Counting finite categories

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Outline

The question we are interested in: how many categories are there with n morphisms? (Up to isomorphism).

- How far can we calculate this number exactly, either by hand or with computer assistance? (And store the relevant categories).
- Can we get a formula, either precise or asymptotic, for the growth of this function?
- Can we say anything about specific types of categories, eg, Cauchy-complete categories?

Purely for interest's sake, no application in mind!

Known results

These were the previous known results:

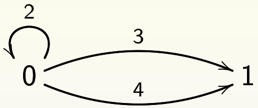
	Total	1 object	2	3	4	5	6
1 arrow	1	1					
2	3	2	1				
3	11	7	3	1			
4	55	35	16	3	1		
5	329	228	77	20	3	1	
6	2858	2,237	485	111	21	3	1
7		31,559					
8		1,668,997					
9		3,685,886,630					
10		$\sim 1.05x10^{15}$					

Finding categories as a CSP

- One can formulate the problem of "finding all categories with n morphisms" as a constraint satisfaction problem: a set of variables and constraints they must satisfy.
- For this, we use the "arrows only" version of category: objects are represented by their identity arrows.
- The category represented in this way is then simply an $n \times n$ table of how the n arrows compose, with each value in the table either another numbered arrow or a dummy value if the pair is non-composable.
- The constraints are the domain/codomain/identity/associativity axioms for a category.

Example category with five morphisms

The category:



with: $2 \circ 2 = 0$, $3 \circ 2 = 4$, $4 \circ 2 = 3$, is represented as:

	0	1	2	3	4
0	0	*	2	3	4
1	*	1	*	*	*
2	2	*	0	4	3
3	*	3	*	*	*
4	*	4	*	*	*

Minion

- There are many constraint satisfaction solvers available.
- We used Minion: a relatively recent constrant satisfaction solver that focuses on speed at the expense of some flexibility.
- Like any constraint satisfaction solver, it has many optimizations to solve CSP's; both algorithmic and hardware related.

Trimming isomorphisms

- One problem: the above process will give us many isomorphic copies of the same category.
- To resolve this problem, we run through the output we are given, and only select the categories which are lexicographically-least in each isomorphism class.
- That is, for each category given to us by Minion, we run through all permutations of the arrows in that category, and re-arrange the original category's table according to that permutation. If the original category is lexicographically greater than the permuted version, we discard it.
- This leaves only one category in each isomorphism class.

Additional optimizations, part I

- This process can find all categories with 7 morphisms in a reasonable amount of time, but then starts to take days for 8 morphisms, so additional optimization is required.
- **First optimization**: count the number of categories with n morphisms and k objects. Can remove all associativity constraints for identity morphisms. This gives all categories with 8 morphisms and most of 9.
- **Second optimization**: count only the connected categories with n morphisms and k objects. Non-connected categories can be found as a function of the previous lower connected counts. This gives counts up to 9 and 10 for all but the 2-object case.

Additional optimizations, part II

- **Third optimization**: count only the categories with 2 objects that have at least a certain number of non-endomorphic arrows (in practice, at least 3 or 4 non-endomorphic arrows).
- This avoids counting, for example, categories with two monoids with a single arrow between them: these can easily be counted by hand, but add a lot of processing time when running Minion.
- We then seperately run Minion instances with specific directed graphs with, say, 3 arrows between two monoids.

Updated table

	1	2	3	4	5	6
1	1					
2	2	1				
3	7	3	1			
4	35	16	3	1		
5	228	77	20	3	1	
6	2237	485	111	21	3	1
7	31559	**4013**	**716**	**127**	**21**	**3**
8	1668997	**47648**	**5623**	**862**	**131**	**21**
9	3.68×10^{10}	**1868157**	**60201**	**6739**	**926**	**132**
10	$\sim 1.05(10^{14})$	$\sim \mathbf{3.69(10^{10})}$	$\sim \mathbf{6(10)^5}$	**65922**	**7349**	**945**

This is as we far as we can go until someone counts the 11 monoids.

Monoids vs. Categories

Arrows	Monoids	All categories	Ratio
1	1	1	1.0
2	2	3	0.66
3	7	11	0.63
4	35	55	0.64
5	228	329	0.69
6	2,237	2,858	0.78
7	31,559	36,440	0.87
8	1,668,997	1,723,286	0.97
9	3,685,886,630	3,687,822,810	0.999
10	1.05986×10^{14}	1.05982×10^{14}	0.9999

Can we explain this?

Almost all semigroups are 3-nilpotent

- Almost all semigroups are 3-nilpotent: there is an element 0 with for any x, $x0 = 0x = 0$ and for any x, y, z, $xyz = 0$ (Kleitman, Rothschild, Spencer 1976).
- Why? These are very easy to construct: to construct such a semigroup on a set A, pick a subset B, an element $0 \in B$, and define xy to be 0 if x or y is in B and an arbitrary element of B otherwise. Such an operation is automatically associative.
- These types of semigroups overwhelm all other possibilities as the order of the semigroups grow.

Semigroup and monoid counts

- One can get a precise count of the number of 3-nilpotent semigroups of order n up to isomorphism (Distler and Mitchell 2012) and thus obtain an asymptotic formulae for the number of semigroups of order n up to isomorphism.
- Moreover, almost all monoids are semigroups with an identity attached (Koubek and Rodl, 1985).
- Thus one has asymptotic counts for both semigroups and monoids of order n.

Conjecture: almost all categories are 3-nilpotent semigroups

- Thus, if we can prove that almost all categories with n morphisms are monoids, then almost all categories will actually be 3-nilpotent semigroups with an identity adjoined.
- The direct known formula for the the number of 3-nilpotent semigroups, and its astonishing growth rate, should allow us to prove this.
- We would thus have an asymptoptic count for the number of categories with n morphisms (either up to isomorphism or up to equivalence).

Cauchy-complete

- Most interest in finite categories comes from looking at their associated presheaf categories.
- Thus, it makes sense to also look at the counts of Cauchy-complete categories.
- This has the additional advantage of removing all monoids which are not groups, and so avoids the ridiculous numbers which come with the 3-nilpotent semigroups.

Cauchy-complete table

	Total	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	2	1	1								
3	4	1	2	1							
4	11	2	6	2	1						
5	25	1	12	9	2	1					
6	63	2	23	25	10	2	1				
7	163	1	45	69	35	10	2	1			
8	451	5	98	178	119	38	10	2	1		
9	1311	2	278	457	371	151	39	10	2	1	

Conclusions

- We have counted and stored all categories with 10 morphisms or less up to isomorphism: this is about the limit with current technology using our techniques.
- One can get an asymptotic count for the number of categories with n morphisms, based on the idea that almost all categories are monoids, and almost all monoids are 3-idempotent semigroups.
- A more interesting question, then, is how many Cauchy-complete categories there are: further investigation needed.

References

References:

- Distler, A. and Mitchell, J. The number of nilpotent semigroups of degree 3. *Electronic Journal of Combinatorics*, Vol. 19 (2), pg. 51–64, 2012.
- Kleitman, D., Rothschild, B., and Spencer, J. The number of semigroups of order n. *Proceedings of the American Mathematical Society*, Vol. 53 (1), pg. 227–232, 1976.
- Koubek, V. and Rodl,V. Note on the number of monoids of order n. *Commentationes Mathematicae Universitatis Carolinae*, Vol. 26 (2), pg. 309–314, 1985.