Latent Fibrations (part II)

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From last time...

Last time, we looked at the following:

- Discussed how the simple fibration and the codomain fibration appear naturally when talking about the derivative and the tangent bundle.
- Saw that a natural restriction version of the simple fibration is *not* a fibration (a similar issue occurs with the codomain fibration).
- Briefly discussed the dual fibration, and why in general a restriction version of it may not exist (though it would be useful to have one for a reverse derivative and/or a cotangent bundle on categories of smooth *partial* maps).
- Introduced latent fibrations: a modification of the fibration notion for restriction categories.

Overview for today

Today: more on latent fibrations:

- How they can be seen as a fibration in a certain 2-category
- Properties that general latent fibrations do and do not enjoy
- Types of latent fibrations and what properties they enjoy
- The dual fibration of the nicest kind of latent fibrations: hyperfibrations.

Restriction categories

Definition

A **restriction category** (Cockett/Lack 2002) is a category $\mathbb{C}$ equipped with an operation which takes a map $f : A \longrightarrow B$ in $\mathbb{C}$ and gives a map $\overline{f} : A \longrightarrow A$ which satisfies four axioms:

[R.1] $\overline{f} f = f$ [R.2] $\overline{f}\, \overline{g} = \overline{g}\, \overline{f}$ [R.3] $\overline{f}\, \overline{g} = \overline{\overline{f} g}$ [R.4] $f \overline{g} = \overline{fg} f$

Examples:

- Sets and *partial* functions
- Smooth *partial* functions between $\mathbb{R}^n$'s
- Smooth *partial* functions between smooth manifolds
- More generally, any partial map category is a restriction category (built by assuming a pullback-stable collection of monics)
- Any category is a restriction category in which for each $f : A \longrightarrow B$ one defines $\overline{f} = 1_A$.

Restriction categories basics

Definition

In a restriction category $\mathbb{C}$:

- A map $f : A \longrightarrow B$ is said to be **total** if $\overline{f} = 1_A$.
- A **partial inverse** of a map $f : A \longrightarrow B$ is a map $g : B \longrightarrow A$ such that $fg = \overline{f}$ and $gf = \overline{g}$. For example, $\frac{1}{x-1} : \mathbb{R} \longrightarrow \mathbb{R}$ does not have an inverse but it does have a *partial* inverse $\frac{1}{x} + 1$.
- We can define a **partial order** on hom-sets of a restriction category: for maps $f, g : A \longrightarrow B$, $f \leq g$ if $\overline{f}g = f$. (g is defined wherever f is, and is equal to f on f's domain). For example,

$$\frac{x^2 - 1}{x - 1} \leq x + 1$$

Restriction idempotents

In a restriction category $\mathbb{C}$:

- A **restriction idempotent** is a map $e : A \longrightarrow A$ such that $\overline{e} = e$.
- For each f, $\overline{f}$ is a restriction idempotent (its "domain of definition").
- You can split restriction idempotents to form a restriction category $\text{Split}_r(\mathbb{C})$, thus "making the domains of definition actual objects".

Restriction functors and semifunctors

If $\mathbb{C}$ and $\mathbb{D}$ are restriction categories:

Definition

A **restriction functor** $F : \mathbb{C} \longrightarrow \mathbb{D}$ is a functor that preserves restrictions, ie., for any f in $\mathbb{C}$, $F(\overline{f}) = \overline{F(f)}$.

Definition

A **restriction semifunctor** $F : \mathbb{C} \longrightarrow \mathbb{D}$ is a map of objects and arrows that preserves composition and restriction (but not necessarily identities).

Note, however, that while restriction semifunctors do not necessarily preserve identities, they do at least send identities to restriction idempotents:

$$F(1_A) = F(\overline{1_A}) = \overline{F(1_A)},$$

so $F(1_A)$ is still a restriction idempotent.

Restriction transformations

Definition

If $F, G : \mathbb{C} \longrightarrow \mathbb{D}$ are restriction semifunctors, a **restriction transformation** $\alpha : F \Rightarrow G$ has components $\alpha_C : FC \longrightarrow GC$ such that the naturality squares commute: for any $f : A \longrightarrow B$,

$$F(f)\alpha_B = \alpha_A G(f)$$

but also, for each object C,

$$\overline{\alpha_C} = F(1_C).$$

(Note that if the restriction semifunctors are actual functors, this forces each component to be total.)

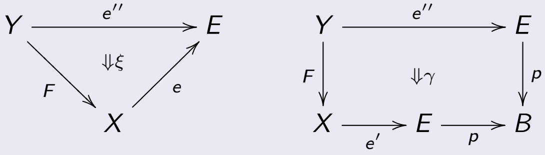
Definition

Let $\mathbf{rCat}_s$ be the 2-category of restriction categories, restriction semifunctors, and restriction transformations.

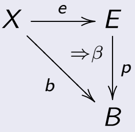
Fibrations in a 2-category

Definition (Street 1974)

For a 1-cell $p : E \longrightarrow B$, in a 2-category, a 2-cell $\alpha : e' \Rightarrow e$ is p-**Cartesian** if for all $F\colon Y \longrightarrow X$ and all 2-cells



such that $p\xi = p\alpha F \cdot \gamma$, there is a unique 2-cell $\zeta\colon e'' \Rightarrow e'F$ such that $\xi = \alpha F \cdot \zeta$ and $p\zeta = \gamma$. $p\colon E \longrightarrow B$ is a **fibration** if every 2-cell

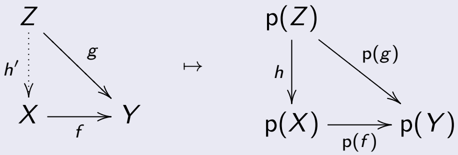


has a p-Cartesian lift: $\alpha\colon e' \Rightarrow e$ Cartesian so that $p\alpha = \beta$.

Latent fibration definition (slightly generalized)

Definition

For a restriction **semi**functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$, a prone arrow is a map $f : X \longrightarrow Y$ in $\mathbb{E}$ so that for any $g : Z \longrightarrow Y$ in $\mathbb{E}$ and $h : \mathrm{p}(Z) \longrightarrow \mathrm{p}(X)$ in $\mathbb{B}$ so that $h\mathrm{p}(f) = \mathrm{p}(g)$ and $\overline{h} = \overline{\mathrm{p}(g)}$ there is a unique $h' : Z \longrightarrow X$ so that $\mathrm{p}(h') = h$, $h'f = g$ and $\overline{h'} = \overline{g}$:



Definition

A restriction **semi**functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a latent fibration if every $\alpha : A \longrightarrow B$ in $\mathbb{B}$ and Y over B **such that** $\alpha = \alpha\mathrm{p}(1_Y)$ there is a prone arrow over α.

Fibrations in $\mathbf{rCat}_s$

- Fibrations in the 2-category $\mathbf{rCat}_s$ are precisely latent fibrations as defined on the previous slide.
- This is not a completely straightforward proof...it does take a fair bit of work translating between the two notions.
- In most of our examples, the latent fibrations will have the projection functor $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ be a restriction functor (not a semi-functor).
- These are not the same as fibrations in the 2-category of restriction categories, restriction functors, and transformations: it is important that the universal property is with respect to restriction *semi*functors, even if the functor itself is not semi.

Examples

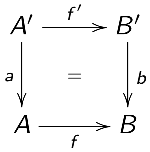
1. If $\mathbb{X}$ has latent products, the **strict simple latent fibration** $\mathbb{X}[\mathbb{X}]$ has maps $(f, f') : (A, A') \longrightarrow (B, B')$ with

$$f : A \longrightarrow B, f' : A \times A' \longrightarrow B'$$

such that $\overline{f'} = \overline{f} \times 1$.

2. The **lax simple latent fibration** $\mathbb{X}(\mathbb{X})$ is as above, except $\overline{f'} \leq \overline{f} \times 1$.

3. For any restriction category with latent pullbacks, the **strict codomain latent fibration** $\mathbb{X}^{\rightarrow}$ has objects maps $a : A' \longrightarrow A$; and maps commuting squares



such that $f'\overline{b} = f'$.

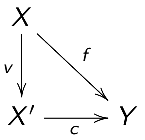
4. The **lax codomain latent fibration** $\mathbb{X}^{\rightsquigarrow}$ is as above but with $f'b \leq af$ (as opposed to equality).

More examples

1. For any functor $F: \mathbb{X} \longrightarrow \mathsf{Set}$ one may form the category of elements as an (ordinary) discrete fibration $\partial_F : \mathsf{Elt}(F) \longrightarrow \mathbb{X}$. If $\mathbb{X}$ is a restriction category then $\mathsf{Elt}(F)$ is also a restriction category and ∂_F is a latent fibration.
2. The category of restriction idempotents in $\mathbb{X}$, $\mathcal{O}(\mathbb{X})$ is a latent fibration over $\mathbb{X}$: it has objects pairs (X, e) where e is a restriction idempotent on X and maps $f : (X, e) \longrightarrow (X', e')$ such that $e \leq \overline{fe'}$.
3. The restriction-idempotent splitting $\mathsf{Split}_r(\mathbb{X})$ of $\mathbb{X}$ is a latent fibration over $\mathbb{X}$ (which is a genuine semifunctor).
4. For any restriction category $\mathbb{Y}$, the projection $\mathbb{X} \times \mathbb{Y} \longrightarrow \mathbb{X}$ is a latent fibration.
5. Nester defined a latent fibration of *assemblies* (see his thesis)
6. Any ordinary fibration is a latent fibration (with respect to the trivial restriction structures).

Some basic results about latent fibrations

- Identities and isomorphisms are prone.
- The composite of two prones maps is prone.
- Latent fibrations are closed under composition and pullback.
- The correct generalization of vertical maps are **subvertical** maps: maps $v : X \longrightarrow X'$ such that $p(v)$ is a *restriction idempotent* (not necessarily an identity).
- If $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a latent fibration, any map $f : X \longrightarrow Y$ in $\mathbb{E}$ uniquely factors as a **sub**vertical v followed by a prone c:



such that $\overline{f} = \overline{v}$ and $\overline{\mathrm{p}(c)} = \mathrm{p}(v) = \overline{\mathrm{p}(v)}$.

Indexed restriction categories

Is there an indexed version of a latent fibration?

- The correct maps in the fibre over $B \in \mathbb{B}$ is the collection of **sub**vertical maps over B.
- These give restriction categories $\mathrm{p}^{(-1)}(B)$, and we get a pseudofunctor
$$p^{(-1)} : \mathbb{B}^{\mathrm{op}} \longrightarrow \mathbf{rCat}_s.$$
- However, to go back (from one of these objects to a latent fibration) we need additional data related to this pseudofunctor, so the "indexed" version of latent fibrations are not quite as nice as the ordinary case.

Some basic **non**-results about latent fibrations

However, some things are not true about arbitrary latent fibrations:

- *Partial* isomorphisms need not be prone.
- Restriction idempotents need not be prone.
- The (latent) pullback of a subvertical and a prone need not exist (recall that the pullback of a vertical and Cartesian does always exist for an ordinary fibration).
- The restriction-idempotent splitting of a latent fibration need not be a latent fibration.
- The dual of a latent fibration need not exist (as noted last time, this is not surprising since the dual of a restriction category is usually not a restriction category).

Separated fibrations

Definition

A restriction semifunctor $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is said to be **separated** if for any restriction idempotents e, e' on X in $\mathbb{E}$, $\mathsf{p}(e) = \mathsf{p}(e')$ implies $e = e'$.

Some nice results follow from this assumption:

Theorem

For any restriction semifunctor $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$, the following are equivalent:

1. p *is separated;*
2. *all partial isomorphisms in* $\mathbb{E}$ *are* p*-prone;*
3. *all restriction idempotents in* $\mathbb{E}$ *are* p*-prone.*

Theorem

If $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a separated latent fibration, there is a latent pullback of any sub-vertical along any prone in $\mathbb{E}$*.*

Admissible latent fibrations

Definition

A restriction semifunctor $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is said to be **admissible** if for any X in $\mathbb{E}$ and any e a restriction idempotent on $\mathsf{p}(X)$ in $\mathbb{B}$ such that $e\mathsf{p}(1_X) = e$, there is a prone restriction idempotent e^* on X over e.

While splitting doesn't preserve general latent fibrations, we do have:

Theorem

If $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ *is a separated latent fibration, then*

$$\mathsf{Split}_r(\mathsf{p}) : \mathsf{Split}_r(\mathbb{E}) \longrightarrow \mathsf{Split}_r(B)$$

is also a separated latent fibration.

Hyperconnected latent fibrations

Definition

A restriction semifunctor $\mathsf{p}\colon \mathbb{E} \longrightarrow \mathbb{B}$ is said to be **hyperconnected** if for any $X \in \mathbb{E}$, the map

$$\mathsf{p}_{|\mathcal{O}(X)}\colon \mathcal{O}(X) \longrightarrow \{d \in \mathcal{O}(\mathsf{p}(X)) : d\mathsf{p}(1_X) = d\}$$

sending $e \in \mathcal{O}(X)$ to $\mathsf{p}(e)$ is an isomorphism.

Theorem

A semifunctor $\mathsf{p}\colon \mathbb{E} \longrightarrow \mathbb{B}$ *is hyperconnected if and only if it is separated and admissible.*

So hyperconnected latent fibrations (which we'll call **hyperfibrations**) are especially nice.

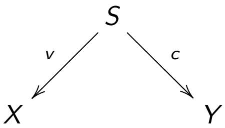
Examples of types

Latent fibration	Admissible	Separated	Hyperfibration
(strict) $\mathbb{X}[\mathbb{X}] \to \mathbb{X}$	✓	✓	✓
(lax) $\mathbb{X}(\mathbb{X}) \to \mathbb{X}$	✓	×	×
(strict) $\mathbb{X}^{\to} \to \mathbb{X}$	✓	✓	✓
(lax) $\mathbb{X}^{\leadsto} \to \mathbb{X}$	✓	×	×
$\mathcal{O}(\mathbb{X}) \to \mathbb{X}$	×	✓	×
$\mathsf{Elt}(F) \to \mathbb{X}$	×	✓	×
$\mathsf{Asm}(F) \to \mathbb{X}$	✓	✓	✓
$\mathsf{Split}_r(\mathbb{X}) \to \mathbb{X}$	✓	✓	✓
$\mathbb{X} \times \mathbb{Y} \to \mathbb{X}$	✓	×	×
Ordinary fibration	✓	✓	✓

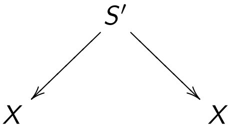
Dual latent fibrations

In general, a latent fibration will not have a dual. But latent *hyper*fibrations will.

- Given a latent fibration $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$, we want to define a new latent fibration with objects as before, but with a map $X \longrightarrow Y$ an (an equivalence class of) a pair (v, c):



  where v is **sub**vertical and c is prone.
- One immediate problem is how to define the restriction of such a map! It would need to be a span



  But both $\overline{v}$ and $\overline{c}$ are endomorphisms on S!

Restriction for the dual

- However, since

$$S \xrightarrow{v} X$$

is subvertical, $\mathsf{p}(v)$ is a restriction idempotent on $\mathsf{p}(S) = \mathsf{p}(X)$.

- So if $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is admissible, then (by definition) there is a prone restriction idempotent $\hat{v} : X \longrightarrow X$ over $\mathsf{p}(v)$, so we can define the restriction of

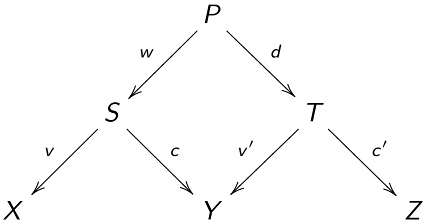
$$X \xleftarrow{v} S \xrightarrow{c} Y$$

to be

$$X \xleftarrow{\hat{v}} X \xrightarrow{\hat{v}} X$$

Composition for the dual

- Recall that composition in the (ordinary) dual fibration is by pullback:



- This works in the restriction setting if $p : \mathbb{E} \longrightarrow \mathbb{B}$ is separated: as mentioned before, for p separated, one can form the latent pullback of a subvertical along a prone.

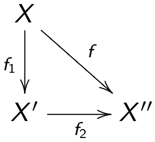
Results about the dual

Thus, for any latent *hyper*fibration $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$, there is an associated dual latent hyperfibration $\mathsf{p}^* : \mathbb{E}^* \longrightarrow \mathbb{B}$.

- There's quite a bit to check, but it all works out, in some cases using other results that follow from the

$$\text{hyperfibration} = \text{separated} + \text{admissible}$$

  condition.
- For example, one another important result that is useful for proving results about the dual is that if $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is separated and



  commutes such that $\overline{f_1} = \overline{f}$ and f and f_2 are both prone, then so is f_1 (this generalizes another well-known result in the ordinary case.)
- One can also prove other nice properties of the dual fibration such as

$$(\mathsf{p}^* : \mathbb{E}^* \longrightarrow B)^* \cong (\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}).$$

Concluding thoughts on latent fibrations

I think there's a surprising amount of nice results and theory around latent fibrations.

- The definition can naturally be seen as a fibration in a particular 2-category.
- There are a variety of examples of latent fibrations.
- General latent fibrations satisfy some, but not all, of the analagous properties of ordinary fibrations.
- There are several natural conditions one could ask of a latent fibration, and each of these produce nice results, including being able to build duals for latent hyperfibrations.

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