The dual fibration, part two: partial case

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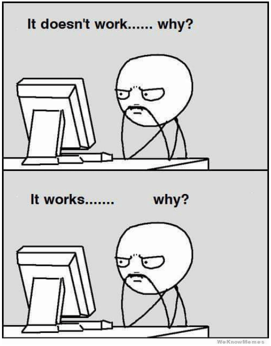

Overview

Last time we reviewed how to define the *dual* fibration to any fibration

$$\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}.$$

- This construction produces interesting examples of fibrations, ones which involve maps going both forwards and backwards.
- Many applied situations seem to involve such maps, eg., lenses, learners, open games, reverse derivatives, etc.
- Today the goal is to see how we can work with fibrations and the dual fibration in categories of partial maps, ie., restriction categories.

Motivation: how can this even be possible?

How can we construct dual fibrations of restriction categories when you can't take the opposite of a restriction category?

In particular, in what sense is the partial simple fibration, with maps

$$(A, X) \xrightarrow{(f, f')} (B, Y)$$

where

$$A \xrightarrow{f} B, \; A \times X \xrightarrow{f'} Y \text{ with } \overline{f'} = \overline{f} \times 1$$

"dual" to the category with maps

$$(A, X) \xrightarrow{(g, g')} (B, Y)$$

where

$$A \xrightarrow{g} B, \; A \times Y \xrightarrow{g'} X \text{ with } \overline{g'} = \overline{g} \times 1 \text{ ?}$$

Restriction categories

Definition

A **restriction category** (Cockett/Lack 2002) is a category $\mathbb{C}$ equipped with an operation which takes a map $f : A \longrightarrow B$ in $\mathbb{C}$ and gives a map $\overline{f} : A \longrightarrow A$ which satisfies four identities:

[R.1] $\overline{f} f = f$ [R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ [R.3] $\overline{f}\,\overline{g} = \overline{\overline{f} g}$ [R.4] $f \overline{g} = \overline{fg} f$

- The prototypical restriction category is the category of sets and partial maps, where $\overline{f}(x)$ is defined to be x when $f(x)$ is defined, and undefined otherwise.
- The category whose objects are $\mathbb{R}^n$'s and whose maps are smooth *partial* functions is similarly a restriction category.

Note: an arrow $f : A \longrightarrow B$ need not have a "domain object" on which it is fully defined! The partiality of f is encoded in the arrow $\overline{f}$ (a "restriction idempotent") not in an object.

Partial order and partial inverses

We'll need a few basic concepts from restriction categories.

Definition

For maps $f, g : A \longrightarrow B$ in a restriction category $\mathbb{C}$, write $f \leq g$ if $\overline{f} g = f$.

- This captures the idea that f is less defined than g, but they are equal where they are both defined (eg., $\frac{x^2-1}{x-1} \leq x+1$).
- This gives a partial order on each homset, and these partial orders are compatible with composition.

Definition

A map $f : A \longrightarrow B$ has a **partial inverse** $g : B \longrightarrow A$ if

$$fg = \overline{f} \text{ and } gf = \overline{g}.$$

- For example, $\frac{1}{2x}$ does not have an inverse (in the ordinary sense) but it does have a *partial* inverse $\frac{2}{x}$.

Restriction functors and semifunctors

If $\mathbb{C}$ and $\mathbb{D}$ are restriction categories:

Definition

A **restriction functor** $F : \mathbb{C} \longrightarrow \mathbb{D}$ is a functor that preserves restrictions, ie., for any g in $\mathbb{C}$, $F(\overline{g}) = \overline{F(g)}$.

Definition

A **restriction semifunctor** $F : \mathbb{C} \longrightarrow \mathbb{D}$ is a map of objects and arrows that preserves composition and restriction (but not necessarily identities).

Note, however, that for restriction semifunctors

$$F(1_A) = F(\overline{1_A}) = \overline{F(1_A)},$$

so $F(1_A)$ is still a restriction idempotent (just not necessarily the identity).

Restriction transformations

Definition

If $F, G : \mathbb{C} \longrightarrow \mathbb{D}$ are restriction functors,

- a **restriction transformation** $\alpha : F \Rightarrow G$ is a natural transformation for which each component α_C is total;
- a **lax restriction transformation** $\alpha : F \Rightarrow G$ has total components $\alpha_C : FC \longrightarrow GC$ such that for any $f : A \longrightarrow B$,

$$F(f)\alpha_B \leq \alpha_A G(f).$$

- These give 2-categories **rCat** and $\mathbf{rCat}_l$.

Definition

If F, G are restriction semifunctors, a **lax restriction transformation** $\alpha : F \Rightarrow G$ has components $\alpha_C : FC \longrightarrow GC$ such that $\overline{\alpha_C} = F(1_C)$ and for any $f : A \longrightarrow B$, $F(f)\alpha_B \leq \alpha_A G(f)$.

- This gives a 2-category $\mathbf{rCat}_{sl}$.

Limits in restriction categories?

Recall that $\mathbb{C}$ has limits of shape $\mathbb{D}$ if the diagonal $\Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{D}}$ has a right adjoint in **Cat**. What is the correct notion for restriction categories?

- Asking for a right adjoint in **rCat** is definately not correct. If $\mathbb{C}$ has even a terminal object T in this sense, then it has unique total maps $t_A : A \longrightarrow T$; this forces every map $f : A \longrightarrow B$ in $\mathbb{C}$ to be total:

$$\overline{f} = \overline{f 1_B} = \overline{f \overline{t_B}} = \overline{f t_B} = \overline{t_A} = 1_A.$$

Limits in restriction categories?

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- Asking for a right adjoint in $\mathbf{rCat}_l$ gives **restriction limits**. However, these force splittings: eg., the diagram

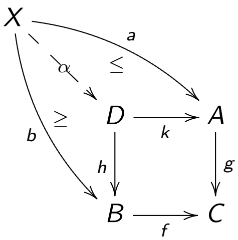
$$A \xrightarrow{f} B$$

has a restriction limit if and only $\overline{f}$ is a split restriction idempotent (ie., f has a "domain").

- For our purposes, a right adjoint in $\mathbf{rCat}_{sl}$, which is known as a **latent limit** (Cockett/Hofstra/Guo 2012), is the most useful.

Latent pullbacks

For example, a **latent pullback** has the following:



with

$$\overline{h} = \overline{k} = \overline{hf} = \overline{kg}$$

and

$$\overline{\alpha} = \overline{bf} = \overline{ag}, \quad \alpha\overline{h} = \alpha\overline{k} = \alpha.$$

Note: the projections h and k need not be total! (They must be for *restriction* pullbacks).

Latent pullback example

For example, for an $f : A \longrightarrow B$, in general the diagram

$$
\begin{array}{ccc}
 & & B \\
 & & \big\downarrow{\scriptstyle 1_B} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

need not have a restriction pullback, but it always has a latent pullback:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \overline{f}}\big\downarrow & & \big\downarrow{\scriptstyle 1_B} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

Note that this also means in general the latent pullback of a total arrow need not be total!

Restriction version of the simple fibration

Recall from last time, we want to consider a restriction version of the simple fibration:

Definition

For a restriction category $\mathbb{C}$ with latent products, let $\mathbb{C}[\mathbb{C}]$ denote the restriction category with:

- objects pairs (A, X);
- morphisms $(f, f') : (A, X) \longrightarrow (B, Y)$ are

$$A \xrightarrow{f} B, \;\; A \times X \xrightarrow{f'} Y \text{ with } \overline{f'} = \overline{f} \times 1$$

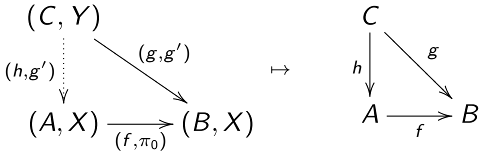
- composition as before: $(f, f') \circ (g, g') := (fg, \langle \pi_0 f, f' \rangle g')$;
- restriction $\overline{(f, f')} := (\overline{f}, \overline{f'}\pi_1 = \overline{\pi_0 f})$.

Recall that one motivation for the condition $\overline{f'} = \overline{f} \times 1$ comes from derivatives:

$$\overline{D[f]} = \overline{f} \times 1.$$

The restriction simple fibration is not a fibration

As noted last time, this is **not** a fibration over $\mathbb{C}$!

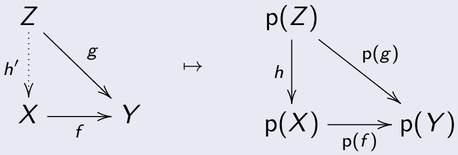


- We need $\overline{g'} = \overline{h} \times 1$, but only have $\overline{g'} = \overline{g} \times 1$.
- There is no reason why $\overline{g} = \overline{h}$.

Latent fibration definition (Nester 2017)

Definition

For a restriction functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$, a **prone arrow** is a map $f : X \longrightarrow Y$ in $\mathbb{E}$ so that for any $g : Z \longrightarrow Y$ in $\mathbb{E}$ and $h : \mathrm{p}(Z) \longrightarrow \mathrm{p}(X)$ in $\mathbb{B}$ so that $h\mathrm{p}(f) = \mathrm{p}(g)$ **and** $\overline{h} = \overline{\mathrm{p}(g)}$ there is a unique $h' : Z \longrightarrow X$ so that $\mathrm{p}(h') = h$, $h'f = g$ **and** $\overline{h'} = \overline{g}$:



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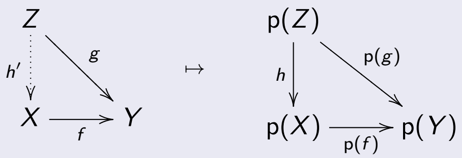

Definition

A restriction functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is said to be a **latent fibration** if for any $\alpha : A \longrightarrow B$ in $\mathbb{B}$, and any Y such that $\mathrm{p}(Y) = B$, there is a prone arrow $\alpha^* : X \longrightarrow Y$ over α, i.e., such that $\mathrm{p}(\alpha^*) = \alpha$.

Latent fibration examples

1. The restriction version of the simple fibration is a latent fibration over $\mathbb{C}$.
2. There is a lax version of the simple fibration (arrows (f, f') have $\overline{f'} \leq \overline{f} \times 1$); this is also a latent fibration over $\mathbb{C}$.

For any restriction category $\mathbb{C}$, let $\mathbb{C}^{\rightarrow}$ be the restriction category whose objects are arrows of $\mathbb{C}$, with morphisms "semi-precise squares": commutative squares

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[g]{} & Y \end{array}$$

such that $\overline{f} = \overline{f\overline{y}}$. Let $\mathbb{C}^{\rightsquigarrow}$ similar to $\mathbb{C}^{\rightarrow}$ but with $xg \geq fy$ rather than equality.

3. For any $\mathbb{C}$ with latent pullbacks, the codomain functors $\mathbb{C}^{\rightarrow} \longrightarrow \mathbb{C}$ and $\mathbb{C}^{\rightsquigarrow} \longrightarrow \mathbb{C}$ are latent fibrations. (In fact, these are latent fibrations if and only if $\mathbb{C}$ has latent pullbacks!)

Latent fibration examples continued

4. For any restriction category $\mathbb{C}$, let $\mathcal{O}(\mathbb{C})$ denote the restriction category whose objects are pairs (A, e) where e is a restriction idempotent on A and whose morphisms $f : (A, e) \longrightarrow (A', e')$ are maps $f : A \longrightarrow A'$ such that $e \leq \overline{fe'}$; this is a latent fibration over $\mathbb{C}$.
5. More generally, if we let $\mathbb{C}^{\leq}$ be the arrow category with now $xg \leq fy$ (with no "semi-precise" requirement), the domain functor $\mathbb{C}^{\leq} \longrightarrow \mathbb{C}$ is a latent fibration. (There is a faithful functor from the previous category to this one).
6. (Nester, 2017) Builds a category of "assemblies" $\mathrm{Asm}(F)$ out of any restriction functor $F : \mathbb{C} \longrightarrow \mathbb{X}$ such that $\mathbb{X}$ has restriction products (generalizing a construction of assemblies from a partial combinatory algebra); this also gives a latent fibration over $\mathbb{X}$.

Theory of latent fibrations

Much of the theory of fibrations works for a latent fibration $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$!

- The composite of two prone maps is again prone.
- Every map in $\mathbb{E}$ factors as a **sub**vertical map followed by a prone map (g is subvertical if $\mathsf{p}(g)$ is a restriction idempotent).
- More generally, factorization systems in $\mathbb{B}$ lift to factorization systems in $\mathbb{E}$ (though one has to define factorization systems in a restriction category first...)

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Also:

- The composite of two latent fibrations is a latent fibration.
- The pullback of a latent fibration along any restriction functor is a latent fibration.

Indexed version

While there is an indexed category version of latent fibrations, it is much more complicated.

- For a latent fibration $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$, we take the fibre over A to be objects over A and *subvertical* maps over A (the usual fibre will not work).
- This produces a pseudofunctor

$$P : \mathbb{B}^{op} \longrightarrow \mathsf{SRest},$$

  where SRest is the 2-category of restriction categories, semifunctors, and semifunctor transformations.
- Unfortunately, to go back from this to a latent fibration, we need more data ("bounding maps").

Warning one: restriction idempotents need not pronely lift

If $e : A \longrightarrow A$ is a restriction idempotent in $\mathbb{B}$, then there may not be a prone restriction idempotent in $\mathbb{E}$ over e!

- The latent fibration property guarantees *a* prone map over e, but it may not necessarily be a restriction idempotent.
- For example, in $\mathcal{O}(C)$, the prone lift of $e : A \longrightarrow A$ over (A, e') is

  $$(A, \overline{ee'}) \xrightarrow{e} (A, e')$$

  which is no longer even an endomorphism!
- In fact, unless $e' \geq e$, it is not possible to find *any* restriction idempotent over e to (A, e'), let alone a prone one.
- There is a similar problem with the domain latent fibration.

Admissible latent fibrations

Definition

Say that a latent fibration $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is **admissible** if for any restriction idempotent $e : A \longrightarrow A$ in $\mathbb{B}$ and any X in $\mathbb{E}$ over A, there is a prone restriction idempotent $e^* : X \longrightarrow X$ in $\mathbb{E}$ over e.

- All the previous examples except the propositions example and the domain example are admissible.
- In general, the splitting of a latent fibration p is a latent fibration only if p is admissible.

Warning two: partials isos need not be prone

In an (ordinary) fibration, every isomorphism is Cartesian. Unfortunately, in a latent fibration, every partial isomorphism need not be prone.

Definition

Say a latent fibration $p : \mathbb{E} \longrightarrow \mathbb{B}$ is a **hyperfibration** if it is admissible and if every partial isomorphism in $\mathbb{E}$ is prone.

- The lax versions of the simple and codomain fibrations are not hyperfibrations.
- The strict versions of the simple and codomain fibrations **are** hyperfibrations.
- The assemblies fibration is a hyperfibration.

Hyperconnections

Definition

(Cockett/Garner 2014) A restriction functor $F : \mathbb{E} \longrightarrow \mathbb{B}$ is a **hyperconnection** if for each $X \in \mathbb{E}$, the restriction of F to the restriction idempotents $\mathcal{O}(X)$ of X is an isomorphism; that is,

$$F|_{\mathcal{O}(X)} : \mathcal{O}(X) \longrightarrow \mathcal{O}(FX) \text{ is invertible.}$$

For example, the codomain functor $\mathbb{C}^{\rightarrow} \longrightarrow \mathbb{C}$ is a hyperconnection: if (e', e) is a restriction idempotent on $x : C \longrightarrow C$:

$$\begin{array}{ccc} C & \xrightarrow{e'} & C \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ X & \xrightarrow[e]{} & X \end{array}$$

then since the square commutes and is semi-precise,

$$e' = \overline{e'} = \overline{e'\overline{x}} = \overline{e'x} = \overline{xe},$$

so e' is entirely determined by e.

Hyperconnections and hyperfibrations

Theorem

An admissible latent fibration $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ *is a latent hyperfibration if and only if* p *is a hyperconnection.*

- This shows why the simple strict latent fibration and the strict codomain fibration are hyperfibrations.
- Similarly, it can also be used to show that their lax versions are *not* latent hyperfibrations.
- As we shall see, we can build duals to latent hyperfibrations.

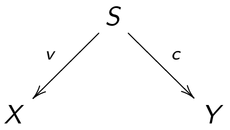
Overview of latent fibrations

In summation:

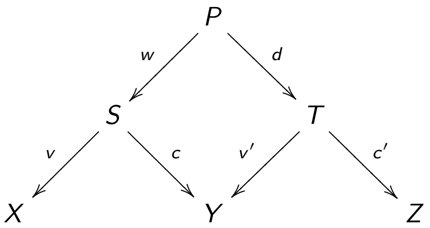
- There is a restriction version of fibrations, with many examples, including some particular to restriction categories.
- Many results for fibrations hold for latent fibrations.
- There is an indexed version of latent fibrations, but it is complex.
- There are strengthenings of the notion of latent fibration which have useful properties.

Idea of the dual fibration

Following the idea for how to define the dual fibration, we would hope that given a latent fibration $p : \mathbb{E} \longrightarrow \mathbb{B}$, we define the dual $\mathbb{E}^*$ to have objects those of $\mathbb{E}$, arrows spans

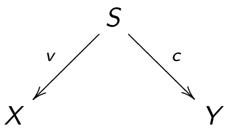


(with v *sub*vertical and c prone) and composition by (latent?) pullback

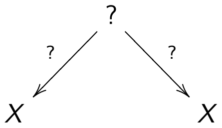


Restriction structure?

But how can this be a restriction category? Given



its restriction $\overline{(v, c)}$ would have to be a span of the form



which doesn't affect the original span when composed with it.
Note: $\overline{v} : C \longrightarrow C$ not of the right type.

Restriction structure

However, suppose $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a latent *hyper*fibration. Then given a span

$$X \xleftarrow{v} S \xrightarrow{c} Y$$

with v subvertical, $p(v)$ is a restriction idempotent in $\mathbb{B}$

$$\mathsf{p}(X) = \mathsf{p}(S) \xrightarrow{\mathsf{p}(v)} \mathsf{p}(S) = \mathsf{p}(X)$$

so since p is a hyperconnection, there is a corresponding unique restriction idempotent on X

$$X \xrightarrow{\hat{v}} X$$

so we can define $\overline{(v, c)}$ to be the span

$$X \xleftarrow{\hat{v}} X \xrightarrow{\hat{v}} X$$

Cospan of a subvertical/prone pair

Moreover, hyperfibrations do have the necessary latent pullbacks:

Lemma

Suppose $p : \mathbb{E} \longrightarrow \mathbb{B}$ is a latent hyperfibration. Then every cospan $c : B \longrightarrow C$, $v : A \longrightarrow C$ with v subvertical and c prone has a corresponding latent pullback:

$$\begin{array}{ccc} U & \xrightarrow{c'} & A \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ B & \xrightarrow[c]{} & C \end{array}$$

where w is subvertical and c' is prone.

Dual of latent hyperfibration

Theorem

If $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ *is a latent hyperfibration, then there is a latent hyperfibration* $\mathsf{p}^* : \mathbb{E}^* \longrightarrow \mathbb{B}^*$ *where*

- $\mathbb{E}^*$ *has the same objects as* $\mathbb{E}$*;*
- *arrows* $(v, c) : X \longrightarrow Y$ *are equivalence classes of spans*

$$X \xleftarrow{\;v\;} S \xrightarrow{\;c\;} Y$$

  with v *subvertical and* c *prone (equivalence is up to vertical partial isomorphism);*
- *restriction is defined as above:* $\overline{(v, c)} := (\hat{v}, \hat{v})$*;*
- *composition is by latent pullback.*

Examples

- The strict simple latent fibration is a hyperfibration, and so has a dual, with maps

$$(A, X) \xrightarrow{(f, f')} (B, Y)$$

of the form

$$A \xrightarrow{f} B, \; A \times Y \xrightarrow{f'} X \text{ such that } \overline{f'} = \overline{f} \times 1$$

One could think of think of this as a "category of partial lenses".

- The strict codomain fibration is a hyperfibration, and so has a dual, which one could think of as a "category of partial dependent lenses".
- The assemblies fibration is a hyperfibration, and so has a dual (though not sure of a good description of it yet).

Concluding thoughts

We began this story simply trying to understand how the dual to the simple fibration worked in restriction categories. This has led to quite a journey, and there's still lots more to understand and do:

- Are categories of partial lenses practically useful? I imagine so, but this needs testing...
- Other examples of latent hyperfibrations and their duals?
- Need a better theoretical understanding of the indexed version of things (in particular, what is the indexed version of this dual?)
- How do latent fibrations relate to fibrations in the various 2-categories of restriction categories?

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