The Jacobi identity for tangent categories

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Tangent category definition

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

A **tangent category** consists of a category $\mathbb{X}$ with:

- an endofunctor $T : \mathbb{X} \to \mathbb{X}$;
- a natural transformation $p : T \to I$;
- for each M, the pullback of n copies of $p_M : TM \to M$ along itself exists (and is preserved by each T^m), call this pullback $T_n M$;
- for each $M \in \mathbb{X}$, $p_M : TM \to M$ has the structure of a commutative monoid in the slice category $\mathbb{X}/M$, in particular there are natural transformation $+ : T_2 \to T$, $0 : I \to T$;

(**Note**: composition will be in diagrammatic order.)

Tangent category definition (continued)

Definition

- (canonical flip) there is a natural transformation $c : T^2 \to T^2$ which preserves additive bundle structure and satisfies $c^2 = 1$;
- (vertical lift) there is a natural transformation $\ell : T \to T^2$ which preserves additive bundle structure and satisfies $\ell c = \ell$;
- various other coherence equations for ℓ and c;
- (universality of vertical lift) elements d of T^2M which have $T(p) = 0$ are uniquely given by elements of T_2M (the second element of T_2M is simply p of d).

Examples

(i) Finite dimensional smooth manifolds with the usual tangent bundle structure.

(ii) Convenient manifolds with the kinematic tangent bundle.

(iii) Any Cartesian differential category is a tangent category, with $T(A) = A \times A$ and $T(f) = \langle Df, \pi_1 f \rangle$.

(iv) The infinitesimally linear objects in any model of synthetic differential geometry.

(v) Both commutative ri(n)gs and its opposite category have tangent structure, as well as various categories in algebraic geometry.

(vi) The category of $C - \infty$-rings has tangent structure.

Some theory

- The following are definable concepts in tangent categories:
  (i) vector bundles;
  (ii) connections;
  (iii) differential forms.
- A tangent category in which T is representable by D has an associated rig R with $R^D \cong R \times R$ (ie., R satisfies the Kock-Lawvere axiom).

Vector fields

A **vector field** on M is simply a section of $p_M : TM \to M$.

- The 0 natural transformation provides for every M a vector field $0_M : M \to TM$.
- Since vector fields have the same projection, one can also add two of them: $x + y := \langle x, y \rangle +$.
- More interesting is that if one has negatives, one can define the Lie bracket of two vector fields x, y, $[x, y]$, by the universal property of the vertical lift:

$$\langle xT(y)-, yT(x)c \rangle +$$

is an element of T^2M with $T(p) = 0$, so $[x, y]$ is defined to be the first part of the corresponding unique element of T_2M.

Some bracket properties

It is relatively easy to prove that:

1. $[x, y]$ is again a vector field.
2. The operation is additive in both variables:

$$[x_1 + x_2, y] = [x_1, y] + [x_2, y] \text{ and } [x, y_1 + y_2] = [x, y_1] + [x, y_2].$$

3. Negation reverses the order:

$$[x, y]- = [y, x].$$

Jacobi identity

But the big problem is determining whether the following Jacobi identity holds:

$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0.$$

Rosický provided a proof which was 80 pages and assumed the existence of additional limits. (Which are potentially problematic in some models).

Jacobi identity in the standard model

In smooth manifolds, vector fields x on M are the same as derivations X on the ring $C^\infty(M)$, and the Lie bracket of X and Y is simply

$$XY - YX$$

So that the Jacobi identity is straightforward:

$$\begin{aligned} & [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] \\ = \; & X[Y,Z] - [Y,Z]X + Z[X,Y] - [X,Y]Z + Y[Z,X] - [Z,X]Y \\ = \; & XYZ - XZY + YZX - ZYX + ZXY - ZYX \\ & -XYZ + YXZ + YZX - YXZ - ZXY + XZY \\ = \; & 0 \end{aligned}$$

But we can't do this in a general tangent category!

Some sample calculations

The calculations quickly get complicated in a tangent category:

- Since the terms are defined by a universal property, it gets tricky to use "parts" of each term to cancel other parts of the other terms.
- Rosický realized that instead of trying to see their universal property, it was easier to post-compose the terms with the lift ℓ:

$$[x, y]\ell = xT(y)T^2(x)T^3(y) - T(-)T(c)\mu_1 T(\mu_1)$$

where

$$\mu_1 = \langle Tp, p\rangle + \text{ is the multiplication of a monad on } T : \mathbb{X} \to \mathbb{X}.$$

- Then post-compose the Jacobi term

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]]$$

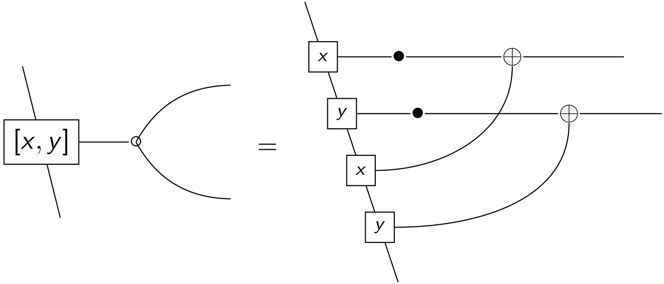
with $\ell\ell$, use the fact that ℓ is a morphism of monads, and try to get the 0 term out.
- What we need is an easier way to manipulate terms like those given above.

Such terms can be represented graphically

We can use the graphical language of the 2-category CAT to do this.

- The object M can be represented as a functor $M : 1 \to \mathbb{X}$.
- A vector field x on M is then a natural transformation $x : M \to MT$.
- Represent $\ell : T \to TT$ by $\circ$.
- Represent $c : TT \to TT$ by a crossing of wires.
- Represent $\mu_1 : TT \to T$ by $\oplus$.
- Negation $- : T \to T$ is represented by $\bullet$.

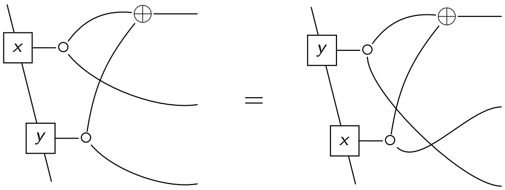
For example, the following diagram



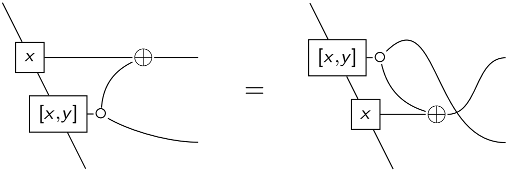
represents $[x, y]\ell = xT(y)T^2(x)T^3(y) - T(-)T(c)\mu_1 T(\mu_1)$.

More graphical examples

Another identity that can be established is that:

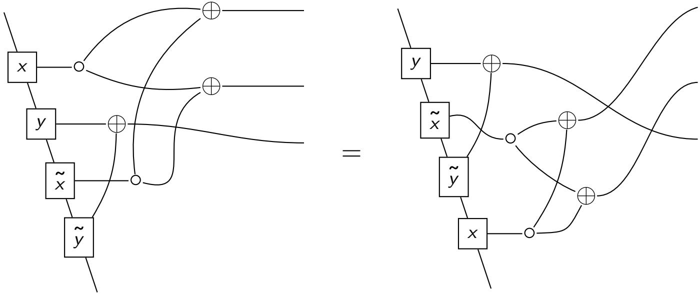


From this identity, one can also prove:



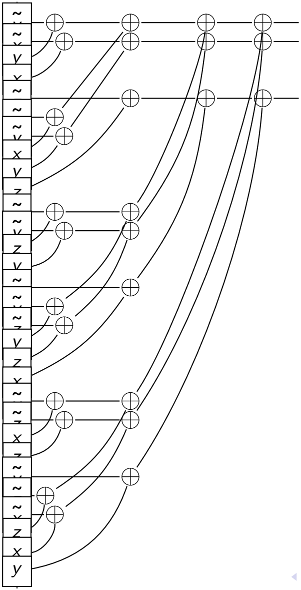
Further graphical examples

And also:



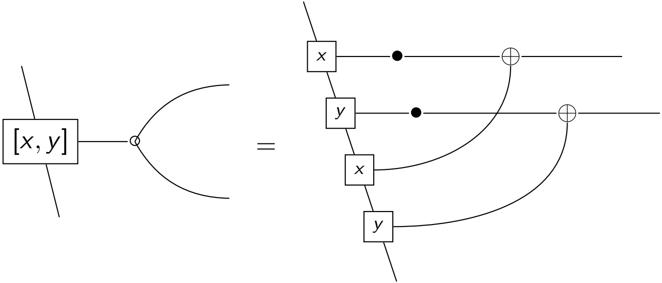
(Where now we write $\tilde{x}$ for the negation of x.)

One expansion of $[x, [y, z]] + [z, [x, y]] + [y, [z, x]]\ell\ell$



Simplifying even further

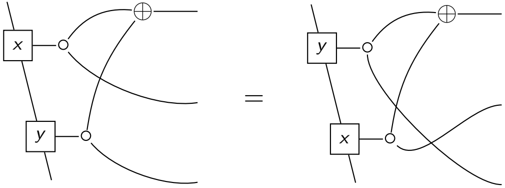
- To simplify further, we use an additional layer of notation.
- We present terms in the graphical calculus as a sequence of vector fields, subscripted by which level they are connected to by ℓ or μ_1.
- For example,



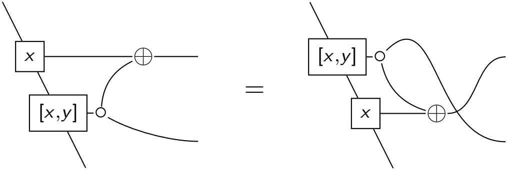
is written as $[x, y]_{12} = \tilde{x}_1 \tilde{y}_2 x_1 y_2$ (1).

Lemmas in this notation

We can represent the other graphical identities in this notation:



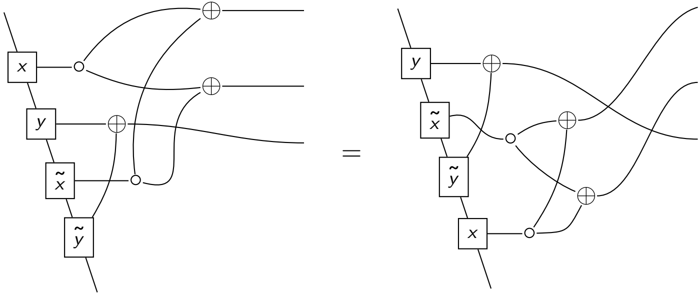
is $x_{12}y_{13} = y_{13}x_{12}$ (2) (two terms lifted to have a level in common commute), and



is $x_1[x, y]_{12} = [x, y]_{12}x_1$ (3) (brackets commute with their constituents);

Lemmas in this notation

and



becomes $x_{12}y_3\tilde{x}_{12}\tilde{y}_3 = y_3\tilde{x}_{12}\tilde{y}_3x_{12}$ (4).

Final version of the proof

In this notation, we can now give a relatively short version of the proof:

$$
\begin{aligned}
& [[x,y],z]_{123}[[y,z],x]_{123}[[z,x],y]_{123} \\
= \quad & [x,y]_{12} z_3 [y,x]_{12} \tilde{z}_3 [y,z]_{23} x_1 [z,y]_{23} \tilde{x}_1 \underline{[x,z]_{31}} \tilde{y}_2 [z,x]_{31} y_2 \text{ (by 1)} \\
= \quad & [x,y]_{12} [x,z]_{31} z_3 [y,x]_{12} \tilde{z}_3 [y,z]_{23} x_1 [z,y]_{23} \tilde{x}_1 \tilde{y}_2 \underline{[z,x]_{31}} y_2 \text{ (by 2,3)} \\
= \quad & [x,y]_{12} [x,z]_{31} z_3 [y,x]_{12} \tilde{z}_3 [y,z]_{23} x_1 [z,y]_{23} \underline{\tilde{x}_1 \tilde{y}_2 x_1 y_2} \tilde{y}_2 \tilde{z}_3 \tilde{x}_1 z_3 y_2 \text{ (by 1)} \\
= \quad & [x,y]_{12} [x,z]_{31} z_3 [y,x]_{12} \tilde{z}_3 [y,z]_{23} x_1 [z,y]_{23} \underline{[x,y]_{12}} \tilde{y}_2 \tilde{z}_3 \tilde{x}_1 z_3 y_2 \text{ (by 1)} \\
= \quad & [x,y]_{12} [x,z]_{31} z_3 [y,x]_{12} \tilde{z}_3 [x,y]_{12} [y,z]_{23} x_1 \underline{[z,y]_{23}} \tilde{y}_2 \tilde{z}_3 \tilde{x}_1 z_3 y_2 \text{ (by 2,3)} \\
= \quad & [x,y]_{12} [x,z]_{31} z_3 [y,x]_{12} \tilde{z}_3 [x,y]_{12} [y,z]_{23} x_1 \tilde{z}_3 \tilde{y}_2 \underline{z_3 y_2 \tilde{y}_2 \tilde{z}_3} \tilde{x}_1 z_3 y_2 \text{ (by 1)} \\
= \quad & [x,y]_{12} [x,z]_{31} z_3 [y,x]_{12} \tilde{z}_3 [x,y]_{12} \underline{[y,z]_{23}} x_1 \tilde{z}_3 \tilde{y}_2 \tilde{x}_1 z_3 y_2 \text{ (negation)} \\
= \quad & [y,z]_{23} [x,y]_{12} [x,z]_{31} \underline{z_3 [y,x]_{12} \tilde{z}_3 [x,y]_{12}} x_1 \tilde{z}_3 \tilde{y}_2 \tilde{x}_1 z_3 y_2 \text{ (by 2,3)}
\end{aligned}
$$

Final version of the proof (continued)

$$
\begin{aligned}
&= [y,z]_{23}[x,y]_{12}[x,z]_{31}[y,x]_{12}\tilde{z}_3[x,y]_{12}z_3x_1\tilde{z}_3\underline{\tilde{x}_1x_1}\tilde{y}_2\tilde{x}_1z_3y_2 \text{ (by 4)}\\
&= [y,z]_{23}[x,y]_{12}[x,z]_{31}[y,x]_{12}\tilde{z}_3[x,y]_{12}\underline{[z,x]_{13}}x_1\tilde{y}_2\tilde{x}_1z_3y_2 \text{ (by 1)}\\
&= [y,z]_{23}[x,y]_{12}\underline{[x,z]_{31}[z,x]_{13}}[y,x]_{12}\tilde{z}_3[x,y]_{12}x_1\tilde{y}_2\tilde{x}_1z_3y_2 \text{ (by 2,3)}\\
&= [y,z]_{23}\underline{[x,y]_{12}[y,x]_{12}}\tilde{z}_3[x,y]_{12}x_1\tilde{y}_2\tilde{x}_1z_3y_2 \text{ (negation)}\\
&= [y,z]_{23}\tilde{z}_3[x,y]_{12}x_1\tilde{y}_2\tilde{x}_1\underline{y_2\tilde{y}_2}z_3y_2 \text{ (negation)}\\
&= [y,z]_{23}\tilde{z}_3\underline{[x,y]_{12}[y,x]_{12}}\tilde{y}_2z_3y_2 \text{ (by 1)}\\
&= [y,z]_{23}\underline{\tilde{z}_3\tilde{y}_2z_3y_2} \text{ (negation)}\\
&= [y,z]_{23}[z,y]_{23} \text{ (by 1)}\\
&= 0_{123} \text{ (negation)}
\end{aligned}
$$

Conclusions

- We have proven Jacobi's identity for tangent categories by making judicious use of the graphical language of 2-categories and then simplifying that further.
- This method may be useful in proving other identities in tangent categories such as the identities of Bianchi and Ricci (these involve connections).
- The result itself may be useful in newly-evolving models of differential geometry (for example, diffeological spaces).
- Is a more conceptual proof possible?