1 e \rangle \pi_0 \\ &= (1 \times e) \pi_0 \\ &= D[e] \end{aligned}$$

so that  $e$  is additive.

(iv) Suppose  $f$  and  $g$  are linear; then consider

$$\begin{aligned}
D[fg] &= \langle D[f], \pi_1 f \rangle D[g] \\
&\geq \overline{\langle \pi_1 f \pi_0 f, \pi_1 f \rangle \pi_1 g \pi_0 g} \text{ since } f \text{ and } g \text{ are linear} \\
&= \overline{\langle \pi_1 f \pi_0 f, \pi_1 f \rangle \pi_1 g \langle \pi_1 f \pi_0 f, \pi_1 f \rangle \pi_0 g} \text{ by R4} \\
&= \overline{\pi_1 f \pi_0 f \pi_1 f g \pi_1 f \pi_1 f \pi_0 f g} \\
&= \overline{\pi_1 f \pi_1 f g \pi_0 f \pi_0 f g} \\
&= \overline{\pi_1 f g \pi_0 f g}
\end{aligned}$$

(v) If  $g \leq f$ , then  $g = \bar{g}f$ ; since restriction idempotents are linear and the composite of linear maps is linear,  $g$  is linear.

(vi) Since  $D[0] = 0 = \pi_0 0$ ,  $0$  is linear. Suppose  $f$  and  $g$  are linear; then consider

$$\begin{aligned}
\overline{\pi_0(f+g)}D[f+g] &= \overline{\pi_0 f + \pi_0 g}(D[f] + D[g]) \\
&= \overline{\pi_0 f \pi_0 g}(D[f] + D[g]) \\
&= \overline{\pi_0 f}D[f] + \overline{\pi_0 g}D[g] \\
&= \overline{\pi_1 f \pi_0 f + \pi_1 g \pi_0 g} \text{ since } f \text{ and } g \text{ are linear} \\
&= \overline{\pi_1 f \pi_1 g \pi_0(f+g)} \\
&\leq \pi_0(f+g)
\end{aligned}$$

as required.

(vii) By D3, projections are linear. Suppose  $f$  and  $g$  are linear; then consider

$$\begin{aligned}
D[\langle f, g \rangle] &= \langle D[f], D[g] \rangle \\
&\geq \overline{\langle \pi_1 f \pi_0 f, \pi_1 g \pi_0 g \rangle} \text{ since } f \text{ and } g \text{ are linear} \\
&= \overline{\pi_1 f \pi_1 g \pi_0 \langle f, g \rangle} \\
&= \overline{\pi_1 f \pi_1 g \pi_0 \langle f, g \rangle} \\
&= \overline{\pi_1 f \pi_1 g \pi_0 \langle f, g \rangle} \\
&= \overline{\pi_1 f \bar{g} \pi_0 \langle f, g \rangle} \text{ by R4} \\
&= \overline{\pi_1 \langle f, g \rangle \pi_0 \langle f, g \rangle}
\end{aligned}$$

as required.

(viii) The proof is identical to that for total differential categories:

$$\begin{aligned}
D[\langle 1, 0 \rangle D[f]] &= \langle D[\langle 1, 0 \rangle], \pi_1 \langle 1, 0 \rangle \rangle D[D[f]] \\
&= \langle \langle \pi_0, 0 \rangle, \langle \pi_1, 0 \rangle \rangle D[D[f]] \\
&= \langle \pi_0, 0 \rangle D[f] \text{ by D6} \\
&= \pi_0 \langle 1, 0 \rangle D[f]
\end{aligned}$$

as required.

(ix) If  $g$  is the partial inverse of a linear map  $f$ , then

$$\begin{aligned}
D[g] &\geq (\bar{g} \times \bar{g})D[g] \\
&= (gf \times gf)D[g] \\
&= (g \times g)(f \times f)D[g] \\
&= (g \times g)\langle \pi_0 f, \pi_1 f \rangle D[g] \\
&= (g \times g)\langle \overline{\pi_1 f} \pi_0 f, \pi_1 f \rangle D[g] \\
&= (g \times g)\langle \overline{\pi_0 f} D[f], \pi_1 f \rangle D[g] \text{ since } f \text{ is linear,} \\
&= (g \times g)\overline{\pi_0 f} \langle D[f], \pi_1 f \rangle D[g] \\
&= (g \times g)\overline{\pi_0 f} D[f]g \text{ by D5,} \\
&= (g \times g)\overline{\pi_0 f} D[\bar{f}] \\
&= (g \times g)\overline{\pi_0 f} (1 \times \bar{f})\pi_0 \text{ by D8,} \\
&= \overline{(g \times g)\pi_0 f} (g \times g)(1 \times \bar{f})\pi_0 \text{ by R4,} \\
&= \overline{\pi_1 \bar{g} \pi_0 g f} (g \times g)\pi_0 \\
&= \overline{\pi_1 \bar{g} \pi_0 \bar{g}} \overline{\pi_1 \bar{g}} \pi_g \\
&= \overline{\pi_1 \bar{g}} \pi_0 g
\end{aligned}$$

as required. □

Note that the join of linear maps need not be linear. Indeed, consider the linear partial maps  $2x : (0, 2) \longrightarrow (0, 4)$  and  $3x : (3, 5) \longrightarrow (9, 15)$ . If their join was linear, then it would be additive. But this is a contradiction, since  $2(1.75) + 2(1.75) \neq 3(3.5)$ . However, the join of linear maps is a standard concept of analysis:

**Definition 3.21** *If  $f$  is a finite join of linear maps, say that  $f$  is **piecewise linear**.*

An interesting result from [Blute et. al. 2008] is the nature of the differential of additive maps. We get a similar result in our context:

**Proposition 3.22** *If  $f$  is additive, then  $D[f]$  is additive and*

$$D[f] \smile \pi_0 \langle 1, 0 \rangle D[f];$$

*if  $f$  is strongly additive, then  $D[f]$  is strongly additive and*

$$D[f] \leq \pi_0 \langle 1, 0 \rangle D[f].$$

PROOF: The proof that  $f$  being (strongly) additive implies  $f$  (strongly) additive is the same as for total differential categories ([Blute et. al. 2008], pg. 19) with  $\smile$  or  $\leq$  replacing  $=$  when one invokes the additivity of  $f$ . The form of  $D[f]$  in each case, however, takes a bit more work. We begin with a short calculation:

$$\langle 0, \pi_1 \rangle \overline{\pi_1 f} = \overline{\langle 0, \pi_1 \rangle \pi_1 f} \langle 0, \pi_1 \rangle = \overline{\pi_1 f} \langle 0, \pi_1 \rangle$$

and

$$\langle \pi_0, 0 \rangle \overline{\pi_1 f} = \overline{\langle \pi_0, 0 \rangle \pi_1 f} \langle \pi_0, 0 \rangle = \overline{0f} \langle \pi_0, 0 \rangle.$$

Now, if  $f$  is additive, we have:

$$\begin{aligned} & \overline{\pi_0 \langle 1, 0 \rangle D[f]} D[f] \\ &= \overline{\langle \pi_0, 0 \rangle \pi_1 f} D[f] \\ &= \overline{0f} \overline{\langle \pi_0, 0 \rangle} D[f] \text{ by the second calculation above,} \\ &= \overline{0f} D[f] \\ &= \overline{0f} \overline{\pi_1 f} D[f] \\ &= \overline{0f} \overline{\pi_1 f} (\langle 0, \pi_1 \rangle + \langle \pi_0, 0 \rangle) D[f] \\ &= (\overline{\pi_1 f} \langle 0, \pi_1 \rangle + \overline{0f} \langle \pi_0, 0 \rangle) D[f] \\ &= (\langle 0, \pi_1 \rangle \overline{\pi_1 f} + \langle \pi_0, 0 \rangle \overline{\pi_1 f}) D[f] \text{ by both calculations above,} \\ &\leq \langle 0, \pi_1 \rangle D[f] + \langle \pi_0, 0 \rangle D[f] \text{ since } D[f] \text{ is additive,} \\ &= \overline{\pi_1 f} 0 + \langle \pi_0, 0 \rangle D[f] \text{ by D2,} \\ &\leq 0 + \langle \pi_0, 0 \rangle D[f] \\ &= \pi_0 \langle 1, 0 \rangle D[f] \end{aligned}$$

so that  $D[f] \smile \pi_0 \langle 1, 0 \rangle D[f]$ , as required. If  $f$  is strongly additive, consider

$$\begin{aligned} & \overline{D[f]} \langle \pi_0, 0 \rangle D[f] \\ &= \overline{\pi_1 f} \langle \pi_0, 0 \rangle D[f] \\ &= \overline{\pi_1 f} 0 + \langle \pi_0, 0 \rangle D[f] \\ &= \langle 0, \pi_1 \rangle D[f] + \langle \pi_0, 0 \rangle D[f] \\ &= (\langle 0, \pi_1 \rangle \overline{\pi_1 f} + \langle \pi_0, 0 \rangle \overline{\pi_1 f}) D[f] \text{ since } D[f] \text{ is strongly additive,} \\ &= \overline{\pi_1 f} \overline{0f} (\langle 0, \pi_1 \rangle + \langle \pi_0, 0 \rangle) D[f] \text{ by the calculations above,} \\ &= \overline{\pi_1 f} \overline{0} (1) D[f] \text{ since } f \text{ strongly additive,} \\ &= \overline{\pi_1 f} D[f] \\ &= D[f] \end{aligned}$$

so that  $D[f] \leq \pi_0 \langle 1, 0 \rangle D[f]$ , as required.  $\square$

Any differential restriction category has the following differential restriction subcategory:

**Proposition 3.23** *If  $\mathbb{X}$  is a differential restriction category, then  $\mathbb{X}_0$ , consisting of the maps which preserve 0 if it is in their domain (i.e., satisfying  $0f \leq 0$ ), is a differential restriction subcategory.*

PROOF: The result is immediate, since the differential has this property:

$$\langle 0, 0 \rangle D[f] = \overline{0f} 0 \leq 0.$$

$\square$

Finally, note that any differential restriction functor preserves additive, strongly additive, and linear maps:

**Proposition 3.24** *If  $F$  is a differential restriction functor, then*

- (i)  $F$  preserves additive maps;
- (ii)  $F$  preserves strongly additive maps;
- (iii)  $F$  preserves linear maps.

PROOF: Since any restriction functor preserves  $\leq$  and  $\smile$ , the result follows automatically.  $\square$

## 4 Rational Functions

Thus far, we have only seen a single, analytic example of a differential restriction category. The following section rectifies this problem by presenting a class of examples of differential restriction categories with a more algebraic flavour. Rational functions over a commutative ring have an obvious formal derivative. Thus, rational functions are a natural candidate for differential structure. Moreover, rational functions have an aspect of partiality: one thinks of a rational function as being undefined wherever the denominator is zero. To capture this partiality, we will construct a category whose maps will consist of a tuple of rational functions, together with a finitely generated set of polynomials representing the partiality of the rational functions. The goal is then to show that, for each commutative ring  $R$ , this category of rational functions over  $R$  is a differential restriction category.

Moreover, we will also show that these categories of rational functions embed into the partial map category of affine schemes with respect to localizations. Thus, we relate each rational function category to a category of traditional interest to algebraic geometrists.

### 4.1 Introduction

Typically, rational functions are defined as a pair of polynomials equipped with an equivalence relation which “reduces” common factors; for example,  $\frac{x}{x}$  is identified with 1. However, keeping track of where the composite of rational functions is undefined requires one to consider composition with factors that have not been reduced. Thus, we first construct a category where maps are given by formal pairs of polynomials, for example if  $y \neq 1$ , then  $\frac{0}{y} \neq \frac{0}{1}$ ; in this category, composition of rational functions and composition of the data tracking their undefinedness is defined in terms of a single homomorphism. We then obtain the category of rational functions by a congruence on the maps of the formal rational function category.

### 4.2 Rig Theory

As mentioned above, we must define the category of rational functions in terms of formal rational functions, and to do so requires rigs, not rings. For example,  $\frac{x}{y} - \frac{x}{y} = \frac{0}{y} \neq \frac{0}{1}$ , so these formal rational functions do not have additive inverses. Thus, we will initially work in the category of commutative rigs **CRig**. The objects of this category are defined below.

**Definition 4.1** *A commutative **rig** (a.k.a. semiring [Golan 1992]) is a quintuple,  $(R, +, \cdot, 0, 1)$ , where  $(R, +, 0)$  is a commutative monoid,  $(R, \cdot, 1)$  is a commutative monoid, and the law of distributivity holds; that is, for all  $a, b, c \in R$ :*

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ and } a \cdot 0 = 0.$$

The maps in the category of commutative rigs must preserve the operations of addition and multiplication as well as the identities of these operations.

We wish to generalize the idea of localization to rigs. Localization at a multiplicatively closed subset of a ring results in a new ring where the elements of the multiplicatively closed set have multiplicative inverses. We recall that multiplicatively closed sets which are factor closed correspond to prime ideals. However, we will take care in the generalization of localization since ring ideal theory does not entirely transfer to rig ideal theory. In rings, multiplicative sets correspond to prime ideals [Dummit and Foote 2004] [Zariski and Samuel 1975], but in general ideals do not identify the same structure in rigs as in rings. In rigs, congruences (equivalence relations compatible with the rig operations) define factor structures, not ideals [Golan 1992]. Yet, the correspondence between prime ideals and multiplicative sets does hold for commutative rigs.

**Definition 4.2** Let  $R$  be a commutative rig. An **ideal**  $I$  is a subset of  $R$  such that for all  $i \in I$  and  $r \in R$ ,  $ir \in I$ .

We note that if  $I, J$  are both ideals of  $R$  and  $a \in R$ , then

- $(a) := \{ar | r \in R\}$ ,
- $I + J := \{i + j | i \in I, j \in J\}$ , and
- $IJ := \left\{ \sum_{l,k} i_l j_k | i_l \in I, j_k \in J \right\}$

are also ideals of  $R$ .

**Definition 4.3** Let  $R$  be a commutative rig. An ideal,  $P$ , is a **prime ideal** if for all  $a, b \in R$ ,  $ab \in P$  implies  $a \in P$  or  $b \in P$ .

Next, we define what we mean by multiplicative sets. Note that these are called saturated sets by some authors [Cohn 2002].

**Definition 4.4** A **multiplicative set**,  $U$ , in a rig  $R$ , is a subset of  $R$  which is closed under multiplication and is **factor closed**; that is, whenever  $u_1 u_2 \in U$ , then  $u_1 \in U$  and  $u_2 \in U$ .

First we prove that any ideal which misses a multiplicatively closed set,  $U$ , sits in a prime ideal,  $P$ , which also satisfies  $U \cap P = \emptyset$ .

**Proposition 4.5** Let  $R$  be a commutative rig and  $U \subseteq R$  be multiplicatively closed. Let  $I$  be an ideal of  $R$  such that  $U \cap I = \emptyset$ . Then there is a prime ideal  $P$  such that  $I \subseteq P$  and  $U \cap P = \emptyset$ .

PROOF: First, let  $A_i$  be a chain of ideals containing  $I$  and disjoint from  $U$ ; i.e.,  $I \subseteq A_{i1} \subseteq \dots$  and  $A_{ij} \cap U = \emptyset$ . Then  $A_i$  has a maximal element since  $T = \bigcup_j A_{ij}$  is an ideal. Let  $A$  be the set of all ideals  $J$  such that  $I \subseteq J$  and  $J \cap U = \emptyset$ . Then Zorn's lemma says that we can take an ideal  $P$  which is maximal such that  $I \subseteq P$  and  $P \cap U = \emptyset$ . Now we argue that  $P$  is prime by contrapositive. Suppose  $a, b \notin P$ , we must show  $ab \notin P$ .

Now,  $P \subseteq (a) + P$  and  $P \subseteq (b) + P$ . Then, the maximality of  $P$  says that  $((a) + P) \cap U \neq \emptyset$  and  $((b) + P) \cap U \neq \emptyset$ . Thus, there are  $c, d \in R$  and  $p_1, p_2 \in P$  such that  $ac + p_1 \in U$  and  $bd + p_2 \in U$ . Since  $U$  is multiplicatively closed,

$$(ac + p_1)(bd + p_2) = acbd + p_1bd + p_2ac + p_1p_2 \in U$$



Next,  $p_1bd, p_2ac, p_1p_2 \in P$ ; thus,  $p_1bd + p_2ac + p_1p_2 = p_3 \in P$ . Thus  $ab(cd) + p_3 \in U$ . Now, if  $ab \in P$  we get a contradiction with the assumption that  $P \cap U = \emptyset$ , since if  $(ab)cd \in P$  then  $ab(cd) + p_3 \in P$ . Thus  $ab \notin P$ .  $\square$

Now we can prove that for commutative rigs, multiplicative sets are the complement of a union of prime ideals. This result is well known for rings, in fact, this argument is similar to [Cohn 2002] proposition 10.2.8.

**Proposition 4.6** *Let  $R$  be a commutative rig. Then  $U \subseteq R$  is a multiplicative set if and only if  $R \setminus U$  is the union of a set of prime ideals.*

PROOF: Let  $U$  be a multiplicative set, and we will first show that  $R \setminus U = \bigcup_i \mathfrak{p}_i$ . for some set of prime ideals  $\mathfrak{p}_i$ . Suppose  $a \notin U$ . Then for every  $b \in R$ ,  $ab \notin U$ . Thus  $(a) \cap U = \emptyset$ . Then by proposition (4.5) there is a prime ideal  $\mathfrak{p}_0$  such that  $a \in \mathfrak{p}_0$  and  $\mathfrak{p}_0 \cap U = \emptyset$ . Since for every  $a \in R \setminus U$  we have such a prime ideal, we can take the union over all such prime ideals. Thus  $R \setminus U \subseteq \bigcup_i \mathfrak{p}_i$ . To show the other inclusion, suppose a set of prime ideals,  $\mathfrak{p}_i$ , are disjoint from  $U$ . Then by construction if  $a \in \bigcup_i \mathfrak{p}_i$  then  $a \notin U$ . Thus  $\bigcup_i \mathfrak{p}_i \subseteq R \setminus U$ .

Conversely, let  $R \setminus U = \bigcup_i \mathfrak{p}_i$ . Let  $a, b \in U$  then  $a, b \notin \bigcup_i \mathfrak{p}_i$ . Then it is immediate that  $ab \notin \bigcup_i \mathfrak{p}_i$ . Thus  $ab \in U$ ; that is,  $U$  is closed to multiplication. Now suppose  $cd \in U$ . Then we will get a contradiction if either  $c$  or  $d$  is assumed to not be in  $U$ . If  $c \in \bigcup_i \mathfrak{p}_i$ , then  $cd \in \bigcup_i \mathfrak{p}_i$  contradicting the assumption that  $cd \in U$ . Similarly if  $d \in \bigcup_i \mathfrak{p}_i$  then  $dc = cd \in \bigcup_i \mathfrak{p}_i$  contradicting the assumption that  $cd \in U$ . Thus both  $c, d$  are in  $U$ . Thus  $U$  is a multiplicative set.  $\square$

We have shown that the desired connection between prime ideals and multiplicative sets holds for rigs, despite the fact that the ideal theory of rigs does not coincide with congruences as it does in rings.

### 4.3 Fractions and Polynomials

In this subsection, we will describe a monad on  $\mathbf{CRig}$  which gives formal fractions, and recall the definition of polynomials as free  $R$ -algebras. Together, these two constructions can be used to define substitution of formal rational functions. As we will see later, this substitution is used to define composition in the category of formal rational functions.

We begin by defining the object part of our monad; that this is again a rig is easy to check.

**Proposition 4.7** *Let  $R$  be a commutative rig, then  $R \times R$  together with the operations*

$$\begin{aligned} (r, s) + (r', s') &= (rs' + r's, ss') & 0 &= (0, 1), \\ (r, s)(r', s') &= (rr', ss') & 1 &= (1, 1) \end{aligned}$$

*is a commutative rig.*

The commutative rig defined in proposition (4.7) will be called the rig of formal fractions, and denoted  $\mathfrak{fr}(R)$ . Think of the above operations as  $\frac{x}{y} + \frac{w}{z} = \frac{xz+wy}{yz}$  and  $\frac{x}{y} \frac{w}{z} = \frac{xw}{yz}$ . Note that  $\frac{0}{a} \neq \frac{0}{b}$  unless  $a = b$ . As noted in the introduction,  $\mathfrak{fr}(R)$  is not a ring, even if  $R$  is. Indeed, if  $\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$  then  $bd = 1$ . In general,  $b, d$  will not be multiplicative inverses of each other, so  $\mathfrak{fr}(R)$  will not have additive inverses.

The formal fraction construction is a functor from **CRig** to **CRig**. The above defines the object part of  $\mathfrak{fr}$ . If  $f : R \rightarrow S$ , then define  $\mathfrak{fr}(f) : \mathfrak{fr}(R) \rightarrow \mathfrak{fr}(S)$  by  $\mathfrak{fr}(f)(r_1, r_2) = (f(r_1), f(r_2))$ . The following proposition is then clear.

**Proposition 4.8**  $\mathfrak{fr} : \mathbf{CRig} \rightarrow \mathbf{CRig}$  is a functor.

The formal fraction construction carries even more structure.  $\mathfrak{fr}$  defines a monad on the category of commutative rigs. The multiplication of the monad,  $\mu$ , is the operation which maps  $\frac{\frac{a}{b}}{\frac{c}{d}}$  to  $\frac{ad}{bc}$ .

**Proposition 4.9** Let  $R, S$  be commutative rigs. Let  $f : R \rightarrow S$  be a commutative rig homomorphism. Let  $r_i, r \in R$ . Let  $\eta_R : R \rightarrow \mathfrak{fr}(R)$  be given by  $\eta_R(r) = (r, 1)$ , and  $\mu_R : \mathfrak{fr}(\mathfrak{fr}(R)) \rightarrow \mathfrak{fr}(R)$  be given by  $\mu_R((a, b), (c, d)) = (ad, bc)$ . Then  $(\mathfrak{fr}, \eta, \mu)$  is a monad, and further  $\eta$  is monic.

PROOF: Clearly,  $\eta$  is monic. We will show that  $\eta$  and  $\mu$  are the natural transformations which make  $(\mathfrak{fr}, \eta, \mu)$  a monad.

**( $\eta$  is natural)**

$$\begin{aligned} \eta_S(f(r)) &= (f(r), 1) \\ &= (f(r), f(1)) \\ &= \mathfrak{fr}(f)(r, 1) \\ &= \mathfrak{fr}(f)(\eta_R(r)) \end{aligned}$$

**( $\mu$  is natural)**

$$\begin{aligned} \mu_S(\mathfrak{fr}(\mathfrak{fr}(f))) &= \mu_S(\mathfrak{fr}(f)(r_1, r_2), \mathfrak{fr}(f)(r_3, r_4)) \\ &= \mu_S((f(r_1), f(r_2)), (f(r_3), f(r_4))) \\ &= (f(r_1)f(r_4), f(r_2)f(r_3)) \\ &= (f(r_1r_4), f(r_2r_3)) \\ &= \mathfrak{fr}(f)(ad, bc) \\ &= \mathfrak{fr}(f)(\mu_R((r_1, r_2), (r_3, r_4))) \end{aligned}$$

**(Unit laws)**

$$\begin{aligned} \mu_R(\eta_{\mathfrak{fr}(R)}(r_1, r_2)) &= \mu((r_1, r_2), (1, 1)) \\ &= (r_1, r_2) \end{aligned}$$

$$\begin{aligned} \mu_R(\mathfrak{fr}(\eta_R)(r_1, r_2)) &= \mu_R(\eta_R(r_1), \eta_R(r_2)) \\ &= \mu_R((r_1, 1), (r_2, 1)) \\ &= (r_1, r_2) \end{aligned}$$

(Associativity)

$$\begin{aligned}
& \mu_R(\text{ft}(\mu_R(((r_1, r_2), (r_3, r_4)), ((r_5, r_6), (r_7, r_8)))))) \\
&= \mu_R(\mu_R((r_1, r_2), (r_3, r_4)), \mu_R((r_5, r_6), (r_7, r_8))) \\
&= \mu_R((r_1 r_4, r_2 r_3), (r_5 r_8, r_6 r_7)) \\
&= (r_1 r_4 r_6 r_7, r_2 r_3 r_5 r_8) = (r_1 r_7 r_4 r_6, r_2 r_8 r_3 r_5) \\
&= \mu_R((r_1 r_7, r_2 r_8), (r_3 r_5, r_4 r_6)) \\
&= \mu_R((r_1, r_2)(r_7, r_8), (r_3, r_4)(r_5, r_6)) \\
&= \mu_R(\mu_{\text{ft}(R)}(((r_1, r_2), (r_3, r_4)), ((r_5, r_6), (r_7, r_8))))
\end{aligned}$$

□

We will use the Kleisli triple presentation of a monad to give a notation for working with formal fractions; this means writing  $\text{ft}(f)\mu$  as  $\#(f)$ . We have the correspondence,

$$\frac{f : R \longrightarrow \text{ft}(S)}{\#(f) : \text{ft}(R) \longrightarrow \text{ft}(S)}.$$

Next, we give the notation we will use for polynomials. Recall that for any commutative rig  $R$ , there is an adjunction between **Sets** and  $R/\mathbf{CRig}$ . The left functor in the adjunction takes a set  $B$  to the free commutative  $R$ -algebra on  $B$ , and we thus have the correspondence

$$\frac{\{x_1, \dots, x_n\} \longrightarrow U(S) \quad \mathbf{Sets}}{R[x_1, \dots, x_n] \longrightarrow S \quad R/\mathbf{CRig}}$$

This correspondence gives the unique homomorphism for substituting  $s_i \in S$  for  $x_i$  in  $R[x_1, \dots, x_n]$  for  $1 \leq i \leq n$  and sending  $r \in R \mapsto r \in R[x_1, \dots, x_n]$ . It is well known that elements of  $R[x_1, \dots, x_n]$  are of the form  $f = \sum_i r_i \mathbf{x}^{\alpha_i}$ , where  $\mathbf{x}^{\alpha_i} = x_1^{\alpha_i^1} \dots x_n^{\alpha_i^n}$ ; see for example [Hungerford 2000].

Next, we use the fact that if  $(T, \eta, \mu)$  is a monad on a category  $\mathbb{X}$ , and  $A$  is an object of  $\mathbb{X}$ , then we have a monad on  $A/\mathbb{X}$ ,  $(T^A, \eta^A, \mu^A)$ , where  $T^A(A \xrightarrow{f} X) = A \xrightarrow{f\eta} T(X)$  on objects, and sends an arrow  $(A \xrightarrow{f} X) \xrightarrow{g} (A \xrightarrow{f'} Y)$  to  $(A \xrightarrow{f\eta} T(X)) \xrightarrow{T(g)} (A \xrightarrow{f'\eta} T(Y))$ . Then  $\eta^A$  and  $\mu^A$  are given respectively by  $\eta$  and  $\mu$  in an obvious way. Specializing this to commutative  $R$ -algebras, and using the formal fraction monad, we obtain a way to substitute formal fractions into formal fractions of polynomials. Formally we have expressed substitution with the following two adjunctions,

$$\frac{\text{ft}(R[x_1, \dots, x_n]) \xrightarrow{\#(f[h])} \text{ft}(S) \quad R/\mathbf{CRig}}{R[x_1, \dots, x_n] \xrightarrow{f[h]} \text{ft}(S) \quad R/\mathbf{CRig}}$$

$$\{x_1, \dots, x_n\} \xrightarrow{h} U(\text{ft}(S)) \quad \mathbf{Sets}.$$

The substitution of  $s_i \in \mathfrak{ft}(S)$  for  $x_i$  into  $\frac{p}{q} \in \mathfrak{ft}(R[x_1, \dots, x_n])$  for each  $1 \leq i \leq n$  will be written as,

$$[s_1/x_1, \dots, s_n/x_n] \frac{p}{q},$$

which we may abbreviate as,

$$[s_i/x_i] \frac{p}{q}.$$

Here is an example of the substitution of three elements of  $\mathfrak{ft}(\mathbb{Z}[x_1, x_2])$  into an element of  $\mathfrak{ft}(\mathbb{Z}[y_1, y_2, y_3])$ :

**Example 4.10**

$$\begin{aligned} & \left[ \frac{5x_1x_2}{x_1}/y_1, \frac{x_1x_2^2}{x_1+x_2}/y_2, \frac{(x_1+x_2)^2}{3x_2}/y_3 \right] \frac{7(y_1+y_3)}{y_1y_2} \\ &= \mu \left( \frac{\left( \frac{105x_1x_2^2+7x_1(x_1+x_2)^2}{3x_1x_2} \right)}{\left( \frac{5x_1^2x_2^3}{x_1(x_1+x_2)} \right)} \right) \\ &= \frac{105x_1x_2^2+7x_1(x_1+x_2)^2x_1(x_1+x_2)}{3x_1x_2 \boxed{5x_1^2x_2^3}}. \end{aligned}$$

Consider the denominator of the resultant expression; there are two pieces  $3x_1x_2$  and  $5x_1^2x_2^3$ . The first piece comes from substituting into the numerator. The second piece, which we boxed to stand out, comes from substituting into the denominator, and is the numerator after this substitution. Thus, the resultant expression is undefined either where one of the expressions being substituted is undefined, or is undefined where the numerator of the substitution into the denominator is zero. Using the definition of addition and  $\mu$ , it is easy to see that these two considerations always capture the undefinedness after substitution. We will use this idea to define the “partiality” of these formal rational functions.

#### 4.4 The Category of Rational Functions

As mentioned in the introduction, we begin by defining a category of formal rational functions. The objects are natural numbers. The maps  $n \rightarrow m$  in this category are  $m$ -tuples of formal rational functions in  $n$  variables equipped with a multiplicative set of polynomials called the **restriction set**. This multiplicative set of polynomials has the property that zeroes of polynomials in this set are the points at which the rational function is undefined. For convenience we will often write  $x_1, \dots, x_n$  as  $\vec{x}^n$ .

**Definition 4.11** *Let  $R$  be a commutative rig. Define  $\text{FRAT}_R$  to be the following*

**Objects:**  $n \in \mathbb{N}$

**Arrows:**  $n \rightarrow m$  given by a pair  $(\vec{x}^n \mapsto (f_i, g_i)_{i=1}^m, \mathcal{U})$  where

- $(f_i, g_i) \in \mathfrak{ft}(R[x_1, \dots, x_n])$  for each  $i$
- $\mathcal{U} \subseteq R[x_1, \dots, x_n]$  is a finitely generated multiplicative set

- Every  $g_i \in \mathcal{U}$

**Identity:**  $(\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \{\}) : n \longrightarrow n$

**Composition:** Given  $(\vec{x}^n \mapsto (f_i, g_i)_{i=1}^m, \mathcal{U}) : n \longrightarrow m$  and  $(\vec{x}^m \mapsto (f'_i, g'_i)_{i=1}^k, \mathcal{U}') : m \longrightarrow k$ , then the composition is given by substitution:

$$\frac{(\vec{x}^n \mapsto (f_i, g_i)_{i=1}^m, \mathcal{U}) \quad (\vec{x}^m \mapsto (f'_i, g'_i)_{i=1}^k, \mathcal{U}')}{(\vec{x}^n \mapsto (a_i, b_i)_{i=1}^k, \mathcal{U}'')}$$

Where,

- $(a_j, b_j) = [(f_i, g_i)/x_i](f'_j, g'_j)$ ,
- $(\alpha_j, \beta_j) = [(f_i, g_i)/x_i] u'_j$  where  $u'_j$  is a generator of  $\mathcal{U}'$ ,
- And  $\mathcal{U}'' = \langle u_1, \dots, u_l, \alpha_1, \dots, \alpha_w \rangle$ , the multiplicative and factor closure of the generators  $u_i, \alpha_i$  where  $u_i \in \mathcal{U}$ .

For an explanation of  $\mathcal{U}''$  see the discussion after example (4.10); also note that for each  $j$ ,  $\beta_j \in \mathcal{U}$ .

We can extend the substitution example above to give an example of composition in  $\text{FRAT}_{\mathbb{Z}}$ . Take the maps

$$\left( x_1, x_2 \mapsto \left( \frac{5x_1x_2}{x_1}, \frac{x_1x_2^2}{x_1+x_2}, \frac{(x_1+x_2)^2}{3x_2} \right), \langle x_1, x_1+x_2, x_2 \rangle \right) : 2 \longrightarrow 3,$$

and

$$\left( x_1, x_2, x_3 \mapsto \left( \frac{7(x_1+x_3)}{x_1x_2}, \frac{x_1}{1} \right), \langle 4+x_3+x_1, x_1, x_2 \rangle \right) : 3 \longrightarrow 2.$$

The composite of the above maps is,

$$\left( x_1, x_2 \mapsto \left( \frac{105x_1x_2^2 + 7x_1^2(x_1+x_2)^3}{15x_1^3x_2^4}, \frac{5x_1x_2}{x_1} \right), \langle \frac{x_1, x_1+x_2, x_2, 5x_1x_2}{15x_1x_2^2+12x_1x_2+x_1(x_1+x_2)^2} \rangle \right) : 2 \longrightarrow 2.$$

Now we will prove that  $\text{FRAT}_R$  is a category.

**Proposition 4.12**  $\text{FRAT}_R$  is a category for each commutative rig  $R$ .

PROOF: Substitution is a homomorphism, so composition is clearly closed and associative. Thus, we must show that the unit laws hold. Since the proofs are sufficiently similar, we provide the proof for the left identity.

Consider

$$(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) : n \longrightarrow m.$$

And take

$$p_j = \sum_l \gamma_l \prod_j x_j^{l_j}$$

$$q_j = \sum_l \delta_l \prod_j x_j^{l_j}$$

After lifting the coefficients appropriately, the following calculation simplifies the result of the substitution,

$$\begin{aligned}
(a_j, b_j) &= [(x_i, 1) / x_i] \sum_l (\gamma_l, 1) x_1^{l_1} \cdots x_n^{l_n} \\
&= \sum_l (\gamma_l, 1) (x_1, 1)^{l_1} \cdots (x_n, 1)^{l_n} \\
&= \sum_l (\gamma_l, 1) (x_1^{l_1}, 1) \cdots (x_n^{l_n}, 1) \\
&= \sum_l (\gamma_l x_1^{l_1} \cdots x_n^{l_n}, 1) \\
&= \left( \sum_l \gamma_l x_1^{l_1} \cdots x_n^{l_n}, 1 \right) \\
&= (p_j, 1)
\end{aligned}$$

We similarly derive

$$(c_j, d_j) = \left( \sum_l \delta_l x_1^{l_1} \cdots x_n^{l_n}, 1 \right) = (q_j, 1)$$

And for  $u_j = \sum_l \tau_l \prod_j x_j^{l_j}$

$$(\alpha_j, \beta_j) = \left( \sum_l \tau_l x_1^{l_1} \cdots x_n^{l_n}, 1 \right) = (u_j, 1)$$

Then the composite is

$$(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) : n \longrightarrow m$$

□

Moreover, the category  $\text{FRAT}_R$  is a restriction category.

**Proposition 4.13** *For every commutative rig  $R$ ,  $\text{FRAT}_R$  has a restriction structure given by*

$$\overline{(\vec{x}^n \mapsto (f_i, g_i)_{i=1}^m, \mathcal{U})} = (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \mathcal{U}).$$

PROOF:

**R1**

$$\begin{aligned}
&(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle u_1, \dots, u_s \rangle) \\
&= (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s, u_1, \dots, u_s \rangle) \\
&= (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle)
\end{aligned}$$

**R2** Immediate, since the ordering of the generators in the multiplicative set does not matter; i.e.,

$$\langle u_1, \dots, u_s, v_1, \dots, v_w \rangle = \langle v_1, \dots, v_w, u_1, \dots, u_s \rangle$$

**R3** Consider first,  $\bar{g}f$

$$\begin{aligned} & (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle v_1, \dots, v_w \rangle) (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) \\ = & (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s, v_1, \dots, v_w \rangle) \end{aligned}$$

Then,

$$\begin{aligned} & \overline{(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle)} \overline{(\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^k, \langle v_1, \dots, v_w \rangle)} \\ = & (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle u_1, \dots, u_s \rangle) (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle v_1, \dots, v_w \rangle) \\ = & (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle u_1, \dots, u_s, v_1, \dots, v_w \rangle) \\ = & \overline{(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s, v_1, \dots, v_w \rangle)} \end{aligned}$$

**R4** First we calculate  $fg$ .

$$\begin{aligned} & (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) (\vec{x}^m \mapsto (p'_i, q'_i)_{i=1}^k, \langle t_1, \dots, t_w \rangle) \\ = & (\vec{x}^n \mapsto (a_i d, b_i c)_{i=1}^k, \langle u_1, \dots, u_s, \alpha_1, \dots, \alpha_w \rangle) \end{aligned}$$

Then a few manipulations show that  $f\bar{g}$  equals  $\bar{f}g$ . Consider,

$$\begin{aligned} & (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) (\vec{x}^m \mapsto (x_i, 1)_{i=1}^m, \langle t_1, \dots, t_w \rangle) \\ = & (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s, \alpha_1, \dots, \alpha_w \rangle) \\ = & (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle u_1, \dots, u_s, \alpha_1, \dots, \alpha_w \rangle) (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) \\ = & \overline{(\vec{x}^n \mapsto (a_i d, b_i c)_{i=1}^k, \langle u_1, \dots, u_s, \alpha_1, \dots, \alpha_w \rangle)} (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \langle u_1, \dots, u_s \rangle) \end{aligned}$$

□

Next, we will define the category of rational functions  $\text{RAT}_R$  by imposing the following quotient on the hom-sets of  $\text{FRAT}_R$ .

**Definition 4.14** Let  $(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}), (\vec{x}^n \mapsto (y_i, z_i)_{i=1}^m, \mathcal{U}) \in \text{FRAT}_R(n, m)$ . Then the relation defining  $\text{RAT}_R$  is

$$(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) \sim (\vec{x}^n \mapsto (y_i, z_i)_{i=1}^m, \mathcal{U})$$

iff for each  $i \leq m$  there is an  $u \in \mathcal{U}$  such that

$$up_i z_i = uq_i y_i \in R[x_1, \dots, x_n]$$

Here is an example showing how maps are identified by this equivalence relation.

**Example 4.15**

$$\left(x \mapsto \frac{x}{x}, \langle x \rangle\right) = (x \mapsto 1, \langle x \rangle),$$

but

$$\left(x \mapsto \frac{x}{x}, \langle x \rangle\right) \neq (x \mapsto 1, \{\}).$$

The relation,  $\sim$ , is clearly an equivalence relation. To obtain a restriction category by quotienting the homsets of  $\text{FRAT}_R$ , we need to show that the above relation is a **restriction congruence**. This means that the equivalence relation is compatible with composition and the restriction operation. To be compatible with composition, we need that if  $(f, \mathcal{U}) \sim (f', \mathcal{U}) : n \longrightarrow m$  then for any  $(h, \mathcal{V}) : k \longrightarrow n$  and  $(g, \mathcal{W}) : m \longrightarrow k$  we have  $(h, \mathcal{V})(f, \mathcal{U}) \sim (h, \mathcal{V})(f', \mathcal{U})$  and  $(f, \mathcal{U})(g, \mathcal{W}) \sim (f', \mathcal{U})(g, \mathcal{W})$ . To be compatible with the restriction operation we need  $(f, \mathcal{U}) \sim (f', \mathcal{U})$

**Lemma 4.16** *The equivalence relation defined above is a restriction congruence.*

PROOF: Let  $\mathcal{U} = \langle \mathcal{U}_1, \dots, \mathcal{U}_k \rangle$ , and assume that

$$(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) \sim (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{U}).$$

First we will show the relation is compatible with composition on the left. Let  $\mathcal{V} = \langle \mathcal{V}_1, \dots, \mathcal{V}_l \rangle$ , and consider  $(\vec{x}^k \mapsto (f_i, g_i)_{i=1}^n, \mathcal{V})$ . We must show that

$$\left( \vec{x}^k \mapsto (f_i, g_i)_{i=1}^n, \mathcal{V} \right) (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) \sim \left( \vec{x}^k \mapsto (f_i, g_i)_{i=1}^n, \mathcal{V} \right) (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{U}).$$

Let  $(\alpha_j, \beta_j) = [(f_i, g_i)/x_i] \mathcal{U}_j$ , and set  $\mathcal{V}' = \langle \mathcal{V}_1, \dots, \mathcal{V}_l, \alpha_1, \dots, \alpha_k \rangle$ . Thus, after computing the above composition, the goal of the proof is to show:

$$\left( \vec{x}^n \mapsto ((f_i, g_i)/x_i) (p_j, q_j)_{j=1}^m, \mathcal{V}' \right) \sim \left( \vec{x}^n \mapsto ((f_i, g_i)/x_i) (p'_j, q'_j)_{j=1}^m, \mathcal{V}' \right).$$

Now, take

$$\begin{aligned} (a_j, b_j) &= [(f_i, g_i)/x_i] p_j; & (c_j, d_j) &= [(f_i, g_i)/x_i] q_j \\ (a'_j, b'_j) &= [(f_i, g_i)/x_i] p'_j; & (c'_j, d'_j) &= [(f_i, g_i)/x_i] q'_j. \end{aligned}$$

Our goal is then to show that there is a  $V_j \in \mathcal{V}'$  such that

$$V_j a_j d_j b'_j c'_j = V_j a'_j d'_j b_j c_j.$$

Now note that if we have  $V_{j1}, V_{j2} \in \mathcal{V}'$  such that

$$(V_{j1}, V_{j2})(a_j, b_j)(c'_j, d'_j) = (V_{j1}, V_{j2})(a_j, b_j)(c'_j, d'_j),$$

then the proof is done, since  $V_{j1} a_j c'_j = V_{j1} a'_j c_j$  and  $V_{j2} b_j d'_j = V_{j2} b'_j d_j$  which imply

$$V_{j1} V_{j2} a_j d_j b'_j c'_j = V_{j1} a_j c'_j V_{j2} b'_j d_j = V_{j1} V_{j2} a'_j d'_j b_j c_j.$$

Note the assumption,  $(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) \sim (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{U})$ , means that for each  $i$ , there exists a  $U_i \in \mathcal{U}$  such that  $U_i p_i q'_i = U_i p'_i q_i$ . For each  $i$ , set  $(\gamma_{i1}, \gamma_{i2}) = [(f_i, g_i)/x_i] U_i$ . We will prove that  $(\gamma_{i1}, \gamma_{i2})$  gives the desired equality.

$$\begin{aligned} & (\gamma_{i1}, \gamma_{i2}) [(f_i, g_i)/x_i] p_j [(f_i, g_i)/x_i] q'_j \\ &= [(f_i, g_i)/x_i] U_j [(f_i, g_i)/x_i] p_j [(f_i, g_i)/x_i] q'_j \\ &= [(f_i, g_i)/x_i] (U_j p_j q'_j) && \text{subst. is a homomorphism} \\ &= [(f_i, g_i)/x_i] (U_j p'_j q_j) && \text{definition of } \sim \\ &= [(f_i, g_i)/x_i] U_j [(f_i, g_i)/x_i] p'_j [(f_i, g_i)/x_i] q_j && \text{subst. is a homomorphism} \\ &= (\gamma_{i1}, \gamma_{i2}) [(f_i, g_i)/x_i] p'_j [(f_i, g_i)/x_i] q_j \end{aligned}$$



Thus, we have shown that  $\sim$  is compatible with composition on the left.

Next, we will show that  $\sim$  is compatible with composition on the right. Consider the map  $(\vec{x}^n \mapsto (y_i, z_i)_{i=1}^k, \mathcal{T})$ . We must show:

$$(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) (\vec{x}^n \mapsto (y_i, z_i)_{i=1}^k, \mathcal{T}) \sim (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{U}) (\vec{x}^n \mapsto (y_i, z_i)_{i=1}^k, \mathcal{T}).$$

Let  $\mathcal{U}' = \langle \mathcal{U} \cup ((p_i, q_i)/x_i] \mathcal{T} \rangle$ , and again, let:

$$\begin{aligned} (a_j, b_j) &= [(p_i, q_i)/x_i] y_j; & (c_j, d_j) &= [(p_i, q_i)/x_i] z_j \\ (a'_j, b'_j) &= [(p'_i, q'_i)/x_i] y_j; & (c'_j, d'_j) &= [(p'_i, q'_i)/x_i] z_j. \end{aligned}$$

Our goal is to show that for each  $j$ , there is a  $U_j \in \mathcal{U}'$  such that  $U_j a_j d_j b'_j c'_j = U_j a'_j d'_j b_j c_j$ . We will find such a  $U_j$  in two parts. First, we will prove that for each  $j$ , there is a  $V_j \in \mathcal{U}'$  such that  $V_j a_j b'_j = V_j a'_j b_j$ . Similarly, the second part is a  $V'_j \in \mathcal{U}'$  such that  $V'_j d_j c'_j = V'_j d'_j c_j$ . Setting  $U_j = V_j V'_j$  will complete the proof.

Now, we construct  $V_j$ . To start the proof, for each  $l$  we have:

$$y_l = \sum_j \gamma_{lj} x_1^{j_1} \cdots x_n^{j_n}.$$

After substituting, we have:

$$\begin{aligned} [(p_i, q_i)/x_i] y_l &= \sum_j \frac{\gamma_{lj} p_1^{j_1} \cdots p_n^{j_n}}{q_1^{j_1} \cdots q_n^{j_n}} \\ &= \frac{\sum_j \gamma_{lj} p_1^{j_1} \cdots p_n^{j_n} \prod_{i \neq j} q_1^{i_1} \cdots q_n^{i_n}}{\prod_i q_i^{i_1} \cdots q_n^{i_n}}, \end{aligned}$$

and

$$[(p'_i, q'_i)/x_i] y_l = \frac{\sum_j \gamma_{lj} p_1'^{j_1} \cdots p_n'^{j_n} \prod_{i \neq j} q_1'^{i_1} \cdots q_n'^{i_n}}{\prod_i q_i'^{i_1} \cdots q_n'^{i_n}}.$$

Now, for each  $i$  there exists a  $U_i \in \mathcal{U}'$  such that  $U_i p_i q'_i = U_i p'_i q_i$ . Let  $w = \max\{i_j\}$ ; we will show  $V_j = \prod_i^n U_i^w$ , consider:

$$\begin{aligned}
& \left( \prod_i^n U_i^w \right) \left( \sum_j \gamma_j p_1^{j_1} \cdots p_n^{j_n} \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \prod_i^n q_i^{i_1} \cdots q_n^{i_n} \\
&= \left( \sum_j \gamma_j U_1^w p_1^{j_1} \cdots U_n^w p_n^{j_n} \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \prod_i^n q_i^{i_1} \cdots q_n^{i_n} \\
&= \left( \sum_j \gamma_j U_1^w p_1^{j_1} \cdots U_n^w p_n^{j_n} \left( \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) q_1^{j_1} \cdots q_n^{j_n} \left( \prod_{i \neq j}^n q_i^{i_1} \cdots q_n^{i_n} \right) \right) \\
&= \sum_j \gamma_j U_1^{w-j_1} U_1^{j_1} p_1^{j_1} q_1^{j_1} \cdots U_n^{w-j_n} U_n^{j_n} p_n^{j_n} q_n^{j_n} \left( \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \left( \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \\
&= \sum_j \gamma_j U_1^{w-j_1} (U_1 p_1 q'_1)^{j_1} \cdots U_n^{w-j_n} (U_n p_n q'_n)^{j_n} \left( \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \left( \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \\
&= \sum_j \gamma_j U_1^{w-j_1} (U_1 p'_1 q_1)^{j_1} \cdots U_n^{w-j_n} (U_n p'_n q_n)^{j_n} \left( \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \left( \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \\
&= \left( \prod_i^n U_i^w \right) \left( \sum_j \gamma_j p_1^{j_1} \cdots p_n^{j_n} \prod_{i \neq j}^n q_1^{i_1} \cdots q_n^{i_n} \right) \prod_i^n q_i^{i_1} \cdots q_n^{i_n}.
\end{aligned}$$

Thus,  $V_j = \prod_i^n U_i^w$  as desired. One can similarly derive  $V_j'$ . Therefore we have shown

$$(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) (\vec{x}^n \mapsto (y_i, z_i)_{i=1}^k, \mathcal{T}) \sim (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{U}) (\vec{x}^n \mapsto (y_i, z_i)_{i=1}^k, \mathcal{T}),$$

as desired.

Finally, we will show that  $\sim$  is compatible with the restriction operation. This is immediate since, in fact,

$$\overline{(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U})} = \overline{(\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{U})}.$$

Therefore, we have proved that  $\sim$  is a restriction congruence.  $\square$

Finally, we define  $\text{RAT}_R$  as  $\text{FRAT}_R / \sim$ , and have the following:

**Proposition 4.17**  *$\text{RAT}_R$  is a restriction category for any commutative rig  $R$ .*

It might seem that breaking the construction of rational functions in two stages is unnecessary, as one could directly define rational functions by generating a ring localized at a multiplicative set of polynomials. However, this allows reductions in the rational functions such as  $\frac{x}{x}$  to 1. In the substitution of the restriction set, the numerator *after* substituting is used; allowing such a reduction causes some elements to be left out of the restriction set.

## 4.5 Embedding of Rational Functions in $\text{Par}(\mathbf{CRig}^{\text{op}}, \text{LOC})$

Before we prove that  $\text{RAT}_R$  has differential structure, we wish to give a different interpretation of the partiality of the maps in  $\text{RAT}_R$ . To that end, the goal of this section is to prove that  $\text{RAT}_R$  embeds into the partial map category of commutative rigs opposite with respect to localizations. The definition of a localization as a map of commutative rigs is a direct generalization of localization for commutative rings, as in [Eisenbud 2004].

**Definition 4.18** *A localization  $\phi$  is a rig homomorphism  $\phi : R \rightarrow S$  such that there exists a multiplicatively closed set,  $U$ , with  $\phi(U) \subseteq \text{units}(S)$ , and for any rig homomorphism  $f : R \rightarrow T$ , with  $f(U) \subseteq \text{units}(T)$ , there is a unique map  $k : S \rightarrow T$  that makes the following diagram commute:*

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ & \searrow f & \vdots k \\ & & T. \end{array}$$

Denote the class of localizations by  $\text{LOC}$ . We would like to show that  $\text{LOC}$  is a stable system of monics in  $\mathbf{CRig}^{\text{op}}$  so that there is a partial map category of commutative rigs opposite with respect to localizations.

If  $R$  is a commutative rig, and  $U$  is a multiplicatively closed set,  $R[U^{-1}]$  is the rig obtained by making all the elements of  $U$  into units. This rig is called the rig of fractions with respect to a multiplicative set  $U$ , as the operations in the rig are defined as for fractions; see for example [Dummit and Foote 2004]. There is a canonical localization,  $l_U : R \rightarrow R[U^{-1}]$ ;  $l_U(r) = \frac{r}{1}$ . It is clear that localizations in  $\mathbf{CRig}$  are epic, contain all isomorphisms, and are closed to composition. Furthermore, we have the following:

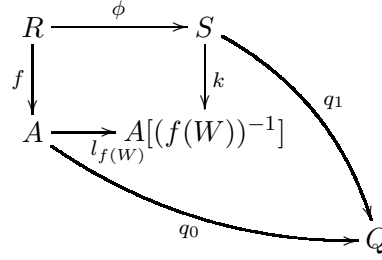
**Proposition 4.19** *The pushout along a localization exists, and is a localization.*

PROOF:

Let  $R, A, S$  be rigs. Let  $\phi : R \rightarrow S$  be a localization, let  $f : R \rightarrow A$  be a rig homomorphism, and let  $W \subseteq R$  be a multiplicatively closed set that  $\phi$  inverts. Then  $f(W)$  is also a multiplicative set, so we can form the canonical localization  $l_{f(W)} : A \rightarrow A[(f(W))^{-1}]$ . This means that  $l_{f(W)}f(W) \subseteq \text{units}(A[(f(W))^{-1}])$ , and so we get a unique  $k : S \rightarrow A[(f(W))^{-1}]$  such that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ f \downarrow & & \downarrow k \\ A & \xrightarrow{l_{f(W)}} & A[(f(W))^{-1}] \end{array}$$

Next, we show the above square gives a pushout. Suppose  $Q$  is a rig, and  $q_0, q_1$  are rig homomorphisms such that the outer square commutes in,



If we can show that  $q_0$  sends  $f(W)$  to units, then we get an arrow from  $A[(f(W))^{-1}] \rightarrow Q$ . Now,  $q_0(f(W)) = q_1(\phi(W))$  by commutativity; thus,  $q_1(\phi(W)) \subset \text{units}(Q)$ , so  $q_0(f(W)) \subset \text{units}(Q)$ . Then we induce a map  $\hat{k} : A[(f(W))^{-1}] \rightarrow Q$ . Next, we must show that  $k\hat{k} = q_1$ . Now,

$$\phi q_1 = f q_0 = f l_{f(W)} \hat{k} = \phi k \hat{k}$$

Since  $\phi$  is epic,  $q_1 = k\hat{k}$ . Moreover, since  $\hat{k}$  is the unique map that makes the bottom triangle commute, the square is indeed a pushout.  $\square$

Thus LOC is a stable system of monics in  $\mathbf{CRig}^{op}$ , and so we can form its partial map category.

**Proposition 4.20** *( $\mathbf{CRig}^{op}, \text{LOC}$ ) is an  $\mathcal{M}$ -category, and  $\text{Par}(\mathbf{CRig}^{op}, \text{LOC})$  is a split restriction category.*

Now we show that for any commutative rig  $R$ ,  $\text{RAT}_R$  embeds into this partial map category.

**Proposition 4.21** *If  $R$  is any commutative rig, then there is a faithful restriction embedding of  $\text{RAT}_R$  into  $\text{Par}(\mathbf{CRig}^{op}, \text{LOC})$ .*

PROOF: Define the following inclusion functor  $\text{RAT}_R \hookrightarrow \text{Par}(\mathbf{CRig}^{op}, \text{LOC})$  where in  $\text{RAT}_R$ , each natural number,  $n$ , is sent to the commutative rig,  $R[x_1, \dots, x_n]$ . Next every map,  $n \xrightarrow{(f, \mathcal{U})} m$ , in  $\text{RAT}_R$  is sent to the cospan

$$\begin{array}{ccc}
& R[x_1, \dots, x_n][\mathcal{U}^{-1}] & \\
l_{\mathcal{U}} \nearrow & & \nwarrow \xi_f \\
R[x_1, \dots, x_n] & & R[x_1, \dots, x_m]
\end{array}$$

where  $\xi_f$  is the substitution homomorphism which does the substitution  $[f_i/x_i]$ .  $\text{RAT}$  is clearly a bijection on objects. Now assume we have two equal maps; by the definition of the partial map category, this means we have

$$\begin{array}{ccc}
& R[x_1, \dots, x_n][\mathcal{U}^{-1}] \xrightarrow{\cong} R[x_1, \dots, x_n][\mathcal{U}'^{-1}] & \\
l_{\mathcal{U}} \nearrow & \xleftrightarrow{l_{\mathcal{U}'}} & \nwarrow \xi_f \\
R[x_1, \dots, x_n] & & R[x_1, \dots, x_m] \\
& \searrow \xi_f & \nearrow \xi_{f'} \\
& & R[x_1, \dots, x_m]
\end{array}$$

Then,

$$(p_i, q_i)\alpha = (p'_i, q'_i) \text{ by assumption.}$$

But since  $\alpha$  is an isomorphism, it is unit reflecting, meaning that  $q_i \in \mathcal{U}'$ . Similarly,  $q'_i \in \mathcal{U}$ . Thus  $\mathcal{U}' = \mathcal{U}$ , so the localization  $l_{\mathcal{U}} = 1l_{\mathcal{U}}$ . By the uniqueness property of localizations, we have that  $\alpha = 1_{R[x_1, \dots, x_n][U^{-1}]}$ ; therefore,  $\xi_f = \xi_{f'}$ . Thus  $f = f'$  which implies  $(f, \mathcal{U}) = (f', \mathcal{U}')$ . Thus the embedding is faithful. Note the embeddings is a restriction functor, since  $(\overrightarrow{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) = (\overrightarrow{x}^n \mapsto (x_i, 1)_{i=1}^n, \mathcal{U})$  is sent to

$$R[x_1, \dots, x_n] \xrightarrow{l_U} R[x_1, \dots, x_n][U^{-1}] \xleftarrow{\xi_{(x_i)_i}} R[x_1, \dots, x_n],$$

but  $\xi_{(x_i)_i} = [x_i/x_i] = l_U$ , which is

$$\overline{R[x_1, \dots, x_n] \xrightarrow{l_U} R[x_1, \dots, x_n][U^{-1}] \xleftarrow{\xi_{(p_i, q_i)_i}} R[x_1, \dots, x_m]},$$

the restriction of the map in the partial map category. This finishes the proof that embedding is a faithful restriction embedding.  $\square$

We started off by defining a restriction category of rational functions from any commutative rig as a direct way to view the partiality implicit in rational functions. The above result gives another, more general, view of the partiality in commutative rigs.

## 4.6 Rational Functions are a Cartesian Restriction Category

Our next goal is to show the category of rational functions is a differential restriction category. The first piece we will need to show is that  $\text{RAT}_R$  is a cartesian restriction category. Doing so will require no assumptions about properties of the rig.

**Proposition 4.22**  *$\text{RAT}_R$  is a cartesian restriction category for each commutative rig  $R$ .*

PROOF:

**Restriction Terminal Object** The object  $0 \in \mathbb{N}$  will be the terminal object, with  $!_n : n \rightarrow 0$  equal to  $(\overrightarrow{x}^n \mapsto (), \{\})$  for each  $n$ . Moreover,  $!_0 = \text{id}_0$ , and  $!_n$  is always total. Then for any  $(\overrightarrow{x}^m \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) : n \rightarrow m$  we have

$$\begin{aligned} & (\overrightarrow{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) (\overrightarrow{x}^m \mapsto (), \{\}) \\ &= (\overrightarrow{x}^n \mapsto (), \mathcal{U}) \\ &= (\overrightarrow{x}^n \mapsto (x_i, 1)_{i=1}^n, \mathcal{U}) (\overrightarrow{x}^n \mapsto (), \{\}) \\ &= \overline{(\overrightarrow{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) (\overrightarrow{x}^n \mapsto (), \{\})} \\ &= \overline{(\overrightarrow{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) (\overrightarrow{x}^m \mapsto (), \{\})} (\overrightarrow{x}^n \mapsto (), \{\}) \\ &= \overline{(\overrightarrow{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) (\overrightarrow{x}^m \mapsto (), \{\})} (\overrightarrow{x}^n \mapsto (), \{\}) \end{aligned}$$

so that  $0$  is indeed the terminal object.

**Restriction Products** The product of two objects in  $\text{RAT}_R$  for any  $R$  will be given by addition.

The total projections are

$$\begin{aligned}\pi_0 &= (\overrightarrow{x}^{n+m} \mapsto (x_i, 1)_{i=1}^n, \{\}) \\ \pi_1 &= (\overrightarrow{x}^{n+m} \mapsto (x_i, 1)_{i=n+1}^{n+m}, \{\})\end{aligned}$$

The pairing is given by the following:

$$\begin{aligned}& \left\langle \left( \overrightarrow{x}^k \mapsto (p_i, q_i)_{i=1}^n, \mathcal{U} \right), \left( \overrightarrow{x}^k \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{V} \right) \right\rangle \\ &= \left( \overrightarrow{x}^k \mapsto (p_1, q_1), \dots, (p_n, q_n), (p'_1, q'_1), \dots, (p'_m, q'_m), \langle \mathcal{U} \cup \mathcal{V} \rangle \right)\end{aligned}$$

Now consider:

$$\begin{aligned}& \left\langle \left( \overrightarrow{x}^k \mapsto (p_i, q_i)_{i=1}^n, \mathcal{U} \right), \left( \overrightarrow{x}^k \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{V} \right) \right\rangle \pi_0 \\ &= \left( \overrightarrow{x}^k \mapsto (p_1, q_1), \dots, (p_n, q_n), (p'_1, q'_1), \dots, (p'_m, q'_m), \langle \mathcal{U} \cup \mathcal{V} \rangle \right) \\ & \quad \left( \overrightarrow{x}^{n+m} \mapsto (x_i, 1)_{i=1}^n, \{\} \right) \\ &= \left( \overrightarrow{x}^k \mapsto (p_i, q_i)_{i=1}^n, \langle \mathcal{U} \cup \mathcal{V} \rangle \right) \\ &= \left( \overrightarrow{x}^k \mapsto (x_i, 1)_{i=1}^k, \langle \mathcal{U} \cup \mathcal{V} \rangle \right) \left( \overrightarrow{x}^k \mapsto (p_i, q_i)_{i=1}^n, \mathcal{U} \right) \\ &= \overline{\left( \overrightarrow{x}^k \mapsto (p_1, q_1), \dots, (p_n, q_n), (p'_1, q'_1), \dots, (p'_m, q'_m), \langle \mathcal{U} \cup \mathcal{V} \rangle \right)} \\ & \quad \left( \overrightarrow{x}^k \mapsto (p_i, q_i)_{i=1}^n, \mathcal{U} \right) \\ &= \overline{\left\langle \left( \overrightarrow{x}^k \mapsto (p_i, q_i)_{i=1}^n, \mathcal{U} \right), \left( \overrightarrow{x}^k \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{V} \right) \right\rangle} \left( \overrightarrow{x}^k \mapsto (p_i, q_i)_{i=1}^n, \mathcal{U} \right)\end{aligned}$$

so that  $\pi_0$  is indeed a projection. The same proof, *mutatis mutandis*, works for  $\pi_1$ .

□

It may be helpful to describe the product of two maps. Suppose we have

$$h = \left( \overrightarrow{x}^n \mapsto (h_{ni}, h_{di})_{i=1}^k, \mathcal{V} \right) : n \longrightarrow k \text{ and } g = \left( \overrightarrow{x}^m \mapsto (g_{ni}, g_{di})_{i=1}^j, \mathcal{W} \right) : m \longrightarrow j,$$

and we want to consider the product,  $(h \times g) : n + m \longrightarrow k + j$ . Let  $\vec{x} = (x, \dots, x_n)$ ,  $\vec{x}' = (x_{n+1}, \dots, x_{n+m})$ . Let  $h'_i(\vec{x}, \vec{x}') = h_i(\vec{x})$  and  $g'_i(\vec{x}, \vec{x}') = g_i(\vec{x}')$ . Then it is straightforward to see that the product of the maps must be

$$h \times g = \left( \overrightarrow{x}^{n+m} \mapsto (h'_1, \dots, h'_k, g'_1, \dots, g'_j), \langle \mathcal{V} \cup ([x_{n+j}/x_j] \mathcal{W}_{j=1}^m) \rangle \right)$$

Note that the variables in  $\mathcal{W}$  are shifted to account for the fact that all the  $g'_i$  are phenomenally in variables  $x_{n+1}$  to  $x_{n+m}$ .

#### 4.7 Rational Functions have Left Additive Restriction Structure

To be a differential restriction category, the category must have cartesian left additive structure. We will show  $\text{RAT}_R$  has such structure, and this structure exists for any commutative rig  $R$ .

**Proposition 4.23** *For each commutative rig,  $R$ ,  $\text{RAT}_R$  is a left additive restriction category.*

PROOF: Let  $(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}), (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{V}) : n \rightarrow m$ . Define

$$\begin{aligned} & (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) + (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{V}) \\ &= (\vec{x}^n \mapsto (p_i q'_i + p'_i q_i, q_i q'_i)_{i=1}^m, \langle \mathcal{U} \cup \mathcal{V} \rangle) \end{aligned}$$

so we are using the addition defined in  $\text{ft}(R[x_1, \dots, x_n])$ .

**RAT<sub>R</sub>(n,m) is a commutative monoid** Clearly, the set of irreducibles are joined by union, and so that part is commutative. Next, pick any index  $l$ , and each component can be commuted since  $\text{ft}(R[x_1, \dots, x_n])$  is a rig, and so addition is commutative and associative. Thus every piece can be commuted, and the addition as defined gives a commutative monoid on  $\text{RAT}_R(n,m)$  if the additive zero is in the hom-object. Take  $0_{nm} = (\vec{x}^n \mapsto ((0, 1))^m, \{\})$ . Again, since  $\text{ft}(R[x_1, \dots, x_n])$  is a rig, each component will act as an additive identity, and since  $0_{nm}$  is total,  $0_{nm}$  is the additive identity for  $\text{RAT}_R(n,m)$ .

**Additive restriction** This follows immediately,

$$\begin{aligned} & \overline{(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) + (\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{V})} \\ &= (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle \mathcal{U} \cup \mathcal{V} \rangle) \\ &= (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \mathcal{U}) (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \mathcal{V}) \\ &= \overline{(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U})} \overline{(\vec{x}^n \mapsto (p'_i, q'_i)_{i=1}^m, \mathcal{V})} \end{aligned}$$

**Restriction of additive zero** Since  $0_{nm}$  is total,  $\overline{0_{nm}} = 1_n$ .

**Left additivity** It is clear that the restriction sets will be the same. The substitution distributes over sums because substitution is a homomorphism.

□

Since, the addition and pairing of maps is componentwise, it is clear that  $(f \times g) + (h \times k) = (f + h) \times (g + k)$  for any maps  $f, h \in \text{RAT}_R(n, m)$  and  $g, k \in \text{RAT}_R(j, l)$ . It is also a quick calculation to show that  $\pi_0, \pi_1$ , and  $\Delta$  are additive. Thus, we end this subsection with the following proposition:

**Proposition 4.24** *For every commutative rig  $R$ ,  $\text{RAT}_R$  is a cartesian left additive restriction category.*

## 4.8 Differential Structure on Rational Functions

We now define the differential structure of  $\text{RAT}_R$ . We will use formal partial derivatives to define this structure. Formal partial derivatives are used in many places: in Galois theory the formal derivative is used to determine if a polynomial has repeated roots [Stewart 2004], and in algebraic geometry the rank of the formal Jacobian matrix is used to determine if a local ring is regular [Eisenbud 2004]. Thus, the differential structure of  $\text{RAT}_R$  is an important construct.

**Proposition 4.25** *If  $R$  is a commutative ring, then  $\text{RAT}_R$  is a differential restriction category.*

Given a ring,  $R$ , there is a formal partial derivative for elements of  $R[x_1, \dots, x_n]$ . Let  $f = \sum_l a_l x_1^{l_1} \cdots x_n^{l_n}$  be a polynomial. Then the formal partial derivative of  $f$  with respect to the variable  $x_k$  is

$$\frac{\partial f}{\partial x_k} = \sum_l l_k a_l x_1^{l_1} \cdots x_{k-1}^{l_{k-1}} x_k^{l_k-1} x_{k+1}^{l_{k+1}} \cdots x_n^{l_n}$$

Extend the above definition to rational functions, where  $g = \frac{p}{q}$  by

$$\frac{\partial g}{\partial x_k} = \frac{\frac{\partial p}{\partial x_k} q - p \frac{\partial q}{\partial x_k}}{q^2}.$$

From the above observation, one can show that the unit has an additive inverse, and since rigs are multiplicatively closed, every element is forced to have an additive inverse. Thus we need a ring to define the differential structure on rational functions. Now, if we have  $f = (f_1, \dots, f_m) = \left(\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m}\right)$ , an  $m$ -tuple of rational functions in  $n$  variables over  $R$ , then we can define the formal Jacobian at a point of  $R^n$  as the  $m \times n$  matrix

$$J_f(y_1, \dots, y_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(y_1, \dots, y_n) & \cdots & \frac{\partial f_1}{\partial x_n}(y_1, \dots, y_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(y_1, \dots, y_n) & \cdots & \frac{\partial f_m}{\partial x_n}(y_1, \dots, y_n) \end{bmatrix}$$

Finally, consider  $\text{RAT}_R$  where  $R$  is a commutative ring. Then, define the differential structure to be

$$\frac{(\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) : n \longrightarrow m}{(\vec{x}^{2n} \mapsto ((J_{(p_i, q_i)}(x_{n+1}, \dots, x_{2n})) \cdot (x_1, \dots, x_n)), [x_{n+i}/x_i] \mathcal{R}) : 2n = n \times n \longrightarrow m} D[\_]$$

For a quick example, consider  $\text{RAT}_{\mathbb{Z}}$  and the map  $(x_1, x_2 \mapsto \left(\frac{1}{x_1}, \frac{x_1^2}{1+x_2}\right), \langle x_1, 1+x_2 \rangle)$ . Then the differential of this map is

$$\left(x_1, x_2, x_3, x_4 \mapsto \left(\frac{-x_1}{x_3}, \frac{2x_3x_1(x_4+1) - x_3^2x_2}{(x_4+1)^2}\right), \langle x_3, 1+x_4 \rangle\right)$$

PROOF: In [Blute et. al. 2008], the category of smooth functions between finite dimensional  $\mathbb{R}$  vector spaces spaces is established as an example of a cartesian differential category using the Jacobian as the differential structure. The proof for showing that  $\text{RAT}_R$  is a differential restriction category is much the same, so we will highlight pieces where the axioms have changed, and consider the new axioms. Consider the map  $f = (\vec{x}^n \mapsto (f)_i, \mathcal{V})$ .

**[DR.2]** We will show the second part of **[DR.2]** since it has changed in light of restriction; that is, we will show  $\langle 0, g \rangle D[f] = \overline{gf} 0$ . Consider the map  $(\vec{x}^k \mapsto (g_i, g'_i)_{i=1}^n, \mathcal{U})$ . It is immediate that the rational functions in both maps are all 0. Set  $\mathcal{V}' = [x_{n+i}/x_i] \mathcal{V}$ . Using that the zero map is total, it is clear that we must show

$$[0/x_i, (g_i, g'_i)/x_{n+i}] \mathcal{V}' = [(g_i, g'_i)/x_i] \mathcal{V}$$

A proof by induction on  $n$  works here, but the idea is simple.  $\mathcal{V}'_j$  is just  $\mathcal{V}_j$  with variable indices shifted by  $n$ . Thus the substitution of  $(\vec{0}, \vec{g})_i$  for  $x_i$  into  $\mathcal{U}'_j$  is the same as the substitution of  $(\vec{0}, \vec{g})_{n+i} = \vec{g}_i$  for  $x_i$  in  $\mathcal{U}_j$ .



**[DR.6]** Consider the maps  $(\vec{x}^k \mapsto (g_i, g'_i)_{i=1}^n, \mathcal{U})$ ,  $(\vec{x}^k \mapsto (h_i, h'_i)_{i=1}^n, \mathcal{W})$ , and  $(\vec{x}^k \mapsto (k_i, k'_i)_{i=1}^n, \mathcal{V})$ . The restriction set for  $D[f]$  is  $\mathcal{V}' = [x_{n+i}/x_i]\mathcal{V}$ , and the restriction set for  $D[D[f]]$  is  $\mathcal{V}'' = [x_{2n+j}/x_j]\mathcal{V}' = [x_{3n+l}/x_l]\mathcal{V}$ . For **[DR.6]**, we must prove,

$$\begin{aligned} & \left( \vec{x}^k \mapsto ((g_1, g'_1), \dots, (g_n, g'_n), 0, \dots, 0, (h_1, h'_1), \dots, (h_n, h'_n), (k_1, k'_1), \dots, (k_n, k'_n)), \langle \mathcal{U} \cup \mathcal{W} \cup \mathcal{T} \rangle \right) D[D[f]] \\ &= \overline{(\vec{x}^k \mapsto (h_i, h'_i)_{i=1}^n, \mathcal{T})} \left( \vec{x}^k \mapsto ((g_1, g'_1), \dots, (g_n, g'_n), (k_1, k'_1), \dots, (k_n, k'_n)), \langle \mathcal{U} \cup \mathcal{W} \rangle \right) D[f]. \end{aligned}$$

A couple manipulations shows the rational functions of the maps are the same. Below, we understand the substitution to be performed and then the numerator taken. It remains to prove that the below equality holds:

$$\begin{aligned} & \langle (U \cup W \cup T) \cup [(g_i, g'_i)/x_i, 0_i/x_{n+i}, (h_i, h'_i)/x_{2n+i}, (k_i, k'_i)/x_{3n+i}] \mathcal{V}'' \rangle \\ &= \langle U \cup (W \cup T \cup [(g_i, g'_i)/x_i, (k_i, k'_i)/x_{n+i}] \mathcal{V}') \rangle \end{aligned}$$

Which boils down to showing that

$$[(g_i, g'_i)/x_i, 0_i/x_{n+i}, (h_i, h'_i)/x_{2n+i}, (k_i, k'_i)/x_{3n+i}] \mathcal{V}'' = [(g_i, g'_i)/x_i, (k_i, k'_i)/x_{n+i}] \mathcal{V}'.$$

The argument for the above equality is the same as for **[D.2]**; we proceed by induction on  $n$ . For  $n = 1$ ,

$$\begin{aligned} & [(g_n, g_d)/x_1, (0, 1)/x_2, (h_n, h_d)/x_3, (k_n, k_d)/x_4] \mathcal{V}'' \\ &= [(g_n, g_d)/x_1, (0, 1)/x_2, (h_n, h_d)/x_3, (k_n, k_d)/x_4] ([x_4/x_1]\mathcal{V}) \\ & \quad \text{So the substitution is void for } x_1, x_2, x_3, \text{ and we proceed} \\ &= [(k_n, k_d)/x_4] ([x_4/x_1]\mathcal{V}) \\ &= [(k_n, k_d)/x_1] \mathcal{V} \\ &= [(k_n, k_d)/x_2] ([x_2/x_1]\mathcal{V}) \\ &= [(g_n, g_d)/x_1, (k_n, k_d)/x_2] ([x_2/x_1]\mathcal{V}) \\ &= [(g_n, g_d)/x_1, (k_n, k_d)/x_2] \mathcal{V}' \end{aligned}$$

Similar computations work for the general case of  $n > 1$ , so **[DR.6]** holds.

**[DR.8]** Let  $f = (\vec{x}^n \mapsto (f_i)_{i=1}^n, \mathcal{V}) : n \rightarrow m, \vec{x} = (x_1, \dots, x_n)$ , and  $\vec{x}' = (x_{n+1}, \dots, x_{2n})$ . Then

$$\begin{aligned} (1 \times \bar{f})\pi_0 &= (\vec{x}^{2n} \mapsto (x_i)_{i=1}^{2n}, [x_{n+i}/x_i]\mathcal{V}) \pi_0 \\ &= (\vec{x}^{2n} \mapsto (x_i)_{i=1}^n, [x_{n+i}/x_i]\mathcal{V}) \\ &= (\vec{x}^{2n} \mapsto I_{n \times n} \vec{x}, [x_{n+i}/x_i]\mathcal{V}) \\ &= (\vec{x}^{2n} \mapsto (J_{(x_i)_i}(\vec{x}')) \cdot \vec{x}, [x_{n+i}/x_i]\mathcal{V}) \\ &= D[\bar{f}]. \end{aligned}$$

**[DR.9]** Considering  $f = (\vec{x}^n \mapsto (f_i)_{i=1}^n, \mathcal{V})$ , we have

$$\begin{aligned} 1 \times \bar{f} &= (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \{\}) \times (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \mathcal{V}) \\ &= \left( \vec{x}^{2n} \mapsto (x_i, 1)_{i=1}^{2n}, \langle [x_{n+i}/x_i] \mathcal{V} \rangle \right) \\ &= \overline{D[f]}. \end{aligned}$$

□

## 4.9 Further properties of $\text{RAT}_R$

In this section we will describe three aspects of  $\text{RAT}_R$ . First we will prove that  $\text{RAT}_R$  has nowhere defined maps for each  $R$ . Next, after briefly introducing the definition of 0-unitariness for restriction categories, we will show that if  $R$  is an integral domain, then  $\text{RAT}_R$  is a 0-unitary restriction category. Finally, we will show that  $\text{RAT}_R$  does not in general have joins.

Recall from section 2.3 that a restriction category  $\mathbb{X}$  has **nowhere defined maps**, if for each  $\mathbb{X}(A, B)$  there is a map  $\emptyset_{AB}$  which satisfies **J1** and **J2**. We will show that  $\text{RAT}_R$  always has nowhere defined maps. Intuitively, a nowhere defined rational function should be one whose restriction set  $\mathcal{U}$  is the entire rig  $R[x_1, \dots, x_n]$ . This can be achieved with a finitely generated set by simply considering the set generated by 0, since any such polynomial is in the factor closure of 0.

**Proposition 4.26** *For any commutative rig  $R$ ,  $\text{RAT}_R$  has nowhere defined maps given by*

$$(\vec{x}^n \mapsto (1, 1)_{i=1}^m, \langle 0 \rangle).$$

PROOF:

**J1** First, note

$$\overline{(\vec{x}^n \mapsto (1, 1)_{i=1}^m, \langle 0 \rangle)} = (\vec{x}^n \mapsto (x_i, 1)_{i=1}^n, \langle 0 \rangle) = (\vec{x}^n \mapsto (1, 1)_{i=1}^n, \langle 0 \rangle)$$

since  $0x_i = 0$ . Next, note that  $R[x_1, \dots, x_n] = \langle 0 \rangle = \langle \langle 0 \rangle \cup \mathcal{U} \rangle$ . Let  $(a_i, b_i) = [(1, 1)/x_i](p_i, q_i)$ ; clearly for each  $i$ ,

$$0 = 0a_i = 0b_i.$$

Thus, the following equalities are clear:

$$\begin{aligned} & (\vec{x}^n \mapsto (1, 1)_{i=1}^n, \langle 0 \rangle) (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) \\ &= (\vec{x}^n \mapsto (a_i, b_i)_{i=1}^m, \langle \langle 0 \rangle \cup \mathcal{U} \rangle) \\ &= (\vec{x}^n \mapsto (1, 1)_{i=1}^m, \langle 0 \rangle). \end{aligned}$$

**J2** Consider,

$$\begin{aligned} & (\vec{x}^n \mapsto (p_i, q_i)_{i=1}^m, \mathcal{U}) (\vec{x}^m \mapsto (1, 1)_{i=1}^k, \langle 0 \rangle) \\ &= (\vec{x}^n \mapsto (1, 1)_{i=1}^k, \langle \mathcal{U} \cup \langle 0 \rangle \rangle) \\ &= (\vec{x}^n \mapsto (1, 1)_{i=1}^k, \langle 0 \rangle), \end{aligned}$$

which completes the proof that  $\text{RAT}_R$  has nowhere defined maps.

□

Now, if  $R$  is an integral domain, we would expect that whenever two rational functions agree on some common restriction idempotent, then they should be equal wherever they are both defined. To make this idea explicit, we will introduce the concept of 0-unitary for restriction categories<sup>2</sup>.

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<sup>2</sup>This is related to the concept of 0-unitary from inverse semigroup theory; the relationship will be explored in detail in a future paper.

Let  $\mathbb{X}$  be a restriction category with nowhere defined maps. To define 0-unitariness, we first define a relation  $\leq_0$  on parallel arrows, called the **0-density relation**, as follows:

$$f \leq_0 g \text{ if } f \leq g \text{ and } hf = \emptyset \text{ implies } hg = \emptyset.$$

$\mathbb{X}$  is a **0-unitary** restriction category when for any  $f, g, h$ :

$$f \geq_0 h \leq_0 g \text{ implies } f \smile g.$$

**Lemma 4.27** *Let  $\mathbb{X}$  be a restriction category with nowhere defined maps, and assume  $h \leq_0 f$ . Then if  $f$  or  $h$  equals  $\emptyset$ , then both  $f$  and  $h$  equal  $\emptyset$ .*

PROOF: Since  $h \leq_0 f$ , we have  $h = \bar{h}f$ , and whenever  $kh = \emptyset$ ,  $kf = \emptyset$ .

First assume that  $f = \emptyset$ . Then  $h = \emptyset$  since

$$h = \bar{h}f = \bar{h}\emptyset = \emptyset.$$

Next, assume that  $h = \emptyset$ . Then by 0-unitariness,

$$1h = h = \emptyset \text{ implies } 1f = \emptyset,$$

which completes the proof.  $\square$

Now we prove that  $\text{RAT}_R$  is a 0-unitary restriction category when  $R$  is an integral domain.

**Proposition 4.28** *Let  $R$  be an integral domain. Then  $\text{RAT}_R$  is a 0-unitary restriction category.*

PROOF: Consider the maps  $(\vec{x}^n \mapsto (f_i, f'_i)_{i=1}^m, \mathcal{U})$ ,  $(\vec{x}^n \mapsto (g_i, g'_i)_{i=1}^m, \mathcal{V})$ , and  $(\vec{x}^n \mapsto (h_i, h'_i)_{i=1}^m, \mathcal{W})$ . Assume:

$$(\vec{x}^n \mapsto (h_i, h'_i)_{i=1}^m, \mathcal{W}) \leq_0 (\vec{x}^n \mapsto (f_i, f'_i)_{i=1}^m, \mathcal{U}) \text{ and } (\vec{x}^n \mapsto (h_i, h'_i)_{i=1}^m, \mathcal{W}) \leq_0 (\vec{x}^n \mapsto (g_i, g'_i)_{i=1}^m, \mathcal{V}).$$

Now if any of the above maps are  $\emptyset$ , then lemma (4.27) says that all three of the above equal  $\emptyset$ ; therefore,

$$(\vec{x}^n \mapsto (f_i, f'_i)_{i=1}^m, \mathcal{U}) \smile (\vec{x}^n \mapsto (g_i, g'_i)_{i=1}^m, \mathcal{V}).$$

Thus, suppose all three are not  $\emptyset$ . Then  $0 \notin \mathcal{U}, \mathcal{V}$ , or  $\mathcal{W}$ . Then we have

$$\begin{aligned} (\vec{x}^n \mapsto (f_i, f'_i)_{i=1}^m, \langle \mathcal{W} \cup \mathcal{U} \rangle) &= \overline{(\vec{x}^n \mapsto (h_i, h'_i)_{i=1}^m, \mathcal{W})} (\vec{x}^n \mapsto (f_i, f'_i)_{i=1}^m, \mathcal{U}) \\ &= \overline{(\vec{x}^n \mapsto (h_i, h'_i)_{i=1}^m, \mathcal{W})} (\vec{x}^n \mapsto (g_i, g'_i)_{i=1}^m, \mathcal{V}) \text{ since } \bar{h}f = h = \bar{h}g, \\ &= (\vec{x}^n \mapsto (g_i, g'_i)_{i=1}^m, \langle \mathcal{W} \cup \mathcal{V} \rangle). \end{aligned}$$

Now, since  $R$  is an integral domain, the product of two nonzero elements is nonzero. Thus,  $0 \notin \langle \mathcal{W} \cup \mathcal{U} \rangle$ . Thus for each  $i$ , there is a  $W_i \neq 0 \in \langle \mathcal{W} \cup \mathcal{U} \rangle$  such that  $W_i f_i g'_i = W_i f'_i g_i$ . Moreover, the fact that  $R$  is an integral domain also gives the cancellation property: if  $a \neq 0$ ,  $ac = ab$  implies  $c = b$ . Thus, we have that  $f_i g'_i = f'_i g_i$ , which proves

$$(\vec{x}^n \mapsto (f_i, f'_i)_{i=1}^m, \mathcal{U}) \smile (\vec{x}^n \mapsto (g_i, g'_i)_{i=1}^m, \mathcal{V}).$$

Thus, when  $R$  is an integral domain,  $\text{RAT}_R$  is a 0-unitary restriction category.  $\square$

It may seem natural to ask if  $\text{RAT}_R$  has finite joins, especially if  $R$  has unique factorization. If  $R$  is a unique factorization domain, it is easy to show that any two compatible maps in  $\text{RAT}_R$  will have the form

$$(\vec{x}^n \mapsto (P_i, Q_i)_{i=1}^m, \mathcal{U}) \smile (\vec{x}^n \mapsto (P_i, Q_i)_{i=1}^m, \mathcal{V}),$$

where  $\gcd(P_i, Q_i) = 1$ . Thus  $Q_i \in U, V$ , so  $Q_i \in \langle \mathcal{U} \cap \mathcal{V} \rangle$ . Thus from the order theoretic nature of joins, the only candidate for the join is  $(\vec{x}^n \mapsto (P_i, Q_i)_{i=1}^m, \langle \mathcal{U} \cap \mathcal{V} \rangle)$ . However, reducing the restriction sets of compatible maps by intersection does not define a join restriction structure on  $\text{RAT}_R$ , as stability will not always hold. For a counterexample, consider the maps

$$(1, \langle x - 1 \rangle) \smile (1, \langle y - 1 \rangle).$$

By the above discussion,  $(1, \langle \langle x - 1 \rangle \cap \langle y - 1 \rangle \rangle)$  must be  $(1, \langle 1 \rangle)$ . We will show that  $s(f \vee g) \neq sf \vee sg$ . Consider the map  $((x^2, x^2), \{\})$ . Then

$$((x^2, x^2), \{\}) (1, \langle 1 \rangle) = (1, \langle 1 \rangle).$$

However,

$$((x^2, x^2), \{\}) (1, \langle x - 1 \rangle) = (1, \langle x + 1, x - 1 \rangle)$$

and

$$((x^2, x^2), \{\}) (1, \langle y - 1 \rangle) = (1, \langle x + 1, x - 1 \rangle).$$

The “join” of the latter two maps is  $(1, \langle x + 1, x - 1 \rangle) \neq (1, \langle 1 \rangle)$ . So, the join defined in this way satisfies **J.1-J.5**, but **J.6** fails. Thus, in general  $\text{RAT}_R$  does not have joins.

## 5 Join completion and differential structure

In the final two sections of the paper, our goal is to show that one adds joins or relative complements of partial maps, differential structure is preserved. These are important results, as they show that one can add more logical operations to the maps of a differential restriction category, while retaining the differential structure.

### 5.1 The join completion

As we have just seen, a restriction category need not have joins, but there is a universal construction which freely adds joins to any restriction category. We show in this section that if the original restriction category has differential structure, then so does its join completion. By join completing  $\text{RAT}_R$ , we thus get a restriction category which has both joins and differential structure, but is very different from the differential restriction category of smooth functions defined on open subsets of  $\mathcal{R}^n$ .

The join completion we describe here was first given in this form in [Cockett and Manes 2009], but follows ideas of Grandis from [Grandis 1989].

**Definition 5.1** *Given a restriction category  $\mathbb{X}$ , define  $\mathbf{Jn}(\mathbb{X})$  to have:*

- *objects: those of  $\mathbb{X}$ ;*

- an arrow  $X \xrightarrow{A} Y$  is a subset  $A \subseteq \mathbb{X}(X, Y)$  such that  $A$  is down-closed (under the restriction order), and elements are pairwise compatible;
- $X \xrightarrow{1_X} X$  is given by the down-closure of the identity,  $\downarrow 1_X$ ;
- the composite of  $A$  and  $B$  is  $\{fg : f \in A, g \in B\}$ ;
- restriction of  $A$  is  $\{\bar{f} : f \in A\}$ ;
- the join of  $(A_i)_{i \in I}$  is given by the union of the  $A_i$ .

From [Cockett and Manes 2009], we have the following result:

**Theorem 5.2**  $\mathbf{Jn}(\mathbb{X})$  is a join-restriction category, and is the left adjoint to the forgetful functor from join restriction categories to restriction categories.

Because we will frequently be dealing with the down-closures of various sets, the following lemma will be extremely helpful.

**Lemma 5.3** (Down-closure lemma) Suppose  $A, B \subseteq \mathbb{X}(A, B)$ . Then we have:

- (i)  $\downarrow A \downarrow B = \downarrow(AB)$ ;
- (ii)  $\overline{\downarrow A} = \downarrow(\overline{A})$ ;
- (iii)  $\langle \downarrow A, \downarrow B \rangle = \downarrow \langle A, B \rangle$ ;
- (iv)  $\downarrow A + \downarrow B = \downarrow(A + B)$ ;
- (v)  $D[\downarrow A] = \downarrow D[A]$ .

PROOF:

- (i) If  $h \in \downarrow(AB)$ , then  $\exists f \in A, g \in B$  such that  $h \leq fg$ . So  $\bar{h}fg = h$ , and  $\bar{h}f \in \downarrow A$ ,  $b \in \downarrow B$ , so  $h \in \downarrow A \downarrow B$ . Conversely, if  $mn \in \downarrow A \downarrow B$ , there exists  $f, g$  such that  $m \leq f \in A, n \leq g \in B$ . But composition preserves order, so  $mn \leq fg$ , so  $mn \in \downarrow(AB)$ .
- (ii) Suppose  $h \in \overline{\downarrow A}$ . So there exists  $f \in A$  such that  $h \leq f$ . Since restriction preserves order,  $\bar{h} \leq \bar{f}$ . But since  $h \in \overline{\downarrow A}$ ,  $h$  is idempotent, so we have  $h \leq \bar{f}$ . So  $h \in \downarrow(\overline{A})$ . Conversely, suppose  $h \in \downarrow(\overline{A})$ , so  $h \leq \bar{f}$  for some  $f \in A$ . Then we have  $h = \bar{h} \bar{f} = \bar{h} f$ , so  $h$  is idempotent and  $h \leq f$ , so  $h \in \overline{\downarrow A}$ .
- (iii) Suppose  $h \in \downarrow \langle A, B \rangle$ , so  $h \leq \langle f, g \rangle$  for  $f \in A, g \in B$ . Then  $h = \bar{h} \langle f, g \rangle = \langle \bar{h} f, g \rangle$ , and  $\bar{h} f \in \downarrow A, g \in \downarrow B$ , so  $h \in \langle \downarrow A, \downarrow B \rangle$ . Conversely, suppose  $h \in \langle \downarrow A, \downarrow B \rangle$ , so that  $h = \langle m, n \rangle$  where  $m \leq f \in A, n \leq g \in B$ . Since pairing preserves order,  $h = \langle m, n \rangle \leq \langle f, g \rangle$ , so  $h \in \downarrow \langle A, B \rangle$ .
- (iv) Suppose  $h \in \downarrow A + \downarrow B$ , so  $h = m + n$ , where  $m \leq f \in A, n \leq g \in B$ . Since addition preserves order,  $h = m + n \leq f + g$ , so  $h \in \downarrow(A + B)$ . Conversely, suppose  $h \in \downarrow(A + B)$ . Then there exist  $f \in A, g \in B$  so that  $h \leq f + g$ . Then  $h = \bar{h}(f + g) = \bar{h}f + \bar{h}g$  (by left additivity), so  $h \in \downarrow A + \downarrow B$ .

- (v) Suppose  $h \in D[\downarrow A]$ . Then there exists  $m \leq f \in A$  so that  $h \leq D[m]$ . But differentiation preserves order, so  $h \leq D[m] \leq D[f]$ , so  $h \in \downarrow D[A]$ . Conversely, suppose  $h \in D[A]$ . Then there exists  $f \in A$  so that  $h \leq D[f]$ , so  $h \in D[\downarrow A]$ .

□

## 5.2 Cartesian structure

We begin by showing that cartesianess is preserved by the join completion.

**Theorem 5.4** *If  $\mathbb{X}$  is a cartesian restriction category, then so is  $\mathbf{Jn}(\mathbb{X})$ .*

PROOF: We define  $1$  and  $X \times Y$  as for  $\mathbb{X}$ , the projections to be  $\downarrow\pi_0$  and  $\downarrow\pi_1$ , the terminal maps to be  $\downarrow(!_A)$ , and

$$\langle A, B \rangle := \{ \langle f, g \rangle : f \in A, g \in B \}$$

This is compatible by Proposition 2.14, and down-closed since if  $h \leq \langle f, g \rangle$ , then

$$h = \bar{h} \langle f, g \rangle = \langle \bar{h} f, g \rangle$$

so since  $A$  is down-closed, this is also in  $\langle A, B \rangle$ .

The terminal maps do indeed satisfy the required property, as

$$\bar{A} \downarrow(!_A) = \bar{A} !_A = \{ \bar{f} !_A : f \in A \} = \{ f : f \in A \} = A,$$

as required.

To show that  $\langle -, - \rangle$  satisfies the required property, consider

$$\langle A, B \rangle \downarrow\pi_0 = \{ \langle f, g \rangle \pi_0 : f \in A, g \in B \} = \{ \bar{g} f : f \in A, g \in B \} = \bar{B} A$$

and similarly for  $\downarrow\pi_1$ .

We now need to show that  $\langle -, - \rangle$  is universal with respect to this property. That is, suppose there exists a compatible down-closed set of arrows  $C$  with the property that  $C \downarrow\pi_0 = \bar{B} A$  and  $C \downarrow\pi_1 = \bar{A} B$ . We need to show that  $C = \langle A, B \rangle$ .

To show that  $C \subseteq \langle A, B \rangle$ , let  $c \in C$ . Since  $\downarrow(C\pi_0) = C \downarrow\pi_0 = \bar{B} A$ , there exists  $f \in A, g \in B$  such that  $c\pi_0 = \bar{g} f$ . Then, since  $\downarrow(C\pi_1) = C \downarrow\pi_1 = \bar{A} B$ , there exists a  $c'$  such that  $c'\pi_1 = \bar{f} g$ . Then

$$\bar{c} c\pi_0 = \bar{c} \bar{c} c\pi_0 = \bar{c} \bar{c} \bar{g} f$$

and since  $c \smile c'$ ,

$$\bar{c} c\pi_1 = \bar{c} \bar{c} c'\pi_1 = \bar{c} \bar{c} \bar{f} g$$

Thus, by the universality of  $\bar{c} \bar{c} \langle f, g \rangle$ ,  $\bar{c} c = \bar{c} \bar{c} \langle f, g \rangle$ . Thus

$$c \leq \bar{c} c = \bar{c} \bar{c} \langle f, g \rangle \leq \langle f, g \rangle,$$

so since  $\langle A, B \rangle$  is down-closed,  $c \in \langle f, g \rangle$ .

To show that  $\langle A, B \rangle \subseteq C$ , let  $f \in A, g \in B$ . Then there exists  $c$  such that

$$c\pi_0 = \bar{g} f = \langle f, g \rangle \pi_0.$$

Thus, there exists  $f' \in A, g' \in B$  such that

$$c\pi_1 = \overline{f'} g' = \langle f', g' \rangle \pi_1.$$

Now, we have

$$\overline{\langle f', g' \rangle} \langle f, g \rangle \pi_0 = \overline{\langle f', g' \rangle} \overline{\langle f, g \rangle} \langle f, g \rangle \pi_0 = \overline{\langle f', g' \rangle} \overline{\langle f, g \rangle} c\pi_0$$

and since  $f \smile f'$  and  $g \smile g'$ ,  $\langle f, g \rangle \smile \langle f', g' \rangle$ , so we also get

$$\overline{\langle f', g' \rangle} \langle f, g \rangle \pi_1 = \overline{\langle f', g' \rangle} \overline{\langle f, g \rangle} \langle f', g' \rangle \pi_1 = \overline{\langle f', g' \rangle} \overline{\langle f, g \rangle} c\pi_1.$$

Thus, by the universality of  $\overline{\langle f', g' \rangle} \langle f, g \rangle$ ,

$$\langle f, g \rangle \leq \overline{\langle f', g' \rangle} \langle f, g \rangle = \overline{\langle f', g' \rangle} \overline{\langle f, g \rangle} c \leq c.$$

Since  $C$  is down-closed, this shows  $\langle f, g \rangle \in C$ , as required.  $\square$

### 5.3 Left additive structure

Next, we show that left additive structure is preserved.

**Theorem 5.5** *If  $\mathbb{X}$  is a left additive restriction category, then so is  $\mathbf{Jn}(\mathbb{X})$ , where*

$$0_{\mathbf{Jn}(\mathbb{X})} := \downarrow 0 \text{ and } A + B := \{f + g : f \in A, g \in B\}$$

.

PROOF: By Proposition 3.2,  $A + B$  is a compatible set. For down-closed, suppose  $h \leq f + g$ . Then  $h = \overline{h}(f + g) = \overline{h}f + \overline{h}g$ . Since  $A$  and  $B$  are down-closed,  $\overline{h}f \in A, \overline{h}g \in B$ , so  $h \in A + B$ .

That this gives a commutative monoid structure on each hom-set follows directly from Lemma 5.3, as does  $\overline{0} = \downarrow 1$ . Finally,

$$\overline{A + B} = \overline{\{f + g : f \in A, g \in B\}} = \overline{\{f + g : f \in A, g \in B\}} = \overline{\{f \overline{g} : f \in A, g \in B\}} = \overline{A} \overline{B}.$$

so that  $\mathbf{Jn}(\mathbb{X})$  is a left additive restriction category.  $\square$

**Theorem 5.6** *If  $\mathbb{X}$  is a cartesian left additive restriction category, then so is  $\mathbf{Jn}(\mathbb{X})$ .*

PROOF: Immediate from Theorem 3.11.  $\square$

### 5.4 Differential structure

Finally, we show that differential structure is preserved. There is one small subtlety, however. To define the pairing or addition of maps in  $\mathbf{Jn}(\mathbb{X})$ , we merely needed to add or pair pointwise, as the resulting set was automatically down-closed and pairwise compatible if the original was. However, note that  $A$  being down-closed does not imply  $\{D[f] : f \in A\}$  down-closed. Axiom [D.9] requires that differentials be total in the first component. However, this is not always true of an arbitrary  $h \leq D[f]$ . Thus, to define the differential in the join completion, we make take the down-closure of  $\{D[f] : f \in A\}$ .

**Theorem 5.7** *If  $\mathbb{X}$  is a differential restriction category, then so is  $\mathbf{Jn}(\mathbb{X})$ , where*

$$D[A] := \downarrow\{D[f] : f \in A\}$$

PROOF: Checking the differential axioms is a straightforward application of our down-closure lemma. For example, for [D.1], by the down-closure lemmas,

$$D[0_{\mathbf{Jn}(\mathbb{X})}] = D[\downarrow 0] = \downarrow D[0] = \downarrow 0 = 0_{\mathbf{Jn}(\mathbb{X})}$$

and

$$D[A + B] = \downarrow\{D[f + g] : f \in A, g \in B\} = \downarrow\{D[f] + D[g] : f \in A, g \in B\} = D[A] + D[B].$$

Similarly, to check [D.5]:

$$D[AB] = \downarrow\{D[fg] : f \in A, g \in B\} = \downarrow\{\langle D[f], \pi_1 f \rangle D[g] : f \in A, g \in B\} = \langle D[A], \downarrow \pi_1 A \rangle DB$$

where the last equality follows from several applications of the down-closure lemmas. All other axioms similarly follow.  $\square$

Finally, it is easy to see the following:

**Proposition 5.8** *The unit  $\mathbb{X} \longrightarrow \mathbf{Jn}(\mathbb{X})$ , which sends  $f$  to  $\downarrow f$ , is a differential restriction functor.*

PROOF: The result immediately follows, given the additive, cartesian, and differential structure of  $\mathbf{Jn}(\mathbb{X})$ .  $\square$

Thus, by Proposition 3.24, we have

**Corollary 5.9** *If  $\mathbb{X}$  is a differential restriction category, and  $f$  is additive/strongly additive/linear, then so is  $\downarrow f$  in  $\mathbf{Jn}(\mathbb{X})$ .*

## 6 Classical completion and differential structure

In our final section, we show that differential structure is preserved when we add relative complements to a join restriction category. This process will greatly expand the possible domains of definition for differentiable maps, even in the standard example. The standard example (smooth maps on open subsets) does not have relative complements. By adding them in, we add smooth maps between any set which is the complement of an open subset inside some other open subset. Of course, this includes closed sets, and so by applying this construction, we have a category of smooth maps defined on all open, closed and half open-half-closed sets. This includes smooth functions defined on points; as we shall see below, this captures the notion of the germ of a smooth function.

### 6.1 The classical completion

The notion of classical restriction category was defined in [Cockett and Manes 2009] as an intermediary between arbitrary restriction categories and the boolean restriction categories of [Manes 2006].

**Definition 6.1** *A restriction category  $\mathbb{X}$  with restriction zeroes is a **classical restriction category** if*



1. the homsets are locally Boolean posets (under the restriction order), and for any  $W \xrightarrow{f} X, Y \xrightarrow{g} Z$ ,

$$\mathbb{X}(X, Y) \xrightarrow{f \circ (-) \circ g} \mathbb{X}(W, Z)$$

is a locally Boolean morphism;

2. for any disjoint maps  $f, g$  (that is,  $\overline{f} \overline{g} = \emptyset$ ),  $f \vee g$  exists.

**Example 6.2** Sets and partial functions form a classical restriction category.

For our purposes, the following alternate characterization of the definition, which describes classical restriction categories as join restriction categories with relative complements, is more useful.

**Definition 6.3** If  $f' \leq f$ , the **relative complement** of  $f'$  in  $f$ , denoted  $f \setminus f'$ , is the unique map such that

- $f \setminus f' \leq f$ ;
- $g \wedge (f \setminus f') = \emptyset$ ;
- $f \leq g \vee (f \setminus f')$ .

The following can be found in [Cockett and Manes 2009]:

**Proposition 6.4** A classical restriction category is a join restriction category with relative complements  $f \setminus f'$  for any  $f' \leq f$ .

Just as one can freely add joins to an arbitrary restriction category, so too can one freely add relative complements to a join restriction category. We will first describe this completion process, then show that cartesian, additive, and differential structure is preserved when classically completing. This is of great interest, as classically completing adds in a number of new maps, even to the standard examples.

**Definition 6.5** Let  $\mathbb{X}$  be a join restriction category. A **classical piece** of  $\mathbb{X}$  is a pair of maps  $(f, f') : A \longrightarrow B$  such that  $f' \leq f$ .

One thinks of a classical piece as a formal relative complement.

**Definition 6.6** Two classical pieces  $(f, f'), (g, g')$  are **disjoint**, written  $(f, f') \perp (g, g')$ , if  $\overline{f} \overline{g} = \overline{f'} \overline{g'} \vee \overline{f} \overline{g'}$ . A **raw classical piece** consists of a finite set of classical pieces,  $(f_i, f'_i)$  that are pairwise disjoint, and is written

$$\bigsqcup_{i \in I} (f_i, f'_i) : A \longrightarrow B.$$

One defines an equivalence relation on the set of raw classical maps by:

- **Breaking:**  $(f, f') \equiv (ef, ef') \sqcup (f, f' \vee fe)$  for any restriction idempotent  $e = \overline{e}$ ,
- **Collapse:**  $(f, f) \equiv \emptyset$ .

The first part of the equivalence relation says that if we have some other domain  $e$ , then we can split the formal complement  $(f, f')$  into two parts: the first part,  $(ef, ef')$ , inside  $e$ , and the second,  $(f, f' \vee fe)$ , outside  $e$ . The second part of the equivalence is obvious: if you formally take away all of  $f$  from  $f$ , the result should be nowhere defined.

**Definition 6.7** A classical map is an equivalence class of raw classical maps.

**Proposition 6.8** Given a join restriction category  $\mathbb{X}$ , there is a classical restriction category  $\mathbf{Cl}(\mathbb{X})$  with

- objects those of  $\mathbb{X}$ ,
- arrows classical maps,
- composition by

$$\bigsqcup_{i \in I} (f_i, f'_i) \bigsqcup_{j \in J} (g_j, g'_j) := \bigsqcup_{i,j} (f_i g_j, f'_i g'_j \vee f_i g'_j),$$

- restriction by

$$\overline{\bigsqcup_{i \in I} (f_i, f'_i)} := \bigsqcup_{i \in I} (\overline{f_i}, \overline{f'_i}),$$

- disjoint join is simply  $\sqcup$  of classical pieces,
- relative complement is

$$(f, f') \setminus (g, g') := (f, f' \vee \overline{g} f) \sqcup (\overline{g'} f, \overline{g'} f').$$

In [Cockett and Manes 2009], this process is shown to give a left adjoint to the forgetful functor from classical restriction categories to join restriction categories.

We make one final point about the definition. We defined  $(f, f') \perp (f_0, f'_0)$  if  $\overline{f} \overline{f_0} = \overline{f'} \overline{f_0} \vee \overline{f} \overline{f'_0}$ . Note, however, that it suffices that we have  $\leq$ , since

$$\overline{f'} \overline{f_0} \vee \overline{f} \overline{f'_0} \leq \overline{f} \overline{f_0} \vee \overline{f'} \overline{f'_0} = \overline{f} \overline{f_0}$$

We will often use this alternate form of  $\perp$  when checking whether maps we give are well-defined.

## 6.2 Cartesian Structure

Our goal is to show that if  $\mathbb{X}$  has differential restriction structure, then so does  $\mathbf{Cl}(\mathbb{X})$ . We begin by showing that cartesian structure is preserved.

**Definition 6.9** Given a join restriction category  $\mathbb{X}$ , define

$$\langle \prod_i (f_i, f'_i), \prod_j (g_j, g'_j) \rangle := \prod_{i,j} (\langle f_i, g_j \rangle, \langle f'_i, g'_j \rangle \vee \langle f_i, g'_j \rangle)$$

**Lemma 6.10** The above describes a well-defined map in  $\mathbf{Cl}(\mathbb{X})$ .

PROOF: First, we need to check

$$(\langle f, g \rangle, \langle f', g \rangle \vee \langle f, g' \rangle)$$

defines a classical piece. Indeed, since  $f' \smile f$  and  $g \smile g'$ , the two maps being joined are compatible, so we can take the join. Also, since  $f' \leq f$  and  $g' \leq g$ , the right component is less than or equal to the left component.

Now, we need to check that

$$\prod_{i,j} (\langle f_i, g_j \rangle, \langle f'_i, g_j \rangle \vee \langle f_i, g'_j \rangle)$$

defines a raw classical map. That is, we need to check that the pieces are disjoint. That is, we need to show that if

$$(f, f') \perp (f_0, f'_0) \text{ and } (g, g') \perp (g_0, g'_0)$$

then

$$(\langle f, g \rangle, \langle f', g \rangle \vee \langle f, g' \rangle) \perp (\langle f_0, g_0 \rangle, \langle f'_0, g'_0 \rangle \vee \langle f_0, g'_0 \rangle).$$

Consider:

$$\begin{aligned} & \overline{\langle f, g \rangle \langle f_0, g_0 \rangle} \\ &= \overline{f \overline{f_0} \overline{g} \overline{g_0}} \\ &= \overline{(f' \overline{f_0} \vee \overline{f} \overline{f'_0}) (\overline{g'} \overline{g_0} \vee \overline{g} \overline{g'_0})} \\ &= \overline{f' \overline{f_0} \overline{g'} \overline{g_0} \vee \overline{f} \overline{f'_0} \overline{g'} \overline{g_0} \vee \overline{f'} \overline{f_0} \overline{g} \overline{g'_0} \vee \overline{f} \overline{f'_0} \overline{g} \overline{g'_0}} \\ &\leq \overline{f \overline{g'} \overline{f_0} \overline{g_0} \vee \overline{f} \overline{g} \overline{f_0} \overline{g_0} \vee \overline{f'} \overline{g} \overline{f_0} \overline{g_0} \vee \overline{f} \overline{g} \overline{f_0} \overline{g'_0}} \\ &= \overline{(\overline{f'} \overline{g} \vee \overline{f} \overline{g'}) (\overline{f_0} \overline{g_0}) \vee (\overline{f} \overline{g}) (\overline{f'_0} \overline{g_0} \vee \overline{f_0} \overline{g'_0})} \\ &= \overline{\langle f', g \rangle \vee \langle f, g' \rangle \langle f_0, g_0 \rangle \vee \langle f, g \rangle \langle f'_0, g_0 \rangle \vee \langle f_0, g'_0 \rangle} \end{aligned}$$

so that

$$(\langle f, g \rangle, \langle f', g \rangle \vee \langle f, g' \rangle) \perp (\langle f_0, g_0 \rangle, \langle f'_0, g'_0 \rangle \vee \langle f_0, g'_0 \rangle),$$

as required.

Finally, we need to check that this is a well-defined classical map. Thus, we need to check it is well-defined with respect to collapse and breaking. For collapse, consider

$$\langle (f, f'), (g, g) \rangle = \langle (f, g), \langle f', g \rangle \vee \langle f, g \rangle \rangle = \langle (f, g), \langle f, g \rangle \rangle \equiv \emptyset$$

as required.

For breaking, suppose we have

$$(g, g') \equiv (g, g' \vee eg) \perp (eg, eg')$$

Then

$$\begin{aligned} & \langle (f, f'), (g, g' \vee eg) \perp (eg, eg') \rangle \\ &= \langle (f, g), \langle f', g \rangle \vee \langle f, g' \vee eg \rangle \rangle \perp \langle (f, eg), \langle f', eg \rangle \vee \langle f, eg' \rangle \rangle \\ &= \langle (f, g), \langle f', g \rangle \vee e \langle f, g' \vee eg \rangle \rangle \perp \langle e \langle f, g \rangle, e \langle \langle f', g \rangle \vee \langle f, g' \rangle \rangle \rangle \\ &\equiv \langle (f, g), \langle f', g \rangle \vee \langle f, g' \rangle \rangle \\ &= \langle (f, f'), (g, g') \rangle \end{aligned}$$

as required. Thus, the above is a well-defined classical map.  $\square$

We now give some lemmas about our definition. Note that once we show that this pairing does define cartesian structure on  $\mathbf{Cl}(\mathbb{X})$ , these lemmas follow automatically, as they are true in any cartesian restriction category (see Lemma 2.14) However, we will need these lemmas to establish that this does define cartesian structure on  $\mathbf{Cl}(\mathbb{X})$ .

**Lemma 6.11** *For  $f, g, e = \bar{e}$  in  $\mathbf{Cl}(\mathbb{X})$ ,  $e\langle f, g \rangle = \langle ef, g \rangle = \langle f, eg \rangle$ .*

PROOF: It suffices to show the result for classical pieces. Thus, consider

$$\begin{aligned}
& \langle (e, e')(f, f'), (g, g') \rangle \\
&= \langle (ef, e'f \vee ef'), (g, g') \rangle \\
&= \langle \langle ef, g \rangle, \langle e'f \vee ef', g \rangle \vee \langle ef, g' \rangle \rangle \\
&= (e\langle f, g \rangle, e'\langle f, g \rangle \vee e\langle f', g \rangle \vee e\langle f, g' \rangle) \\
&= (e, e')(\langle f, g \rangle, \langle f', g \rangle \vee \langle f, g' \rangle) \\
&= (e, e')\langle (f, f'), (g, g') \rangle
\end{aligned}$$

as required. Putting the  $e$  in the right component is similar.  $\square$

**Lemma 6.12** *For any  $c$  in  $\mathbf{Cl}(\mathbb{X})$ ,  $\langle c\pi_0, c\pi_1 \rangle = c$ .*

PROOF: It suffices to show the result for classical pieces. Thus, consider

$$\begin{aligned}
& \langle (c, c')(\pi_0, \emptyset), (c, c')(\pi_1, \emptyset) \rangle \\
&= \langle (c\pi_0, c'\pi_0), (c\pi_1, c'\pi_1) \rangle \\
&= \langle \langle c\pi_0, c\pi_1 \rangle, \langle c'\pi_0, c\pi_1 \rangle \vee \langle c\pi_0, c'\pi_1 \rangle \rangle \\
&= (c, \bar{c}' \langle c\pi_0, c\pi_1 \rangle \vee \bar{c}' \langle c\pi_0, c\pi_1 \rangle) \\
&= (c, \bar{c}' c \vee \bar{c}' c) \\
&= (c, c')
\end{aligned}$$

as required.  $\square$

It will be most helpful if we can give an alternate characterization of when two classical maps are equivalent. To that, we prove the following result:

**Theorem 6.13** *In  $\mathbf{Cl}(\mathbb{X})$ ,  $(f, f') \equiv (g, g')$  if and only if there exist restriction idempotents  $e_1, \dots, e_n$  such that for any  $I \subseteq \{1, \dots, n\}$ , if we define*

$$e_I := \left( \bigcirc_{i \in I} e_i, \left( \bigcirc_{i \in I} e_i \right) \left( \bigvee_{j \notin I} e_j \right) \right)$$

then for each such  $I$ ,

$$e_I(f, f') = e_I(g, g')$$

or they both collapse to the empty map.

PROOF: As discussed in [Cockett and Manes 2009], breaking and collapse form a system of rewrites, so that if two maps are equivalent, they can be broken into a series of pieces, each of which are either equal or both collapse to the empty map. Thus, it suffices to show that the above is what occurs after doing  $n$  different breakings along the idempotents  $e_1, \dots, e_n$ . To this end, note that the two pieces left after breaking  $(f, f')$  by  $e$  are given by pre-composing with  $(e, \emptyset)$  and  $(1, e)$ ; indeed:

$$(e, \emptyset)(f, f') = (ef, ef') \text{ and } (1, e)(f, f') = (f, ef \vee f')$$

Thus, if  $n = 1$ , the result holds. Now assume by induction that the result holds for  $k$ . Then for any subset  $I \subseteq \{1, \dots, n\}$ , breaking  $e_I$  by  $(e_{k+1})$  gives the pieces

$$(e_{n+1}, \emptyset)(\circ e_i, (\circ e_i)(\vee e_j)) = (e_{n+1} \circ e_i, (e_{n+1} \circ e_i)(\vee e_j))$$

and

$$(1, e_{n+1})(\circ e_i, (\circ e_i)(\vee e_j)) = (\circ e_i, (\circ e_i)(e_{n+1}) \vee (\circ e_i)(\vee e_j)) = (\circ e_i, (\circ e_i)(e_{n+1} \vee e_j))$$

Thus, we get all possible idempotents  $e_{I'}$ , where  $I' \subseteq \{1, \dots, n+1\}$ , as required.  $\square$

**Theorem 6.14** *If  $\mathbb{X}$  is a cartesian restriction category, then so is  $\mathbf{CI}(\mathbb{X})$ .*

PROOF: Define the terminal object  $T$  as for  $\mathbb{X}$ , and the unique maps by  $!_A := (!_A, \emptyset)$ . Then for any classical map  $\coprod(f_i, f'_i)$ , we have

$$\coprod(f_i, f'_i) = \coprod(!_A \bar{f}_i, !_A \bar{f}'_i) = \left( \coprod(\bar{f}_i, \bar{f}'_i) \right) (!_A, \emptyset)$$

as required. So  $\mathbf{CI}(\mathbb{X})$  has a partial final object.

We define the product objects  $A \times B$  as for  $\mathbb{X}$ , the projections by  $(\pi_0, \emptyset)$  and  $(\pi_1, \emptyset)$ , and the product map as above. To show that our putative product composes well with the projections, consider

$$\begin{aligned} & \langle (f, f'), (g, g') \rangle (\pi_0, \emptyset) \\ &= \langle \langle f, g \rangle, \langle f', g \rangle \vee \langle f, g' \rangle \rangle (\pi_0, \emptyset) \\ &= \langle \langle f, g \rangle \pi_0, \langle f', g \rangle \pi_0 \vee \langle f, g' \rangle \pi_0 \rangle \\ &= \langle \bar{g} f, \bar{g} f' \vee \bar{g}' f \rangle \\ &= \langle \bar{g}, \bar{g}' \rangle (f, f') \\ &= \overline{(g, g')} (f, f') \end{aligned}$$

as required. Composing with  $\pi_1$  is similar.

Finally, we need to show that the universal property holds. It suffices to show that if  $c\pi_0 \leq f$  and  $c\pi_1 \leq g$ , then  $c \leq \langle f, g \rangle$ . Suppose we have the first two inequalities, so that

$$\bar{c}f \equiv c\pi_0 \text{ by breaking with idempotents } (e_1, \dots, e_n)$$

and

$$\bar{c}g \equiv c\pi_1 \text{ by breaking with idempotents } (d_1, \dots, d_m).$$

We claim that  $\bar{c}\langle f, g \rangle \equiv c$  by breaking with idempotents  $(e_1, \dots, e_n, d_1, \dots, d_m)$ . By the previous theorem, it suffices to show they are equal (or both collapse to the empty map) when composing

with an element of the form in the theorem for an arbitrary subset  $K \subseteq \{1, \dots, n, n+1, \dots, n+m\}$ . However, if  $I = K \cap \{1, \dots, n\}$  and  $J = K \cap \{n+1, \dots, n+m\}$ , then such an element can be written as

$$(e_I, e_I e_{I'}) (d_J, d_J d_{J'})$$

since that equals

$$(e_I d_J, (e_I d_J)(e_{I'} \vee d_{J'}))$$

which is  $e_K$ . Thus, writing  $e$  for  $(e_I, e_I e_{I'})$  and  $d$  for  $(d_J, d_J d_{J'})$ , it suffices to show that  $ed\bar{c}\langle f, g \rangle = edc$  (or they both collapse to the empty map). However, we know that

$$e\bar{c}f = ec\pi_0 \text{ and } d\bar{c}g = dc\pi_1$$

(or one or the other collapses to the empty map). Pairing the above equalities, we get

$$\langle e\bar{c}f, d\bar{c}g \rangle = \langle ec\pi_0, dc\pi_1 \rangle$$

which, by lemma 2.14, reduces to

$$(ed)\bar{c}\langle f, g \rangle = edc$$

as required. If either equality has both sides collapsing to the empty map, then both sides of the above collapse to the empty map, since we showed earlier that pairing is well-defined when applied to collapsed maps. Thus, we have the required universal property, and  $\mathbf{CI}(\mathbb{X})$  is cartesian.  $\square$

### 6.3 Left additive structure

Next, we show that left additive structure is preserved.

**Definition 6.15** *Suppose that  $\mathbb{X}$  is a left additive restriction category with joins. Given classical maps  $\coprod (f_i, f'_i)$  and  $\coprod (g_j, g'_j)$ , define their addition to be the map*

$$\coprod_{i,j} (f_i + g_j, (f'_i + g'_j) \vee (f_i + g'_j))$$

**Lemma 6.16** *The above is a well-defined classical map.*

PROOF: The proof is nearly identical to that for showing that our pairing definition gives a well-defined classical map.  $\square$

**Theorem 6.17** *If  $\mathbb{X}$  has the structure of a left additive restriction category, then so does  $\mathbf{CI}(\mathbb{X})$ , where addition of maps is defined as above, and the zero map is given by  $(0, \emptyset)$ .*

PROOF: It is easily checked that the addition and zero give each homset the structure of a commutative monoid. For the restriction axioms,

$$\begin{aligned} & \overline{(f, f') + (g, g')} \\ &= \overline{(f + g, (f' + g) \vee (f + g'))} \\ &= \overline{(\overline{f + g}, \overline{f' + g} \vee \overline{f + g'})} \\ &= \overline{(\overline{f} \overline{g}, \overline{f'} \overline{g} \vee \overline{f} \overline{g'})} \\ &= \overline{(\overline{f}, \overline{f'})} (\overline{g}, \overline{g'}) \\ &= \overline{(f, f')} \overline{(g, g')} \end{aligned}$$

and clearly  $(0, \emptyset)$  is total. For the left additivity, consider

$$\begin{aligned}
& (f, f')(g, g') + (f, f')(h, h') \\
&= (fg, f'g \vee fg') + (fh, f'h \vee fh') \\
&= (fg + fh, ((f'g \vee fg') + fh) \vee (fg + (f'h \vee fh'))) \\
&= (fg + fh, (f'g + fh) \vee (fg' + fh) \vee (fg + f'h) \vee (fg + fh')) \\
&= (fg + fh, \overline{f'}(fg + fh) \vee \overline{f'}(fg + fh) \vee (fg' + fh) \vee (fg + fh')) \text{ since } f' \leq f \\
&= (f(g + h), \overline{f'}f(g + h) \vee f(g' + h) \vee f(g + h')) \\
&= (f(g + h), f'(g + h) \vee f(g' + h) \vee f(g + h')) \\
&= (f, f')(g + h, (g' + h) \vee (g + h')) \\
&= (f, f')((g, g') + (h, h'))
\end{aligned}$$

as required. Thus  $\mathbf{Cl}(\mathbb{X})$  is a left additive restriction category.  $\square$

**Theorem 6.18** *If  $\mathbb{X}$  has the structure of a cartesian left additive restriction category, then so does  $\mathbf{Cl}(\mathbb{X})$ .*

PROOF: Immediate from Theorem 3.11.  $\square$

## 6.4 Differential Structure

Finally, we show that if  $\mathbb{X}$  has differential restriction structure, so does  $\mathbf{Cl}(\mathbb{X})$ .

**Definition 6.19** *If  $\mathbb{X}$  is a differential join restriction category, and  $\coprod(f_i, f'_i)$  is a classical map, define its differential to be*

$$\coprod(D[f_i], D[f'_i])$$

**Lemma 6.20** *The above is a well-defined classical map.*

PROOF: If  $f' \leq f$ , then  $D[f'] \leq D[f]$ , so it is a well-defined classical piece. If  $(f, f') \perp (g, g')$ , then

$$\begin{aligned}
& \overline{Df} \overline{Dg} \\
&= (1 \times \overline{f})(1 \times \overline{g}) \\
&= 1 \times \overline{f} \overline{g} \\
&= 1 \times (\overline{f'} \overline{g} \vee \overline{f} \overline{g'}) \text{ since } (f, f') \perp (g, g') \\
&= (1 \times \overline{f'} \overline{g}) \vee (1 \times \overline{f} \overline{g'}) \\
&= (1 \times \overline{f'}) (1 \times \overline{g}) \vee (1 \times \overline{f}) (1 \times \overline{g'}) \\
&= \overline{Df'} \overline{Dg} \vee \overline{Df} \overline{Dg'}
\end{aligned}$$

so  $(Df, Df') \perp (Dg, Dg')$ , so it is a well-defined raw classical map.

That this is well-defined under collapsing is obvious. For breaking, suppose we have

$$(f, f') \equiv (f, f' \vee ef) \perp (ef, ef')$$

for some restriction idempotent  $e = \bar{e}$ . Then consider

$$\begin{aligned}
& D[(f, f' \vee ef) \perp (ef, ef')] \\
&= (Df, Df' \vee D(ef)) \perp (D(ef), D(ef')) \\
&= (Df, Df' \vee (1 \times e)Df) \perp ((1 \times e)Df, (1 \times e)Df') \text{ by lemma 3.16} \\
&\equiv (Df, Df') \text{ by breaking the restriction idempotent } (1 \times e).
\end{aligned}$$

Thus the map is well-defined under collapsing and breaking, so is a well-defined classical map.  $\square$

**Theorem 6.21** *If  $\mathbb{X}$  is a differential join restriction category, then so is  $\mathbf{Cl}(\mathbb{X})$ .*

PROOF: Most axioms involve a straightforward calculation and use of the lemmas we have developed. We shall demonstrate the two most involved calculations: [D2] and [D5]. For [D2], consider

$$\begin{aligned}
& \langle (g, g'), (k, k') \rangle D(f, f') + \langle (h, h'), (k, k') \rangle D(f, f') \\
&= \langle \langle g, k \rangle, \langle g', k \rangle \vee \langle g, k' \rangle \rangle (Df, Df') + \langle \langle h, k \rangle, \langle h', k \rangle \vee \langle h, k' \rangle \rangle (Df, Df') \\
&= \langle \langle g, k \rangle Df, \langle g', k \rangle Df \vee \langle g, k' \rangle Df \vee \langle g, k \rangle Df \rangle + \langle \langle h, k \rangle Df, \langle h', k \rangle Df \vee \langle h, k' \rangle Df \vee \langle h, k \rangle Df \rangle \\
&= \langle \langle g, k \rangle Df + \langle h, k \rangle Df, [\langle g', k \rangle Df + \langle h, k \rangle Df] \vee [\langle g, k' \rangle Df + \langle h, k \rangle Df] \vee [\langle g, k \rangle Df + \langle h, k \rangle Df] \rangle \\
&\quad \vee [\langle g, k \rangle Df + \langle h', k \rangle Df] \vee [\langle g, k \rangle Df + \langle h, k' \rangle Df] \vee [\langle g, k \rangle Df + \langle h, k \rangle Df'] \rangle
\end{aligned}$$

We can simplify a term like  $\langle g, k' \rangle Df$  as follows:

$$\langle g, k' \rangle Df = \langle g, \bar{k}' k \rangle Df = \bar{k}' \langle g, k \rangle Df$$

And for a term like  $\langle g, k \rangle Df'$ , we can simplify it as follows:

$$\langle g, k \rangle Df' = \langle g, k \rangle D(\bar{f}' f) = \langle g, k \rangle (1 \times \bar{f}') Df = \langle g, k \bar{f}' \rangle Df = \langle g, \bar{k} \bar{f}' k \rangle Df = \bar{k} \bar{f}' \langle g, k \rangle Df$$

Thus, continuing the calculation above, we get

$$\begin{aligned}
&= \langle \langle g, k \rangle Df + \langle h, k \rangle Df, [\langle g', k \rangle Df + \langle h, k \rangle Df] \vee \bar{k}' [\langle g, k \rangle Df + \langle h, k \rangle Df] \vee \bar{k} \bar{f}' [\langle g, k \rangle Df + \langle h, k \rangle Df] \rangle \\
&\quad \vee [\langle g, k \rangle Df + \langle h', k \rangle Df] \vee \bar{k}' [\langle g, k \rangle Df + \langle h, k' \rangle Df] \vee \bar{k} \bar{f}' [\langle g, k \rangle Df + \langle h, k \rangle Df'] \rangle \\
&= \langle \langle g, k \rangle Df + \langle h, k \rangle Df, [\langle g', k \rangle Df + \langle h, k \rangle Df] \vee [\langle g, k \rangle Df + \langle h', k \rangle Df] \rangle \\
&\quad \vee \bar{k}' [\langle g, k \rangle Df + \langle h, k \rangle Df] \vee \bar{k} \bar{f}' [\langle g, k \rangle Df + \langle h, k \rangle Df] \rangle \\
&= \langle \langle g, k \rangle Df + \langle h, k \rangle Df, [\langle g', k \rangle Df + \langle h, k \rangle Df] \vee [\langle g, k \rangle Df + \langle h', k \rangle Df] \rangle \\
&\quad \vee [\langle g, k' \rangle Df + \langle h, k' \rangle Df] \vee [\langle g, k \rangle Df' + \langle h, k \rangle Df'] \rangle \text{ using the above calculations in reverse} \\
&= \langle \langle g + h, k \rangle Df, \langle g' + h, k \rangle Df \vee \langle g + h', k \rangle Df \vee \langle g + h, k' \rangle Df \vee \langle g + h, k \rangle Df' \rangle \text{ by [D2] for } \mathbb{X} \\
&= \langle \langle g + h, k \rangle, \langle g' + h, k \rangle \vee \langle g + h', k \rangle \vee \langle g + h, k' \rangle \rangle (Df, Df') \\
&= \langle \langle g + h, (g' + h) \vee (g + h') \rangle, (k, k') \rangle (Df, Df') \\
&= \langle \langle g, g' \rangle + \langle h, h' \rangle, (k, k') \rangle D(f, f')
\end{aligned}$$



as required. For [D5], consider

$$\begin{aligned}
& \langle D(f, f'), (\pi_1, \emptyset)(f, f') \rangle D(g, g') \\
&= \langle (Df, Df'), (\pi_1 f, \pi_1 f') \rangle (Dg, Dg') \\
&= (\langle Df, \pi_1 f \rangle, \langle Df', \pi_1 \rangle \vee \langle Df, \pi_1 f' \rangle) (Dg, Dg') \\
&= (\langle Df, \pi_1 f \rangle Dg, \langle Df', \pi_1 f \rangle Dg \vee \langle Df, \pi_1 f' \rangle Dg \vee \langle Df, \pi_1 f \rangle Dg')
\end{aligned}$$

Now, we can simplify

$$\langle Df', \pi_1 f \rangle = \langle D(\overline{f'} f), \pi_1 f \rangle = \langle (1 \times \overline{f'}) Df, \pi_1 f \rangle = (1 \times \overline{f'}) \langle Df, \pi_1 f \rangle$$

(where the second equality is by Lemma 3.16), and

$$\langle Df, \pi_1 f' \rangle = \langle Df, \pi_1 \overline{f'} f \rangle = \langle Df, (1 \times \overline{f'}) \pi_1 f \rangle = (1 \times \overline{f'}) \langle Df, \pi_1 f \rangle$$

where the second equality is by lemma 2.14. Thus, the above becomes

$$\begin{aligned}
&= (\langle Df, \pi_1 f \rangle Dg, (1 \times \overline{f'}) \langle Df, \pi_1 f \rangle Dg \vee \langle Df, \pi_1 f \rangle Dg') \\
&= (D(fg), (1 \times \overline{f'} D(fg) \vee D(fg'))) \text{ by [D5] for } \mathbb{X} \\
&= (D(fg), D(f'g) \vee D(fg')) \text{ by Lemma 3.16} \\
&= D(fg, f'g \vee fg') \\
&= D((f, f')(g, g'))
\end{aligned}$$

as required.  $\square$

Now that we know that the classical completion of a differential restriction category is again a differential restriction category, it will be interesting to see what type of maps are in the classical completion of the standard model. For example, consider  $f'(x) = 2x$  defined everywhere but  $x = 5$ , and  $f(x) = 2x$  defined everywhere. Taking a relative complement would give us a map defined *only* at  $x = 5$ , and has the value  $2x = 10$  there. But if differential structure is retained, in what sense is this map “smooth”?

Of course, this map is really an equivalence class of maps. In particular, imagine we have a restriction idempotent  $e = \overline{e}$  (that is, an open subset), which includes 5. Then we have

$$(f, f') \equiv (ef, ef') \sqcup (f, f' \vee ef) = (ef, ef') \sqcup (f, f) \equiv (ef, ef')$$

So that this map is actually equivalent to any other map defined on an open subset which includes 5. This is precisely the definition of the *germ* of a function at 5. Thus, the classical completion process adds germs of functions at points.

Of course, it also allows us to take joins of germs and regular maps, so that for example we could take the join of the above map, and something like  $\frac{x-1}{x-5}$ , giving a total map which has “repaired” the discontinuity of the second map at 5. The fact that this restriction category is a differential restriction category is perhaps now much more surprising. Clearly, this will be an example that will need to be explored further.

Finally, given the additive, cartesian, and differential structure of  $\mathbf{Cl}(\mathbb{X})$ , the following is immediate:

**Proposition 6.22** *The unit  $\mathbb{X} \longrightarrow \mathbf{Cl}(\mathbb{X})$ , which sends  $f$  to  $(f, \emptyset)$ , is a differential restriction functor.*

And as a result, we have the following:

**Corollary 6.23** *Suppose  $\mathbb{X}$  is a differential restriction category with joins, and  $f' \leq f$ . Then:*

- (i) *if  $f$  is additive in  $\mathbb{X}$ , then so are  $(f, \emptyset)$  and  $(f, f')$  in  $\mathbf{Cl}(\mathbb{X})$ ;*
- (ii) *if  $f$  is strongly additive in  $\mathbb{X}$ , then so is  $(f, \emptyset)$  in  $\mathbf{Cl}(\mathbb{X})$ ;*
- (iii) *if  $f$  is linear in  $\mathbb{X}$ , then so are  $(f, \emptyset)$  and  $(f, f')$  in  $\mathbf{Cl}(\mathbb{X})$ .*

PROOF: By Proposition 3.24,  $(f, \emptyset)$  retains being additive/strongly additive/linear, and since  $(f, f')$  is a relative complement,  $(f, f') \leq f$ , so is additive/linear if  $f$  is.  $\square$

## 7 Conclusion

There are a number of different expansions of this work that are possible. The first deals with smooth manifolds. A similar construction to that found in [Grandis 1989] allows one to build a new restriction category of manifolds out of any join restriction category. For example, applying this construction to continuous functions defined on open subsets of  $\mathcal{R}^n$  gives one the usual category of real manifolds. An obvious expansion of the present theory is to understand what happens when we apply this construction to a differential restriction category with joins. Clearly, this will build categories of smooth maps between smooth manifolds. In general, however, one should not expect this to again be a differential restriction category, as the derivative of a smooth manifold map  $f : M \longrightarrow N$  is not a map  $M \times M \longrightarrow N$ , but instead a map  $TM \longrightarrow TN$ , where  $T$  is the tangent bundle functor. Thus, we must show that one can describe the tangent bundle of any object in the manifold completion of a differential restriction category. This is the subject of a future paper, and will allow for closer comparisons between the theory presented here and synthetic differential geometry.

Another avenue for research is the links between this theory and classical work in algebraic and differential geometry. Given that the rational functions example embeds into affine schemes, there is clearly a connection with the work of Grothendieck. Given that the classical completion of the standard model involves germs of smooth functions, there is clearly a connection with differential geometry.

Finally, the definition of an “integral category” still remains to be defined, and the links with this theory should lead to interesting results.

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