Differential categories and differential algebra

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- See how differential rings arise from instances of tangent categories.

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- Cartesian differential categories.
- Its close relation, tangent categories.
- See how differential rings arise from instances of tangent categories.
- Propose an alternative for a general "algebraic" framework to discuss tangent and differential structure.

Differential categories

Definition (Blute/Cockett/Seely 2007)

A **Cartesian differential category** consists of a category with finite products and an addition on hom-sets which has, for each map $f : X \longrightarrow Y$, a map $D[f] : X \times X \longrightarrow Y$ satisfying seven axioms (chain rule, D preserves addition, symmetry of partial derivatives, etc.)

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- Think of Df as the Jacobian of f.
- As an example of the axioms, the chain rule is given by asking that $D(gf) = D(g)\langle D(f), f\pi_1\rangle$.

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- Cockett: Cartesian differential structure exists on polynomial functors.
- Blute, Erhard and Tasson showed convenient vector spaces and smooth maps are a Cartesian differential category.

Tangent category definition

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- for each M, the pullback of n copies of $TM \xrightarrow{p_M} M$ along itself exists (and is preserved by T), call this pullback $T_n M$;
- such that for each $M \in \mathbb{X}$, $TM \xrightarrow{p_M} M$ has the structure of a commutative monoid in the slice category $\mathbb{X}/M$, in particular there are natural transformation $T_2 \xrightarrow{+} T$, $I \xrightarrow{0} T$;

Tangent category definition continued...

Definition

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- various other coherence equations for ℓ and c;
- (universality of vertical lift) the map

$$T_2 M \xrightarrow{v := T(+)\langle \ell\pi_1, 0_T\pi_2\rangle} T^2 M$$

is the equalizer of

$$T^2 M \overset{T(p)}{\underset{0pT(p)}{\rightrightarrows}} TM.$$

Analysis examples

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- Any Cartesian differential category $\mathbb{X}$ has associated tangent structure:

$$TM := M \times M, Tf := \langle Df, f\pi_1 \rangle$$

  with:
  - $p := \pi_1$;
  - $T_n(M) := M \times M \ldots \times M$ ($n + 1$ times);
  - $+(\langle x_1, x_2, x_3 \rangle) := \langle x_1 + x_2, x_3 \rangle$, $0(x_1) := \langle 0, x_1 \rangle$;
  - $\ell(\langle x_1, x_2 \rangle) := \langle \langle x_1, 0 \rangle, \langle 0, x_2 \rangle \rangle$;
  - $c(\langle \langle x_1, x_2 \rangle, \langle x_3, x_4 \rangle \rangle) := \langle \langle x_1, x_3 \rangle, \langle x_2, x_4 \rangle \rangle$.
  - $v(\langle x_1, x_2, x_3 \rangle) = \langle \langle x_1, 0 \rangle, \langle x_2, x_3 \rangle \rangle$;

Manifold and algebraic examples

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- Similarly, convenient vector spaces have tangent structure, as do manifolds built on convenient vector spaces.
- The category **cRing** of commutative rings is a tangent category with:

$$TA := A[\epsilon] = \{a + b\epsilon : a, b \in A, \epsilon^2 = 0\},$$

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  and natural transformations as for Cartesian differential categories.
- $\mathbf{cRing}^{\mathrm{op}}$ is a tangent category as, with

$$TA := A^{\mathbb{Z}[\epsilon]} = S(\Omega_A)$$

  (symmetric ring of the Kahler differentials of A).

SDG examples

Recall that a model of SDG consists of a topos with an internal commutative ring R that satisfies the Kock-Lawvere axiom: if we define

$$D := \{d \in R : d^2 = 0\},$$

then the canonical map

$$\phi : R \times R \longrightarrow R^D,$$

given by $\phi(a, b)(d) := a + b \cdot d$, is invertible.

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- The full subcategory of microlinear objects in a model of SDG is a tangent category, with

$$TM := M^D.$$

- Any tangent category with a representable tangent functor produces a model of SDG (uses the universality of vertical lift).

Cartesian Tangent to Cartesian Differential

Every Cartesian tangent category has an associated Cartesian differential category:

Definition

For an object A in a Cartesian tangent category, **differential structure on** A consists of a commutative monoid structure $+ : A \times A \longrightarrow A$, $0 : 1 \longrightarrow A$ on A together with a map $\hat{p} : TA \longrightarrow A$ such that

$$A \xleftarrow{\hat{p}} TA \xrightarrow{p} A$$

is a product diagram and $\hat{p}$ is compatible with $+$ and 0.

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Theorem (Cockett/Cruttwell)

The differential objects in a Cartesian tangent category form a Cartesian differential category, where, for $f : A \longrightarrow B$, we define

$$D(f) := A \times A \cong TA \xrightarrow{T(f)} TB \xrightarrow{\hat{p}} B,$$

Tangent spaces and differential objects

Definition

For a point $1 \xrightarrow{a} M$ of an object of a tangent category, say that **the tangent space at** a **exists** if the pullback of a along p_M exists:

$$\begin{array}{ccc} T_a(M) & \xrightarrow{i} & TM \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle p_M} \\ 1 & \xrightarrow[a]{} & M \end{array}$$

and this pullback is preserved by T.

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Theorem (Cockett/Cruttwell)

Tangent spaces correspond to differential objects.

(The proof uses the universality of vertical lift.)

Differential and tangent categories

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Theorem (Cockett/Cruttwell)

There is an adjunction between small Cartesian differential categories and small Cartesian tangent categories (with appropriate morphisms):

$$\mathbf{cartDiffCats} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{cartTanCats}$$

This provides additional examples of Cartesian differential categories.

Two general constructions of tangent categories

Theorem (Cockett/Cruttwell)

If $(\mathbb{X}, T)$ is a tangent category in which T has a left adjoint L, then $(\mathbb{X}^{op}, L)$ is also a tangent category.

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Theorem (Rosicky)

If $(\mathbb{X}, T)$ is a tangent category, then the category of functors from $\mathbb{X}$ to **set** *which preserve the equalizers and pullbacks of tangent structure is a tangent category, with tangent functor $T_*(F) := FT$.*

Theory: vector fields in a tangent category

Definition

If $(\mathbb{X}, T)$ is a tangent category with an object $X \in \mathbb{X}$, a **vector field on** X is a map $X \xrightarrow{v} TX$ with $pv = 1$.

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- Rosicky showed how to use the universal property of vertical lift to define the Lie bracket of two vector fields in a tangent category with negatives.
- Cockett/Cruttwell showed in any tangent category, T and T^2 are monads; with the Kleisli category of T containing vector fields and their addition.

Differential rings

Definition

A **differential ring** consists of a ring R with a map $\partial : R \longrightarrow R$ such that for $r, s \in R$,

$$\partial(0) = 0, \partial(r+s) = \partial(r) + \partial(s), \text{ and } \partial(rs) = \partial(r)s + r\partial(s).$$

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For example, if M is a smooth manifold and v a vector field on M, the set of smooth functions $C^\infty(M, \mathbb{R})$ can be given the structure of a differential ring.

Ring objects in tangent categories

Definition

Let $(\mathbb{X}, T)$ be a tangent category. A **tangent ring object** is a ring object $R \in \mathbb{X}$ such that:

- R has a map $\hat{p} : TR \longrightarrow R$ making it into a differential object with respect to its addition;
- the map $\hat{p} : TR \longrightarrow R$ is also compatible with the multiplication of R.

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For example, if $\mathbb{X}$ is the coKleisli category of a monoidal differential category, then the monoidal unit I will be a tangent ring object for the associated tangent category. (For example, $\mathbb{R}$ in the standard example or in convenient vector spaces).

Differential rings for tangent categories

Proposition

Suppose $(\mathbb{X}, T)$ *is a tangent category,* R *a tangent ring object, and* v *a vector field on* $X \in \mathbb{X}$*. Then the hom-set* $\mathbb{X}(X, R)$ *can be given the structure of a differential ring, with differential* $\partial_v : \mathbb{X}(X, R) \to \mathbb{X}(X, R)$ *defined by mapping* $f : X \to R$ *to*

$$X \xrightarrow{v} TX \xrightarrow{Tf} TR \xrightarrow{\hat{p}} R$$

This includes potentially new interesting examples of differential rings; for example, any vector field on a convenient manifold gives a differential ring.

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- Using Hadamard's lemma, for a smooth manifold M, differentials on $\mathbf{C}^\infty(M, \mathbb{R})$ **bijectively** correspond to vector fields on M.
- But, for a general Cartesian differential category or tangent category $\mathbb{X}$ there is not necessarily a correspondence between differentials on $\mathbb{X}(X, R)$ and vector fields on X.

General duality between analysis and algebra

For any category $\mathbb{X}$, there is a fundamental adjoint pair:

$$\mathbf{set}^{\mathbb{X}^{\mathrm{op}}} \underset{\mathrm{Spec}}{\overset{\mathrm{Alg}}{\underset{\perp}{\rightleftarrows}}} (\mathbf{set}^{\mathbb{X}})^{\mathrm{op}}$$

where

$$\mathrm{Alg}(F)(X) = \mathbf{set}^{\mathbb{X}^{\mathrm{op}}}(F, \gamma(X)) \text{ and } \mathrm{Spec}(G)(X) = \mathbf{set}^{\mathbb{X}}(G, \gamma'(X)).$$

From Lawvere's paper *Taking categories seriously* (1986), "The conjugacies [above] are the first step toward expressing the duality between space and quantity fundamental to mathematics".

The analysis/algebra pair for the canonical example

- For the standard Cartesian differential category, these categories are fundamentally important:
- the category on the left includes smooth manifolds and diffeological spaces;
- when restricted to those functors which preserve finite products, the category on the right is the category of C^∞ algebras.

Tangent structure lifts to the algebraic category

Moreover, tangent structure from $\mathbb{X}$ lifts to tangent structure on $(\mathbf{set}^{\mathbb{X}})^{\mathsf{op}}$:

- If $\mathbb{X}$ is a Cartesian differential category, then by the earlier result, the subcategory of $\mathbf{set}^{\mathbb{X}}$ whose elements preserve products has tangent structure T_*, where $T_*(F) = FT$.

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- These tangent functors have been implicitly used in the literature on C^∞ algebras, but not explicitly identified as tangent structure.
- Moreover, this works for **any** Cartesian differential category $\mathbb{X}$ (or tangent category, if we restrict to functors which preserve the limits of tangent structure).

Vector fields on $T_!(C^\infty(M))$

For $A \in (\mathbf{set}^{\mathbb{X}})^{\mathsf{op}}$, we have the following equivalences:

$$\begin{aligned} & A \longrightarrow T_! A \in (\mathbf{set}^{\mathbb{X}})^{\mathsf{op}} \\ \Leftrightarrow \quad & T_! A \longrightarrow A \in \mathbf{set}^{\mathbb{X}} \\ \Leftrightarrow \quad & A \longrightarrow T_* A \in \mathbf{set}^{\mathbb{X}} \end{aligned}$$

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In the particular case of $A = C^\infty(M)$ for some smooth manifold M, in particular we get a map

$$C^\infty(M, \mathbb{R}) \to C^\infty(M, T\mathbb{R})$$

$$\Leftrightarrow C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R})$$

which, being a vector field, in particular simply consists of a map

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Furthermore, the naturality of this map with respect to $+, \cdot : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ makes this operation into a derivation on $C^\infty(M, \mathbb{R})$.

Tangent functors on diffeological spaces

- This means that for a general “space” $X \in \mathbf{set}^{\mathbb{X}^{op}}$, $T_!(Alg(X))$ is a good “algebraic” version of the tangent bundle of X.
- The replacement for differential rings is vector fields on $T_!(Alg(X))$.

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- The replacement for differential rings is vector fields on $T_!(Alg(X))$.
- We then have three possibilities for a "tangent bundle" functor on objects $X \in \mathbf{set}^{\mathbb{X}^{op}}$:

$$T_!(X), \mathrm{Spec}(T_!(\mathrm{Alg}(X)), T_!(\mathrm{Spec}(\mathrm{Alg}(X)).$$

- For the canonical Cartesian differential category $\mathbb{X}$ and X a smooth manifold, these are all the same; but they are distinct for more general $\mathbb{X}$ (say, convenient manifolds).
- The functor $T_!$ (on $\mathbf{set}^{\mathbb{X}^{op}}$) is a standard definition of the tangent bundle for a diffeological space.

Future work

- Determine relationships between the various tangent functors above.
- Are they tangent structure on the category of spaces $\mathbf{set}^{\mathbb{X}^{op}}$?
- Under what circumstances does the adjunction between "smooth spaces" and "smooth algebras" restrict to an equivalence?

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