Curvature and torsion without negatives

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Overview

- Tangent categories provide an abstract framework for unifying many disparate notions of "derivative" and "tangent bundle".
- Examples include smooth manifolds, SDG, schemes, Cartesian differential categories, Abelian functor calculus, potentially Goodwillie functor calculus (perhaps a 2 or infinity tangent category), tropical geometry...
- To encompass a variety of different examples, tangent categories do not assume one can negate tangent vectors.
- Many aspects of differential geometry have been developed in this setting: vector bundles, connections, differential forms, de Rham cohomology, vector fields, flows, Lie brackets...

Overview

- However, some of these definitions have required assuming the existence of negatives, meaning they won't apply to all examples.
- One example has been curvature and torsion of a connection. For example, the standard definitions (for a covariant derivative on a smooth manifold) use negatives:

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

$$T(x,y) = \nabla_x y - \nabla_y x - [x,y]$$

- In this talk, we'll recall how to define curvature and torsion of a connection on an object in a tangent category, and then see how to re-work the definition so that negatives are not required.

Tangent category definition

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

A **tangent category** consists of a category $\mathbb{X}$ with:

- **tangent bundle functor**: an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- **projection of tangent vectors**: a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of p_M along itself exists; call this pullback $T_n M$ (the "space of n tangent vectors at a point")
- **addition and zero tangent vectors**: for each $M \in \mathbb{X}$, p_M has the structure of a commutative monoid in the slice category $\mathbb{X}/M$;

Tangent category definition (continued)

Definition

- **symmetry of mixed partial derivatives**: a natural transformation $c : T^2 \longrightarrow T^2$;
- **linearity of the derivative**: a natural transformation $\ell : T \longrightarrow T^2$;
- "the vertical bundle of the tangent bundle is trivial";
- various coherence equations for ℓ and c.

Say that tangent category *has negatives* if the monoid structure of each $p_M : TM \longrightarrow M$ is actually a group.

Examples

(i) Finite dimensional smooth manifolds with the usual tangent bundle.
(ii) Convenient manifolds with the kinematic tangent bundle.
(iii) Any Cartesian differential category (includes all Fermat theories by a result of MacAdam, and Abelian functor calculus by a result of Bauer et. al.).
(iv) The microlinear objects in a model of synthetic differential geometry (SDG).
(v) Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.
(vi) The category of C^∞-rings.
(vii) With additional pullback assumptions, tangent categories are closed under slicing.

Note: Building on work of Leung, Garner has shown how tangent categories are a type of enriched category.

Intuitive idea of a connection

Idea: a **connection** on a "bundle" $q : E \longrightarrow M$ is a choice of a horizontal and vertical co-ordinate system for TE (see diagram).

Vertical bundle

Definition

If $q : E \longrightarrow M$ is a bundle, its **vertical bundle**, $V(E)$, is the following pullback:

$$\begin{array}{ccc} V(E) & \xrightarrow{\quad i \quad} & T(E) \\ \downarrow & & \downarrow T(q) \\ M & \xrightarrow[\quad 0 \quad]{} & T(M) \end{array}$$

Horizontal bundle

Definition

If $q : E \longrightarrow M$ is a bundle, its **horizontal bundle**, $H(E)$, is the following pullback:

$$\begin{array}{ccc} H(E) & \longrightarrow & T(M) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_M} \\ E & \underset{q}{\longrightarrow} & M \end{array}$$

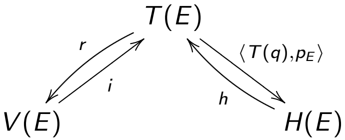
Associated maps

A bundle then has the following diagram of maps:

$$V(E) \xrightarrow{\;i\;} T(E) \xrightarrow{\;\langle T(q), p_E \rangle\;} H(E)$$

General connection

A **connection** on such a bundle is then required to have maps r, h:



satisfying various axioms.

Connection on a vertically trivial bundle

- For vector bundles, the vertical bundle VE is trivial, in the sense that it is a fibred product: $VE \cong E \times_M E$ (this is essentially how we *define* vector bundles in a tangent category).
- In this case, the vertical part of a connection is simply given by a map $K : TE \longrightarrow E$.
- In particular, we axiomatically assume that the vector bundle of the tangent bundle is trivial, and so in this case the vertical part of a connection is given by a map $T^2M \longrightarrow TM$; the horizontal part is given by a map $H : T_2M \longrightarrow T^2M$.
- We shall write (K, H) for a connection on the tangent bundle of M.

Torsion

Definition

A connection (K, H) on M is **torsion-free** if $c_M K = K$:

$$\begin{array}{ccc} T^2M & \xrightarrow{c_M} & T^2M \\ & \searrow^{K} & \downarrow K \\ & & TM \end{array}$$

(Standard definition: for all x, y, $\nabla_x y - \nabla_y x - [x, y] = 0$.)

Definition

In a tangent category with negatives, the **torsion** of a connection is the difference

$$T^2M \xrightarrow{cK - K} TM.$$

Curvature

Definition

A connection (K, H) on M is **flat** (curvature-free) if $c_{TM} T(K)K = T(K)K$:

$$
\begin{array}{ccccc}
T^3M & \xrightarrow{c_{TM}} & T^3M & \xrightarrow{T(K)} & T^2M \\
& \searrow^{T(K)} & & & \downarrow K \\
& & T^2M & \xrightarrow[K]{} & TM
\end{array}
$$

(Standard definition: for all u, v, w, $\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w = 0$.)

Definition

In a tangent category with negatives, the **curvature** of a connection is the difference

$$T^3M \xrightarrow{cT(K)K - T(K)K} T^2M.$$

Problems

There are several problems with these definitions:

- The torsion and curvature maps require negatives.
- Seems to be "higher-order" than the ordinary definitions (eg., torsion goes from T^2M instead of T_2M).
- Neither definition uses H.

Higher order?

- If these definitions really are higher-order, they should have more information than the standard definition. What is this extra information?
- However, when I actually did some calculations with what these notions told me for connections on simple smooth manifolds (eg., spheres), the higher-order terms always vanished!
- Actually, this holds more generally!

Simplifying torsion

- Recall that if M has a connection K, every element of T^2M is uniquely given determined by its horizontal and vertical parts (see diagram).
- Thus, we can look at what the horizontal and vertical parts of the expression $cK - K$ are.
- **The vertical parts vanish, and the horizontal part of K vanishes**. As a result, all the information in $cK - K$ is contained in the expression

$$T_2M \xrightarrow{H} T^2M \xrightarrow{c_M} T^2M \xrightarrow{K} TM.$$

New torsion definition

Definition

For a connection (K, H) on M, its **torsion** is the map

$$T_2M \xrightarrow{H} T^2M \xrightarrow{c_M} T^2M \xrightarrow{K} TM$$

It is **torsion-free** if this is zero (that is, it equals $\pi_0 p0$).

- This solves all three previous problems simultaneously!

Simplifying curvature

- The curvature is a map out of T^3M: but with a connection, the splitting of T^2M also leads to a splitting of T^3M.
- Applying this splitting to the curvature expression $cT(K)K - T(K)K$ shows that all its information is contained in the expression

$$T_3M \xrightarrow{\langle\langle\pi_0,\pi_1\rangle H, \langle\pi_0,\pi_2\rangle H\rangle} T(T_2M) \xrightarrow{T(H)} T^3M \xrightarrow{c_{TM}} T^3M \xrightarrow{T(K)} T^2M \xrightarrow{K} TM$$

New curvature definition

Definition

For a connection (K, H) on M, its **curvature** is the map

$$T_3M \xrightarrow{\langle\langle\pi_0,\pi_1\rangle H,\langle\pi_0,\pi_2\rangle H\rangle T(H)cT(K)K} TM.$$

It is **flat** (curvature-free) if this is zero (that is, it equals $\pi_0 p0$).

- Again, solves all three problems.

Conclusions

- Curvature and torsion can be defined for tangent-bundle connections in a tangent category without requiring negatives.
- This may lead to new ideas in some of the examples without negatives (eg., tropical geometry, functor calculus).
- Still more work to do understanding curvature for differential bundles and more general bundles.