Introduction to tangent categories

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Tangent Categories and their applications
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Overview

Slogan: tangent categories are a *minimal* categorical setting for differential geometry. In this talk I'll discuss:

- The axioms for a tangent category
- Examples of tangent categories
- Differential objects in a tangent category (the analog of vector spaces)
- Affine connections on an object in a tangent category
- I'll also briefly mention some of the other things you can define in a tangent category, such as vector fields, vector bundles, differential forms...the goal being to eventually translate anything one can find from a differential geometry book into tangent categories!

CDCs are not sufficient for differential geometry

Cartesian differential categories are great! But the category of smooth manifolds is not a (Cartesian) differential category.

- One problem is that the definition of a Cartesian differential category (CDC) has baked into it the assumption that points and tangent vectors are the same type of thing: from $f : A \longrightarrow B$ one asks for

$$D[f] : A \times A \longrightarrow B$$

  but really one of the a's is a "point" and the other a "vector".
- So CDCs already have a hidden assumption that vectors = points.
- For example, a smooth function $f : U \subseteq A \longrightarrow V \subseteq B$ has a derivative, but it is of type

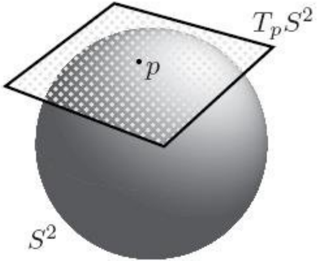
$$D[f] : U \times A \longrightarrow B$$

  not $U \times U \longrightarrow V$.

The tangent bundle

More generally, from any smooth manifold M, one can build its **tangent bundle** TM (another smooth manifold) consisting of "all tangent vectors at all points of M".

- But in general it is **not** true that $TM \cong M \times M$ or even $TM \cong M \times A$ for some vector space A: for example, if M is a sphere - see the hairy ball theorem.



What categorical structure does the tangent bundle possess?

Tangent bundle as a functor

The assignment $M \mapsto TM$ is functorial:

- Given a smooth map $f : M \longrightarrow N$, one can define

  $$T(f) : TM \longrightarrow TN$$

  locally by the formula

  $$(x, v) \mapsto (f(x), D(f)(x, v))$$

  where $D(f)$ is the directional derivative - the operation axiomatized in Cartesian differential categories.
- Functoriality of this operation is precisely the chain rule!

Other basic structure

What other structure does this functor have?

- Every element of TM is "over" some point in M: this gives a natural transformation with components

$$p_M : TM \longrightarrow M$$

- At every point there should be a 0 vector over that point: this gives a natural transformation with components

$$0_M : M \longrightarrow TM$$

- One can add tangent vectors at the same point: for any M the pullback of p_M along itself exists; we write this as T_2M:

$$\begin{array}{ccc} T_2M & \xrightarrow[\pi_0]{} & TM \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p_M} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

and there is a natural transformation with components

$$+ : T_2M \longrightarrow TM.$$

"Tangent spaces have trivial tangent bundle"

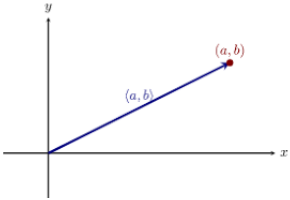
The structure described above is interesting, but doesn't yet capture the full power of the tangent bundle.

- One of its key properties is that each tangent space $T_p(M)$ (the set of all tangent vectors over a fixed point $p \in M$) is a vector space, so that

$$T(T_p(M)) \cong T_p(M) \times T_p(M).$$

  How can we represent this categorically?

- In a vector space V any point of V can also be thought of as a tangent vector at 0 ("draw the vector to that point").



  giving a map $V \longrightarrow TV$.

- Thus there should be a map $TM \longrightarrow T(TM) = T^2M$.

“Tangent spaces have trivial tangent bundle” ctd.

This turns out to be a natural transformation with components

$$\ell : TM \longrightarrow T^2M$$

given locally by the formula

$$(x, v) \mapsto (x, 0, 0, v)$$

- More generally, in a vector space V, given two points v and w, we can turn w into a tangent vector at v by shifting the vector version of w over to v; we can build this map from our ℓ above by the composite

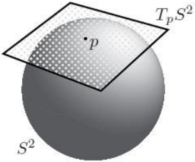
$$T_2M \xrightarrow{\nu := \langle \pi_0 0_M, \pi_1 \ell \rangle T(+)} T^2M$$

which is given locally by

$$(x, v_1, v_2) \mapsto (x, v_1, 0, v_2)$$

Universality of the vertical lift

- This "vertical lift" ℓ and its associated map ν have a special property: they all give elements of T^2M for which $T(p_M)$ is 0: these are known as "vertical vectors".
- Vertical vectors "stay in the plane of the tangent space".



- In fact, all vertical vectors of T^2M arise in this way; in other words, the following diagram is a pullback:

$$\begin{array}{ccc} T_2M & \xrightarrow{\nu} & T^2M \\ {\scriptstyle \pi_0 p_M}\downarrow & & \downarrow{\scriptstyle T(p_M)} \\ M & \xrightarrow[0_M]{} & TM \end{array}$$

Symmetry of mixed partial derivatives

If D is the directional derivative, the symmetry of mixed partial derivatives says that

$$\langle a, b, c, d\rangle D[D[f]] = \langle a, c, b, d\rangle D[D[f]].$$

This gives us a natural transformation

$$c : T^2M \longrightarrow T^2M$$

defined locally by

$$(x, v_1, v_2, w) \mapsto (x, v_2, v_1, w).$$

We now have all the components necessary to define a tangent category!

Tangent category definition

Definition (Rosický 1984, modified Cockett/Cruttwell 2014)

A **tangent category** consists of a category $\mathbb{X}$ with:

- an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of $p_M : TM \longrightarrow M$ along itself exists (and is preserved by each T^m), call this pullback $T_n M$;
- for each $M \in \mathbb{X}$, $p_M : TM \longrightarrow M$ has the structure of a commutative monoid in the slice category $\mathbb{X}/M$, in particular there are natural transformations $+ : T_2 \longrightarrow T$, $0 : 1_{\mathbb{X}} \longrightarrow T$;

Tangent category definition (continued)

Definition

- (canonical flip) there is a natural transformation $c : T^2 \longrightarrow T^2$ which preserves additive bundle structure and satisfies $c^2 = 1$;
- (vertical lift) there is a natural transformation $\ell : T \longrightarrow T^2$ which preserves additive bundle structure and satisfies $\ell c = \ell$;
- various other coherence equations for ℓ and c;
- (universality of vertical lift) "an element of $T^2 M$ which has $T(p) = 0$ is uniquely given by an element of $T_2 M$"; that is, the following is a pullback:

$$\begin{array}{ccc} T_2 M & \xrightarrow{\ \nu\ } & T^2 M \\ {\scriptstyle \pi_0 p_M}\downarrow & & \downarrow{\scriptstyle T(p_M)} \\ M & \xrightarrow[\ 0_M\ ]{} & TM \end{array}$$

where ν is defined as previously.

Examples

- Smooth manifolds with their tangent bundle (the motivating example).
- Convenient manifolds (a certain type of infinite-dimensional manifold) with their "kinematic" tangent bundle.
- The infinitesimally linear objects in a model of synthetic differential geometry (SDG)
- Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.
- The category of C^∞-rings.
- (MacAdam) The category of all small categories with finite limits is a tangent category, where

$$T(\mathbb{X}) = \text{ Beck modules in } \mathbb{X} \text{ (Abelian group objects in } \mathbb{X})$$

There will be more discussion on Wednesday about "tangent infinity-categories" (generalizations of this idea to infinity-categories).

CDCs give tangent categories

Any Cartesian differential category is a tangent category, with

- $TM = M \times M,\ T(f) = \langle \pi_0 f, D[f] \rangle$.
- $p_M(x, v) = x, 0_M(x) = (x, 0), +_M(x, v_1, v_2) = (x, v_1 + v_2)$
- $\ell(x, v) = (x, 0, 0, v)$, and the $\nu(x, v_1, v_2) = (x, v_1, 0, v_2)$.
- $c(x, v_1, v_2, w) = (x, v_2, v_1, w)$.

This includes interesting examples such as the Abelian functor calculus!

Other constructions

- By the previous slide, the coKleisli category of a (monoidal) differential category is a tangent category.
- But also (by Cockett/Lemay/Luchshyn-Wright) the coEilenberg-Moore category of a (monoidal) differential category is a tangent category.
- From a differential restriction category with joins, its "manifold completion" is a tangent category.
- Many other tangent categories can be constructed from a given tangent category $\mathbb{X}$:
  - The collection of objects of $\mathbb{X}$ equipped with a vector field forms a tangent category
  - The collection of objects of $\mathbb{X}$ equipped with a connection forms a tangent category
  - The collection of objects of $\mathbb{X}$ equipped with a flat or torsion-free (or both) connection forms a tangent category

Differential objects and (affine) connections

There are many structures from differential geometry that you can define and work with in an arbitrary tangent category. I'll briefly focus on two in this talk: the analog of vector spaces and of affine connections. Slogan:

- Differential/vector space structure consists of a trivialization of TM
- Affine connection structure consists of a trivialization of T^2M

Differential objects

Definition

A **differential object** in a tangent category consists of a commutative monoid E with a map $\hat{p} : TE \longrightarrow E$ such that

$$E \xleftarrow{\hat{p}} TE \xrightarrow{p_E} E$$

is a product diagram, and such that $\hat{p}$ satisfies various coherences with the tangent structure.

Examples:

- $\mathbb{R}^n$'s in the category of smooth manifolds.
- Convenient vector spaces in the category of convenient manifolds.
- Euclidean R-modules in models of SDG.
- (Beck) Left-additive categories in the tangent structure on small finitely-complete categories.

Tangent spaces are differential objects

Definition

If $p : 1 \longrightarrow M$ is a point of an object in a tangent category, and the pullback

$$\begin{array}{ccc} T_aM & \longrightarrow & TM \\ \downarrow & & \downarrow{\scriptstyle p_M} \\ 1 & \underset{a}{\longrightarrow} & M \end{array}$$

exists, we call T_aM the **tangent space** of M at a.

Theorem

If T_aM exists, then T_aM is a differential object.

The proof uses in an essential way the universality of the vertical lift.

From differential objects to CDCs

If you focus on a collection of differential objects which play nice with products, you get a CDC:

- For $f : A \longrightarrow B$, define $D[f] : A \times A \longrightarrow B$ by

$$D[f] = T(f); \pi_1.$$

- The naturality axioms for the tangent structure give the CDC axioms.

The objects with differential structure are the "nicest" objects in a tangent category.

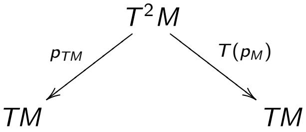
(Affine) connections

Smooth manifolds and their tangent bundles are not enough to do differential *geometry*: for that you need (affine) connections. These are extra structure one can put on a smooth manifold which "gives it a geometry".

- Connections can be overwhelming at first, as there are many equivalent ways of presenting them.
- Here's another one: an affine connection on M is simply a trivilization of the second tangent bundle of M.
- Think of calculus: the second derivative tells you how about how a function curves; the second tangent bundle tells you how a space can be curved.

Projections from the second tangent bundle

The second tangent bundle T^2M always has two projections to the first tangent bundle:



In local co-ordinates/CDCs, these do the following:

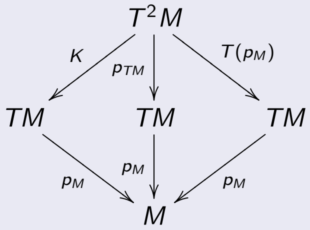
- $p_{TM}(x, v_1, v_2, w) = (x, v_1)$.
- $T(p_M)(x, v_1, v_2, w) = (x, v_2)$.

In a general smooth manifold, there is no canonical way to access the last co-ordinate. This is precisely what a connection provides!

Connections as trivializations

Definition (Lucyshyn-Wright)

An **affine connection** on an object M in a tangent category consists of a map $K : T^2M \longrightarrow TM$ such that



is a limit diagram (and some other coherences hold).

In other words, the second tangent bundle of M can be decomposed as 3 copies of TM (over M). The canonical connection on a differential object

$$K(x, v_1, v_2, w) = (x, w)$$

corresponds to thinking of it as "flat".

Other versions of connections

These can be related to other versions of connections:

- Covariant derivative:

$$\nabla_v u := M \xrightarrow{u} TM \xrightarrow{T(v)} T^2M \xrightarrow{K} TM$$

- Ehresmann connection definition, involving a "lifting"

$$H : T_2M \longrightarrow T^2M$$

  can be defined using the universal property of T^2M given by the connection.

One can also abstractly work with other concepts defined from these connections in any tangent category, such as:

- Curvature
- Torsion
- Geodesics (if your tangent category has the "ability to solve differential equations").

Vector fields

In any tangent category:

Definition

A **vector field** on an object M is a map $V : M \longrightarrow TM$ which is a section of $p_M : TM \longrightarrow M$.

- Vector fields can be added.
- You can define when two vector fields V_1, V_2 on M commute: if

$$V_1 T(V_2) c_M = V_2 T(V_1)$$

- If the tangent category has negatives, one can define a Lie bracket operation for vector fields.
- This uses in an essential way the universality of the vertical lift - you build a vertial element of the second tangent bundle then reduce it to an element of the first tangent bundle!

Other concepts

You can also talk about:

- The analog of vector bundles (see Ben's talk next), called *differential* bundles.
- Connections on such bundles.
- Tangent categories which can "solve differential equations" (sort of like asking for a natural number object!)
- Differential forms
$$T_n M \to E$$
and "sector forms"
$$T^n M \to E$$
for E a differential object.
- Lie groups, groupoids, and algebroids.

Conclusions and references

I've been surprised how many examples of tangent categories have been discovered, and how much one can define in an arbitrary tangent category. And there's lots more yet to be done!

Note that there is also a completely different perspective on tangent categories via Weil algebras: see Richard's talk later today.

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