Latent Fibrations

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BU CatS seminar
April 13th, 2021

Motivation

- One of my main research interests for the past few years has been categorical structures for differentiation.
- Some of these categorical structures (like cartesian differential categories and tangent categories) can be understood in terms of certain fibrations.
- We've extended Cartesian differential categories and tangent categories to categories of partial maps by adding *restriction structure* to the definitions.
- We'd like to understand how the fibrational point of view works in these partial settings.

Motivation continued...

- But as we'll see, some of the structures we'd like to work with in restriction categories look like fibrations but are not.
- Thus (as is usual with restriction categories) we need to modify/generalize the definition of fibration slightly when working with restriction categories.
- We call this modification/generalization *latent* fibrations, and these are the main subject of the talk(s).
- One of the other main themes will be the construction of the *dual* fibration of a given fibration, when this construction can be performed for latent fibrations, and how this construction relates to *reverse* cartesian differential categories (and, potentially, *cotangent* categories).

Overview

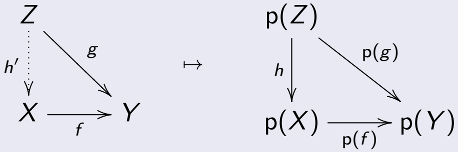
I'll cover the topics in the following order:

1. Review of fibrations, some particular examples we'll be focusing on, and how these relate to derivatives.
2. The construction of the dual fibration (which is not as well-known as it should be!) and how it relates to derivatives.
3. Review of restriction categories.
4. Latent fibrations.
5. Types of latent fibrations, including latent *hyper*fibrations.
6. The dual of a latent hyperfibration.

Fibration definition

Definition

For a functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$, a **Cartesian arrow** is a map $f : X \longrightarrow Y$ in $\mathbb{E}$ so that for any $g : Z \longrightarrow Y$ in $\mathbb{E}$ and $h : \mathrm{p}(Z) \longrightarrow \mathrm{p}(X)$ in $\mathbb{B}$ so that $h\mathrm{p}(f) = \mathrm{p}(g)$[a], there is a unique $h' : Z \longrightarrow X$ so that $\mathrm{p}(h') = h$ and $h'f = g$:



[a](Writing composition in diagrammatic order)

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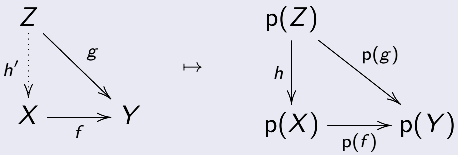

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Definition

A functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is said to be a **fibration** if for any $\alpha : A \longrightarrow B$ in $\mathbb{B}$, and any Y such that $\mathrm{p}(Y) = B$, there is a Cartesian arrow $\alpha^* : X \longrightarrow Y$ over α, i.e., such that $\mathrm{p}(\alpha^*) = \alpha$.

The simple fibration

Definition

For any category $\mathbb{C}$ with binary products, the **simple fibration over** $\mathbb{C}$, $\mathbb{C}[\mathbb{C}]$, is the category with:

- an object is a pair of objects (A, A') from $\mathbb{C}$;
- an arrow from (A, A') to (B, B') is a pair of arrows (f, f') with

$$A \xrightarrow{f} B \text{ and } A \times A' \xrightarrow{f'} B'$$

- the composite of $(f, f') : (A, A') \longrightarrow (B, B')$ with $(g, g') : (B, B') \longrightarrow (C, C')$ is $fg : A \longrightarrow B$ with

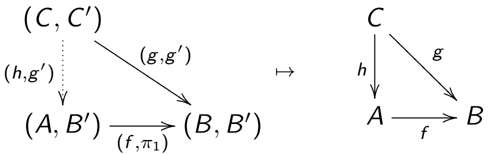
$$A \times A' \xrightarrow{\langle \pi_0 f, f' \rangle} B \times B' \xrightarrow{g'} C'$$

The simple fibration continued

The projection $\mathbb{C}[\mathbb{C}] \longrightarrow \mathbb{C}$ is a fibration: given $f : A \longrightarrow B$ in $\mathbb{C}$ and (B, B') over B, define

$$f^* : (A, B') \longrightarrow (B, B') \text{ by } f^* = (f, \pi_1)$$

This is Cartesian:



The simple fibration and derivatives

Suppose $\mathbb{C}$ is the category of smooth maps between $\mathbb{R}^n$'s.

- Then for any $f : A \longrightarrow B$ in this category, there is a map

$$D[f] : A \times A \longrightarrow B$$

  sending (a, a') to the Jacobian of f at a times the vector a'.
- The chain rule shows that the operation $\mathbb{C} \longrightarrow \mathbb{C}[\mathbb{C}]$ which sends

$$A \mapsto (A, A) \text{ and } f \mapsto (f, D[f])$$

  is a functor (and is a section of the projection $\mathbb{C}[\mathbb{C}] \longrightarrow \mathbb{C}$).

More generally, any Cartesian differential category $\mathbb{C}$ gives a section of its simple fibration.

The codomain fibration

For any category $\mathbb{C}$ with pullbacks, the **codomain fibration over** $\mathbb{C}$ has total category the arrow category of $\mathbb{C}$, $\mathrm{arr}(\mathbb{C})$, so has:

- Objects arrows $a : A' \longrightarrow A$;
- Maps commutative squares

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \end{array}$$

- If we have $f : A \longrightarrow B$ and $b : B' \longrightarrow B$

$$\begin{array}{ccc} & & B' \\ & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \end{array}$$

you can get a Cartesian arrow over f by taking the pullback.

The tangent bundle and the codomain fibration

Suppose $\mathbb{C}$ is the category of smooth manifolds.

- $\mathbb{C}$ does not have all pullbacks, but we can restrict to the full subcategory $\mathrm{arr}_s(\mathbb{C}) \subset \mathrm{arr}(\mathbb{C})$ of submersions (which do have pullbacks along any map into their codomain)
- The tangent bundle then yields a functor $\mathbb{C} \longrightarrow \mathrm{arr}_s(\mathbb{C})$ which sends a map $f : A \longrightarrow B$ to

$$\begin{array}{ccc} TA & \xrightarrow{T(f)} & TB \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_B} \\ A & \xrightarrow{f} & B \end{array}$$

  where p_A is the canonical projection map from the tangent bundle of A to A.
- This point of view helps shed light on the importance of local diffeomorphisms/etale maps: they are precisely the maps $f : A \longrightarrow B$ which get sent by the above functor to a Cartesian arrow.

More generally, any tangent category $\mathbb{C}$ with a display system has similar structure.

The indexed category of a fibration

Recall that if $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a *cloven* fibration (that is, we have chosen Cartesian liftings), then we can build a pseudofunctor

$$p^{-1} : \mathbb{B}^{op} \longrightarrow \mathbf{CAT}$$

as follows:

- Say an arrow $f : X \longrightarrow Y$ in $\mathbb{E}$ is **vertical** if $\mathrm{p}(f)$ is an identity.
- For $A \in \mathbb{B}$, define a category $\mathrm{p}^{-1}(A)$ (the "fibre over A"') whose objects are the objects in $\mathbb{E}$ over A and whose arrows are the vertical arrows over 1_A.
- Each $\alpha : A \longrightarrow B$ in $\mathbb{B}$ gives a functor

$$\alpha^* : \mathrm{p}^{-1}(B) \longrightarrow \mathrm{p}^{-1}(A).$$

This is the *indexed category* associated to the (cloven) fibration.

Indexed category vs. fibrations

Conversely, given any pseudofunctor

$$F : \mathbb{B}^{op} \longrightarrow \mathbf{CAT}$$

one can build a category $\mathrm{El}(F)$, the *category of elements* (or *Grothendieck construction*) which is a cloven fibration over $\mathbb{B}$.

- This gives an equivalence

$$(\text{Cloven fibrations over } \mathbb{B}) \cong (\text{pseudofunctors } \mathbb{B}^{op} \longrightarrow \mathbf{CAT})$$

- Both sides of this equivalence give important perspectives!

The dual indexed category

Given this perspective on fibrations, there is an obvious construction one can perform on an indexed category: take the opposite of each fibre, ie., post-compose the indexed category F with the (covariant!) functor $()^{op} : \mathbf{CAT} \longrightarrow \mathbf{CAT}$:

$$\mathbb{B}^{op} \xrightarrow{F} \mathbf{CAT} \xrightarrow{()^{op}} \mathbf{CAT}$$

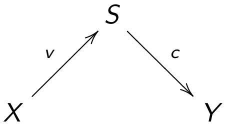
The associated fibration is called the **dual** fibration.

Note: its total category is *not* the opposite of the total category of the original fibration!

The dual fibration

It will be helpful to have a direct description of the dual fibration directly in terms of the original fibration. This idea is originally due to (Bénabou, 1975). Let $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ be a fibration.

- One can show that any arrow $f : X \longrightarrow Y$ in $\mathbb{E}$ uniquely factors as a vertical v followed by a cartesian c:



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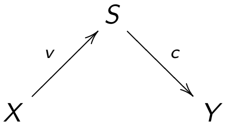
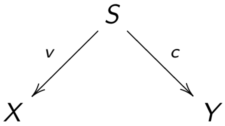


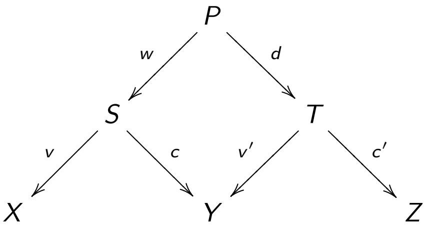
- So to dualize we just reverse the direction of the vertical arrow!
- Define $\mathbb{E}^*$ to have the same objects as $\mathbb{E}$, but an arrow $X \longrightarrow Y$ is (an equivalence class of) a pair (v, c):



The dual fibration continued

How does composition work?

- One can prove that the pullback of a vertical and cartesian with the same codomain always exists.
- Thus,we can define composition by pullback:



- One can show that the resulting functor $\mathbb{E}^* \longrightarrow \mathbb{B}$ is again a fibration, and the fibres of $\mathbb{E}^*$ are the opposites of the fibres of $\mathbb{E}$.

The dual of the simple fibration

Definition

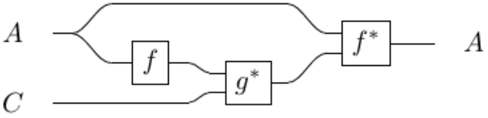
For any category $\mathbb{C}$ with binary products, the dual of the simple fibration over $\mathbb{C}$, $\mathbb{C}[\mathbb{C}]^*$, is the category with:

- an object is a pair of objects (A, A') from $\mathbb{C}$;
- an arrow from (A, A') to (B, B') is a pair of arrows (f, f^*) with

$$A \xrightarrow{f} B \text{ and } A \times B' \xrightarrow{f^*} A'$$

  (*Note the type of f^*!*)
- composite of $(A, A') \xrightarrow{(f, f^*)} (B, B')$ with $(B, B') \xrightarrow{(g, g^*)} (C, C')$ is

$$A \times C' \xrightarrow{\langle \pi_0, (f \times 1) g^* \rangle} A \times B' \xrightarrow{f^*} A'.$$



Dual simple fibration as lenses

- The dual of the simple fibration is sometimes also referred to as the category of **lenses**.
- Lenses as described in database theory form a subcategory of the dual of the simple fibration which is restricted to pairs (A, A).
- In this case, the $f : A \longrightarrow A$ is referred to as the **get** of the lens and the $f^* : A \times B \longrightarrow A$ as the **put** of the lens.
- The pair (f, f^*) are often required to satisfy certain additional equations.
- But the more general arrows in the dual of the simple fibration are also useful in their own right in functional programming and have also been referred to as lenses.
- These maps can also be seen as "generalized learners": f is some action you perform, and f^* is how you update your assumptions.

Reverse derivatives and the dual simple fibration

- Suppose $\mathbb{C}$ is the category of smooth maps between $\mathbb{R}^n$'s.
- For an $f : A \longrightarrow B$, in addition to $D[f] : A \times A \longrightarrow B$, there is also a map called the **reverse** derivative of f

$$R[f] : A \times B \longrightarrow A$$

given by sending (a, b') to the *transpose* of the Jacobian of f at a times the vector b'.

- For example, for $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $D[f] : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by

$$D[f](a, a') = \frac{df}{dx_1}(a_1) \cdot a_1' + \frac{df}{dx_2}(a_2) \cdot a_2'$$

while $R[f] : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2$ is defined by

$$R[f](a, b') = \left[\frac{df}{dx_1}(a_1) \cdot b', \frac{df}{dx_2}(a_2) \cdot b' \right].$$

- This gives a section of the dual of the simple fibration, and more generally any **reverse derivative category** does as well.

Dual of the codomain fibration

- The dual fibration of the codomain fibration $\mathrm{Arr}(\mathbb{C}) \longrightarrow \mathbb{C}$ has been called the *category of dependent lenses* (Spivak, 2020); an arrow from $(a : A' \longrightarrow A)$ to $(b : B' \longrightarrow B)$ consists of

$$f : A \longrightarrow B \text{ (“get”) and}$$

$$f^* : A \times_{f,a} B' \longrightarrow A' \text{ (“put”)}$$

where $A \times_{f,a} Y$ is the pullback of f along a:

$$\begin{array}{ccc} A \times_{f,a} Y & \xrightarrow{\pi_1} & D \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

- In the category of smooth manifolds, the cotangent bundle functor gives a section of the dual of submersion fibration.
- (Currently working on defining “cotangent” categories, these should also give a section of a certain dual fibration...)

Towards partiality

- Thus, one way to view differential/reverse differential/tangent/cotangent structure is as sections to certain fibrations.
- Our goal is to look at versions of these for categories where maps are only partially defined.
- For example, we want to be able to work with the categories of $\mathbb{R}^n$'s or smooth manifolds in which the maps need only be defined on some subset of their domain.
- One nice way to handle partial maps are restriction categories.

Restriction categories

Definition

A **restriction category** (Cockett/Lack 2002) is a category $\mathbb{C}$ equipped with an operation which takes a map $f : A \longrightarrow B$ in $\mathbb{C}$ and gives a map $\overline{f} : A \longrightarrow A$ which satisfies four identities:

[R.1] $\overline{f}f = f$ [R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ [R.3] $\overline{f}\,\overline{g} = \overline{\overline{f}g}$ [R.4] $f\overline{g} = \overline{fg}f$

- The prototypical restriction category is the category of sets and partial maps, where $\overline{f}$ is a *partial identity*: it is defined to be x when $f(x)$ is defined, and undefined otherwise.
- The category whose objects are $\mathbb{R}^n$'s and whose maps are smooth partial functions is similarly a restriction category, as is the category of smooth manifolds and smooth partial functions between manifolds.

Note: an arrow $f : A \longrightarrow B$ need not have a "domain object" on which it is fully defined! The partiality of f is encoded in the arrow $\overline{f}$, not in an object.

Partiality and derivatives

In the category of smooth partial maps between $\mathbb{R}^n$'s:

- If $f : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq R^m$ is only defined on some open subset of U, then its derivative
$$D[f] : U \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
is defined in the first component exactly where f is, but is totally defined in its second component.
- That is, in terms of restriction structure,
$$\overline{D[f]} = \overline{f} \times 1.$$
- Thus a natural choice for maps in a restriction version of the simple fibration would consist of pairs (f, f') such that
$$\overline{f'} = \overline{f} \times 1.$$

Restriction version of the simple fibration

More precisely:

Definition

For a restriction category $\mathbb{C}$ with restriction products, let $\mathbb{C}[\mathbb{C}]$ denote the restriction category with:

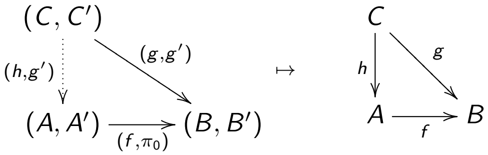
- objects pairs (A, A');
- morphisms $(f, f') : (A, A') \longrightarrow (B, B')$ are

$$A \xrightarrow{f} B,\ A \times X \xrightarrow{f'} Y \textbf{ with } \overline{f'} = \overline{f} \times 1$$

- composition as before: $(f, f') \circ (g, g') := (fg, \langle \pi_0 f, f' \rangle g')$;
- restriction $\overline{(f, f')} := (\overline{f}, \overline{f'}\pi_1 = \overline{\pi_0 f})$.

The restriction simple fibration is not a fibration

Unfortunately, this is not a fibration over $\mathbb{C}$!



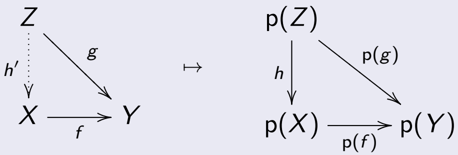
- We need $\overline{g'} = \overline{h} \times 1$, but we only have $\overline{g'} = \overline{g} \times 1$.
- There is no reason why $\overline{g} = \overline{h}$.

So we modify the definition of fibration between restriction categories...

Latent fibration definition

Definition

For a restriction functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$, a **prone arrow** is a map $f : X \longrightarrow Y$ in $\mathbb{E}$ so that for any $g : Z \longrightarrow Y$ in $\mathbb{E}$ and $h : \mathrm{p}(Z) \longrightarrow \mathrm{p}(X)$ in $\mathbb{B}$ so that $h\mathrm{p}(f) = \mathrm{p}(g)$ **and** $\overline{h} = \overline{p(g)}$ there is a unique $h' : Z \longrightarrow X$ so that $\mathrm{p}(h') = h$, $h'f = g$ **and** $\overline{h'} = \overline{g}$:



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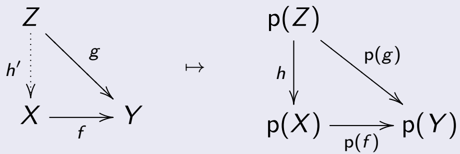

Definition

A restriction functor $\mathrm{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a **latent fibration** if every $\alpha : A \longrightarrow B$ in $\mathbb{B}$ and Y over B there is a prone arrow over α.

Next time...

Presented this way, the definition can be feel a bit ad-hoc. However:

- We'll see next time that there is a nice theoretical explanation for the definition: latent fibrations can be seen as fibrations relative to a certain 2-category of restriction categories.
- Moreover, latent fibrations enjoy many of the nice theoretical properties of ordinary fibrations (partly because of the above fact).
- But some things are subtly different: for example, in general, a latent fibration need not have a dual.
- We'll investigate what structure a latent fibration must have to possess a dual (and this is structure that both the restriction versions of the simple fibration and the codomain fibration enjoy.)

References

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