The dual fibration, part one: total case

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Overview

Today I'll discuss a construction, originally due to ~~Kock~~ Bénabou, of how to build the *dual* fibration to a given fibration, and include some motivation about why this construction is interesting.

- Next time, we'll see how to generalize these ideas to the setting of restriction categories (and why one might want to do this).

The derivative

Recall that for any smooth map $f : U \subseteq \mathbb{R}^n \to V \subseteq R^m$, the derivative of f can be viewed as a map

$$D[f] : U \times R^n \longrightarrow R^m,$$

where $D[f](x, v) := J(f)(x) \cdot v$, the Jacobian of f at x in the direction v.

- This operation satisfies various rules, including the chain rule:

$$U \times \mathbb{R}^n \xrightarrow{D[fg]} \mathbb{R}^k =$$

$$U \times \mathbb{R}^n \xrightarrow{\langle \pi_0 f, D[f] \rangle} V \times \mathbb{R}^m \xrightarrow{D[g]} \mathbb{R}^k$$

- This can be understood as saying that D is a functor from the category **sm** of smooth functions to the simple fibration over **sm**.

The simple fibration

Definition

For any category $\mathbb{C}$ with binary products, the **simple fibration over** $\mathbb{C}$, $\mathbb{C}[\mathbb{C}]$, is the category with:

- an object is a pair of objects (A, X) from $\mathbb{C}$;
- an arrow from (A, X) to (B, Y) is a pair of arrows (f, g) with

$$A \xrightarrow{f} B \text{ and } A \times X \xrightarrow{g} Y$$

- composite of $(A, X) \xrightarrow{(f,g)} (B, Y)$ with $(B, Y) \xrightarrow{(f',g')} (C, Z)$ is

$$A \times X \xrightarrow{\langle \pi_0 f, f' \rangle} B \times Y \xrightarrow{g'} Z$$

Thus the derivative gives a functor from **sm** to **sm**[**sm**]:

- Send $U \subseteq \mathbb{R}^n$ to $(U, \mathbb{R}^n)$;
- Send $f : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ to the pair $(f, D[f])$.

The reverse derivative

There has been much recent interest in the *reverse* derivative of a smooth map $f : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq R^m$.

- It produces a map
$$R[f] : U \times R^m \longrightarrow R^n$$
defined by $R[f](u, w) := [J(f)(x)]^T \cdot w$.
- It satisfies the "reverse" chain rule:
$$U \times R^k \xrightarrow{R[fg]} R^n =$$
$$U \times \mathbb{R}^k \xrightarrow{\langle \pi_0, (f \times 1)R[g] \rangle} U \times \mathbb{R}^m \xrightarrow{R[f]} \mathbb{R}^n$$
- This can be understood as saying that R is a functor from **sm** to the *dual* simple fibration over **sm**.

The dual simple fibration

Definition

For any category $\mathbb{C}$ with binary products, the **dual simple fibration over** $\mathbb{C}$, $\mathbb{C}[\mathbb{C}]^*$, is the category with:

- an object is a pair of objects (A, X) from $\mathbb{C}$;
- an arrow from (A, X) to (B, Y) is a pair of arrows (f, g) with

$$A \xrightarrow{f} B \text{ and } A \times Y \xrightarrow{g} X$$

  (*Note the reversal in direction!*)
- composite of $(A, X) \xrightarrow{(f,g)} (B, Y)$ with $(B, Y) \xrightarrow{(f',g')} (C, Z)$ is

$$A \times Z \xrightarrow{\langle \pi_0, (f \times 1)g' \rangle} A \times Y \xrightarrow{f'} X.$$

  (*A bit strange!*)

Thus the reverse derivative gives a functor from **sm** to **sm**[**sm**]*.

The dual simple fibration as "lenses"

(Spivak, 2019) calls an arrow (f, g) in $\mathbb{C}[\mathbb{C}]^*$ a **lens**.

- Typically, a (state-based) lens involves arrows

$$\text{get} : A \longrightarrow B, \ \ \text{put} : A \times B \longrightarrow A$$

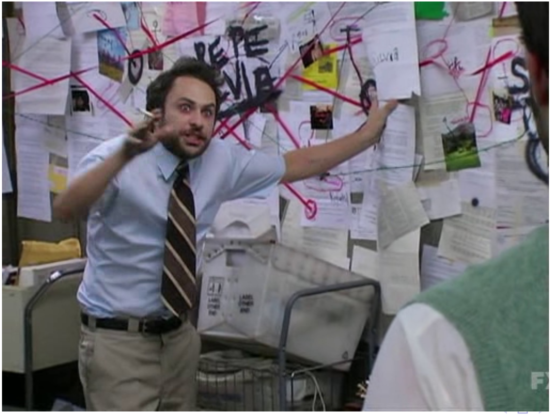
  satisfying three equations.
- The rough idea is that "get" is a view of a database A, and the "put" allows one to make updates to A if one updates the view B.
- A lens in this sense is a morphism

$$(\text{get}, \text{put}) : (A, A) \longrightarrow (B, B)$$

  in $\mathbb{C}[\mathbb{C}]^*$.
- However, the more general morphisms also appear in Haskell as "polymorphic" lenses.

"Lenses" are everywhere

- Moreover, (Hedges, 2018) identifies many other instances of such lenses: backpropagation, learners, open games, the dialectica interpretation, Moore machines...
- Hedges writes "I spent most of the Applied Category Theory workshop in Leiden telling everybody who would listen about all these connections, rather like this:"



The simple fibration vs. the dual simple fibration

To recap:

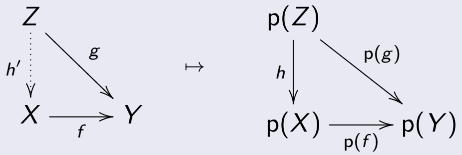
- In $\mathbb{C}[\mathbb{C}]$, an arrow $(f, g) : (A, X) \longrightarrow (B, Y)$ has
$$f : A \longrightarrow B, g : A \times X \longrightarrow Y.$$
(*Think: ordinary derivative*).
- In $\mathbb{C}[\mathbb{C}]^*$, an arrow $(f, g) : (A, X) \longrightarrow (B, Y)$ has
$$f : A \longrightarrow B, g : A \times Y \longrightarrow X.$$
(*Think: reverse derivatives, lenses*).

Note: $\mathbb{C}[\mathbb{C}]^*$ is *not* the opposite category of $\mathbb{C}[\mathbb{C}]$! It is, however, an instance of a more general construction known as the **dual fibration** of a fibration.

Fibration definition

Definition

For a functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$, a **Cartesian arrow** is a map $f : X \longrightarrow Y$ in $\mathbb{E}$ so that for any $g : Z \longrightarrow Y$ in $\mathbb{E}$ and $h : \mathrm{p}(Z) \longrightarrow \mathrm{p}(X)$ in $\mathbb{B}$ so that $h\mathrm{p}(f) = \mathrm{p}(g)$, there is a unique $h' : Z \longrightarrow X$ so that $\mathrm{p}(h') = h$ and $h'f = g$:



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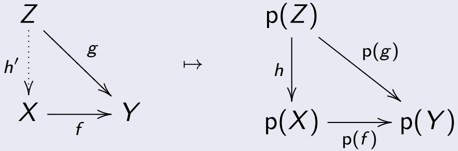


Definition

A functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is said to be a **fibration** if for any $\alpha : A \longrightarrow B$ in $\mathbb{B}$, and any Y such that $\mathrm{p}(Y) = B$, there is a Cartesian arrow

$$\alpha^* : X \longrightarrow Y$$

over α, i.e., such that $\mathrm{p}(\alpha^*) = \alpha$.

The simple fibration as a fibration

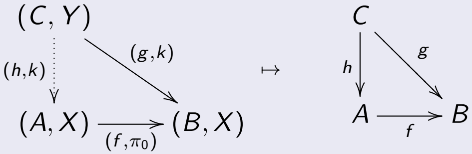
The obvious projection $\mathbb{C}[\mathbb{C}] \longrightarrow \mathbb{C}$ is a fibration.

Proof.

Given $f : A \longrightarrow B$ in $\mathbb{C}$ and (B, X) over B, define

$$f^* : (A, X) \longrightarrow (B, X) \text{ by } f^* = (f, \pi_0).$$

Indeed,



□

Fibration examples

There are *many* examples of fibrations. We'll focus on a few:

1. The simple fibration is a fibration.
2. The dual simple fibration is a fibration.

For any category $\mathbb{C}$, let $\mathrm{Arr}(\mathbb{C})$ be the arrow category: objects are arrows of $\mathbb{C}$, and morphisms are commutative squares

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[g]{} & Y \end{array}$$

3. For any $\mathbb{C}$, the domain functor $\mathrm{Arr}(\mathbb{C}) \longrightarrow \mathbb{C}$ is a fibration.
4. For any $\mathbb{C}$ with pullbacks, the codomain functor $\mathrm{Arr}(\mathbb{C}) \longrightarrow \mathbb{C}$ is a fibration.
5. For any $\mathbb{C}$ with a display system (pullback-closed system of maps), the subcategory of the arrow category consisting of the maps in the display system is a fibration over $\mathbb{C}$.

The indexed category of a fibration

Let $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ be a fibration with chosen Cartesian liftings (ie., "cloven").

- Say an arrow $f : X \longrightarrow Y$ in $\mathbb{E}$ is **vertical** if $\mathsf{p}(f)$ is an identity.
- For $A \in \mathbb{B}$, there is a category $\mathsf{p}^{-1}(A)$ (the "fibre over A") whose objects are the objects in $\mathbb{E}$ over A and whose arrows are the vertical arrows over 1_A.
- Each $\alpha : A \longrightarrow B$ in $\mathbb{B}$ gives a functor

$$\alpha^* : \mathsf{p}^{-1}(B) \longrightarrow \mathsf{p}^{-1}(A).$$

- All together, one gets a pseudofunctor

$$\mathbb{B}^{op} \longrightarrow \mathrm{CAT}$$

(A "$\mathbb{B}$-indexed category")

The indexed category of the simple fibration

For example, for the simple fibration $\mathrm{p} : \mathbb{C}[\mathbb{C}] \longrightarrow \mathbb{C}$ with an $A \in \mathbb{C}$:

- An object of $\mathrm{p}^{-1}(A)$ is a pair (A, X).
- So an object is really just an object X of $\mathbb{C}$.
- An arrow $(f, g) : (A, X) \longrightarrow (A, Y)$ must have $f = 1_A$.
- So an arrow from X to Y is just an arrow $g : A \times X \longrightarrow Y$.

Indexed category vs. fibrations

Conversely, given any pseudofunctor

$$F : \mathbb{B}^{op} \longrightarrow \mathrm{CAT}$$

one can build a category $\mathrm{Gro}(F)$, called the "category of elements" or "Grothendieck construction" which is a fibration over $\mathbb{B}$.

- This gives an equivalence

$$((\text{Cloven}) \text{ Fibrations over } \mathbb{B}) \cong (\text{pseudofunctors } \mathbb{B}^{op} \longrightarrow \mathrm{CAT})$$

- Both sides of this equivalence give important perspectives!

The dual indexed category

The "dual" we want to do is take the opposite in each fibre.

- With the indexed category point of view, it is easy to define this!
- Simply post-compose the indexed category F with the (covariant!) functor $()^{op} : \mathrm{CAT} \longrightarrow \mathrm{CAT}$:

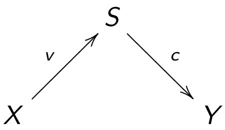
$$\mathbb{B}^{op} \xrightarrow{F} \mathrm{CAT} \xrightarrow{()^{op}} \mathrm{CAT}$$

- Doing this to the simple fibration gives the dual simple fibration.
- But it will be (very) helpful to have a direct description of this in terms of the original fibration.

The dual fibration

This idea is originally due to (Bénabou, 1975). Let $p : \mathbb{E} \longrightarrow \mathbb{B}$ be a fibration.

- One can show that any arrow $f : X \longrightarrow Y$ in $\mathbb{E}$ uniquely factors as a vertical v followed by a cartesian c:



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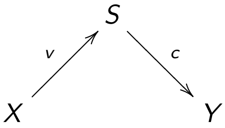
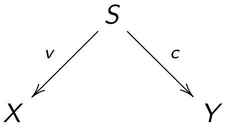


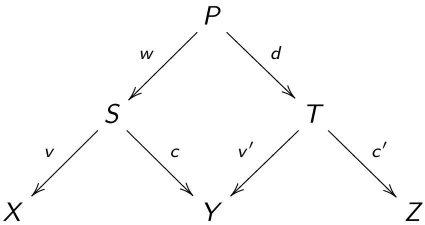
- So to dualize we just reverse the direction of the vertical arrow!
- Define E^* to have the same objects as $\mathbb{E}$, but an arrow $X \longrightarrow Y$ consists of a vertical $v : S \longrightarrow X$, $c : S \longrightarrow Y$:



The dual fibration continued

Wait a minute! Does this actually work?!?

- Fortunately, the pullback of a vertical and cartesian with the same codomain does always exist.
- Thus,we can define composition by pullback:



- One can show that the resulting functor $\mathbb{E}^* \longrightarrow \mathbb{B}$ is again a fibration, and the fibres of $\mathbb{E}^*$ are the opposites of the fibres of $\mathbb{E}$.

Dual fibration examples

Some examples:

1. The dual fibration of the simple fibration is the dual simple fibration ("lenses").
2. The dual fibration of the codomain fibration $\mathrm{Arr}(\mathbb{C}) \longrightarrow \mathbb{C}$ is the "category of dependant lenses": an arrow from $(a : X \longrightarrow A)$ to $(b : Y \longrightarrow B)$ consists of

   $$f : A \longrightarrow B \text{ ("get") and}$$

   $$g : A \times_{f,a} Y \longrightarrow X \text{ ("put")}$$

   where $A \times_{f,a} Y$ is the pullback of f along a:

   $$\begin{array}{ccc} A \times_{f,a} Y & \xrightarrow{\pi_1} & D \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

Dual fibration examples continued

3. In the dual of the display fibration in smooth manifolds (display maps being submersions) a map of the form

$$\begin{array}{ccc} TS & & A \times B \\ \downarrow{\scriptstyle p} & \longrightarrow & \downarrow{\scriptstyle \pi_1} \\ S & & B \end{array}$$

   consists of maps $f : S \longrightarrow B$, $g : S \times A \longrightarrow TS$; these are "open dynamical systems" (see Spivak, 2019).

4. The dual fibration of the *domain* fibration $\mathrm{Arr}(\mathbb{C}) \longrightarrow \mathbb{C}$ is the "twisted arrow category of $\mathbb{C}$": objects are arrows of $\mathbb{C}$, and an arrow from $(a : X \longrightarrow A)$ to $(b : Y \longrightarrow B)$ is a factorization of a through b:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xleftarrow[g]{} & B \end{array}$$

A restriction version of the simple fibration

Our real goal, however, is to look at partial/restriction versions of all this.

- Again, one motivation comes from derivatives.
- If $f : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq R^m$ is only defined on some open subset of U, then its derivative

$$D[f] : U \times \mathbb{R}^n \longrightarrow R^m$$

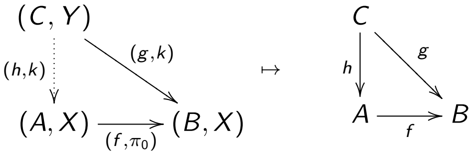
  is defined exactly where f is.
- That is, in terms of restriction categories, $\overline{D[f]} = \overline{f} \times 1$.
- Thus, if $\mathbb{C}$ is a restriction category, a natural restriction version of $\mathbb{C}[\mathbb{C}]$ has maps $(f, g) : (A, X) \longrightarrow (B, Y)$ as before

$$f : A \longrightarrow B, g : A \times X \longrightarrow Y$$

  but now such that $\overline{g} = \overline{f} \times 1$.
- This makes sense from the perspective of "partial lenses" as well.

The restriction simple fibration is not a fibration

Unfortunately, this is not a fibration over $\mathbb{C}$!



- We need $\overline{k} = \overline{h} \times 1$, but only have $\overline{k} = \overline{g} \times 1$.
- There is no reason why $\overline{g} = \overline{h}$.

Towards latent fibrations

Next time, we'll begin by looking at *latent* fibrations, originally due to (Nester, 2017).

- A latent fibration will only ask for liftings of "precise" triangles in the base: triangles where $\overline{g} = \overline{h}$.
- Of course, it's still not clear that we'll even get a dual version of this, as the opposite of a restriction category is not usually a restriction category...
- Nevertheless, we'll see that in many cases of interest, there is a dual fibration of a latent fibration, including for the simple latent fibration described above.

References

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