Affine geometric spaces in tangent categories

R. F. Blute[*], G. S. H. Cruttwell[†], and R. B. B. Lucyshyn-Wright[‡]

July 30, 2018

Abstract

We continue the program of *structural differential geometry* that begins with the notion of a tangent category, an axiomatization of structural aspects of the tangent functor on the category of smooth manifolds. In classical geometry, having an affine structure on a manifold is equivalent to having a flat torsion-free connection on its tangent bundle. This equivalence allows us to define a category of affine objects associated to a tangent category and we show that the resulting category is also a tangent category, as are several related categories. As a consequence of some of these ideas we also give two new characterizations of flat torsion-free connections.

We also consider 2-categorical structure associated to the category of tangent categories and demonstrate that assignment of the tangent category of affine objects to a tangent category induces a 2-comonad.

Finally, following work of Jubin, we consider monads and comonads on the category of affine objects associated to a tangent category. We show that there is a rich theory of monads and comonads in this setting as well as various distributive laws and mixed distributive laws relating these monads and comonads. Even in the category of smooth manifolds, several of these results are new or fill in gaps in the existing literature.

Contents

[*]Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario CANADA
[†]Department of Mathematics and Computer Science, Mount Allison University, Sackville, New Brunswick CANADA
[‡]Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba CANADA

1 Introduction

This paper is part of a broader program of *structural differential geometry.* The idea is to axiomatize structural aspects of the category of smooth manifolds and smooth maps by axiomatizing the tangent functor. A category with an abstract tangent functor is called a *tangent category.* An axiomatization of the tangent functor was first given by Rosický [36], and the concept was elaborated on by Cockett and the second author in [11] with additional ideas introduced in [12, 13, 14, 16].

The axioms of a tangent category are sufficiently strong that one can develop highly nontrivial results, but also general enough to capture a number of different settings where there is a sensible notion of smoothness. For example, models of synthetic differential geometry fit into this framework, as do the convenient manifolds of Frölicher, Kriegl and Michor [20, 29, 9], and a notion of differentiation appearing in the calculus of functors [4].

There is also a strong logical underpinning for tangent categories. *Differential linear logic* [18, 19] and the associated categorical structures of *differential categories* [7] are an extension of linear logic to include an inference rule capturing the operation of taking a directional derivative. To every differential category, one can associate a coKleisli category. Such coKleisli categories are examples of *cartesian differential categories* [8]. Every cartesian differential category has a canonical tangent category structure [11, Section 4.2].

Following this initial work, there has been further development showing the extent to which further ideas of differential geometry can be developed within the abstract setting of tangent categories. See [12, 13, 14, 16]. This paper is another contribution to this program. In particular, we wish to look at affine manifolds (manifolds with an atlas whose transition maps are affine) and how they can be defined in tangent categories. Affine differential geometry is a relatively small subindustry of differential geometry, but affine manifolds have a great deal of interesting structure as well as a variety of examples. See [34, 23, 24, 1, 2].

The work of this paper is in part inspired by the thesis of Jubin [26]. Jubin considers the tangent functor on the category of smooth manifolds and smooth maps. He demonstrates that this tangent functor has a monad structure. Indeed he shows that it has precisely one such and further demonstrates that there are no comonad structures on the tangent functor at all. However he does demonstrate that when one restricts to the subcategory of affine manifolds and affine maps, there are infinite families of monads and comonads. Furthermore there are mixed distributive laws (see, e.g., [33]) between these structures.

While the notion of system of affine charts is not directly amenable to definition in a tangent category, we use a theorem of Auslander and Markus [2] showing that an affine manifold can be defined equivalently as a manifold equipped with a flat torsion-free connection on its tangent bundle (Theorem 2.2 below). Thus we are led to the theory of connections in a tangent category as introduced in [14]. See [17] for the classical theory of connections. Perhaps the best evidence of the strength of the axioms for tangent categories is how well the theory of connections works here. Prompted by the Auslander-Markus characterization of affine manifolds, we define a *geometric space* to be an object equipped with a connection on its tangent bundle and an *affine geometric space* to be a geometric space whose associated connection is flat and torsion-free. It is reasonable to call such objects geometric since the given connection generalizes Riemannian structure and allows one to define such geometric features as curvature and torsion. In particular, such features can be defined for an object with connection in an arbitrary tangent category. Maps in the category of geometric spaces are those maps that commute with the given connections. There has not been much in the way of study of categories with such morphisms, although we do mention [25].

We first show that the various geometric categories that we define remain tangent categories, with the structure lifting from the base category. Along the way, we also look at certain notions of morphism between tangent categories and derive technical lemmas about their lifting to the geometric categories. We also give an alternative characterization of flat torsion-free connections that seems to be new (Theorem 6.20): a flat torsion-free connection K can be seen as a morphism in the category of geometric spaces from $T(TM)$ to TM, where the tangent bundles TM and $T(TM)$ are endowed with geometric structures canonically induced by K. This result alone demonstrates the importance of considering categories whose arrows commute with connections.

We also consider certain 2-categories of tangent categories. We show that there are 2-functors that send each tangent category to its tangent category of geometric spaces or affine geometric spaces. Of course, these results require a careful presentation of the 2-categorical structure of tangent categories. We show that the affine construction induces a 2-comonad on the 2-category of tangent categories. We then define an *affine tangent category* to be an Eilenberg-Moore coalgebra with respect to this comonad and give an alternate characterization of these structures.

Finally, we extend several results of Jubin to a general tangent category. We show that for every non-negative integer there is both a monad and a comonad on the affine category associated to a tangent category and that if one has negatives, the result holds for every integer. We furthermore show that there are mixed distributive laws between these monads and comonads, yielding bimonads and in some instances Hopf monads. As discussed in section 8.3, some of these results are new for the category of smooth manifolds, while one of the results fills in a gap in Jubin's work.

Remark 1.1.

- The authors thank NSERC for its generous support. The third author gratefully acknowledges an AARMS PDF held earlier in the development of this work.

- Note that following previous work on tangent and differential categories [7, 8, 11, 12, 13, 14] we write our compositions in diagrammatic order unless otherwise indicated. However, we write the application of a functor F to a morphism f as $F(f)$, as in the cited works, or as Ff; correspondingly, we write the composite of functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{C}$ in non-diagrammatic order as GF.

- In many of the longer calculations, we omit the subscripts on natural transformations to save space.

2 Affine manifolds

We give an overview of the classical theory of affine manifolds and their associated connections. See [1, 2, 34]. An *affine manifold* is a real manifold whose transition maps are affine (and hence necessarily smooth):

Definition 2.1. *An n-dimensional* ***affine manifold*** *is a real manifold equipped with a specified atlas consisting of charts $\psi_i : U_i \xrightarrow{\sim} V_i \subseteq \mathbb{R}^n$ such that all of the composites $\psi_i \circ \psi_j^{-1} : V_{ji} \to V_{ij}$ are affine maps between the subsets $V_{ji} = \psi_j(U_i \cap U_j)$ and $V_{ij} = \psi_i(U_i \cap U_j)$ of $\mathbb{R}^n$. Here a map $f : V \to W$ between subsets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ is said to be* affine *if it has constant Jacobian, or equivalently, if it is the restriction of a map $F : \mathbb{R}^n \to \mathbb{R}^m$ that is affine in the usual sense, i.e. a composite of a linear map followed by a translation.*

While it is not obvious how one might generalize this notion to the more abstract context of tangent categories, there is a theorem of Auslander and Markus that will enable us to obtain such a generalization. See [2]. See [17] for the classical theory of connections.

Theorem 2.2 (Auslander-Markus)**.** *A manifold is an affine manifold if and only if there is a flat torsion-free connection on its tangent bundle. Moreover, each affine manifold has a canonically associated flat torsion-free connection on its tangent bundle, and every smooth manifold with a flat torsion-free connection on its tangent bundle carries an associated affine structure.*

Of special interest are the *complete affine manifolds*. These are affine manifolds that satisfy geodesic completeness, [17] p. 250. In the affine case, geodesic completeness turns out to be equivalent to being a quotient of an affine space by a discrete group of affine transformations acting on the space. See [24] for further discussion.

3 Tangent categories and connections

We summarize the results of [11, 13, 14] which introduce what could in essence be thought of as *structural differential geometry*. The idea, which originated in the work of Rosický [36], was to axiomatize the structure of the tangent bundle functor on the category of smooth manifolds.

It turns out that this set of axioms is both powerful enough to derive quite strong results, yet general enough to apply to a great many settings. In addition to the category of smooth manifolds being a tangent category, it is shown in [11] that the category of *infinitesimally and vertically linear* objects in any model of synthetic differential geometry [27] is a tangent category, as is any cartesian differential category [8]. A recent example of a very different type comes from the calculus of functors [4].

Definition 3.1. *Let $\mathbb{C}$ be a category. A* ***tangent structure*** *on $\mathbb{C}$ consists of the following structure:*

- *A functor $T\colon \mathbb{C} \to \mathbb{C}$ (thought of as the* ***tangent functor****);*
- *A* ***projection****, i.e. a natural transformation $p\colon T \to id_{\mathbb{C}}$;*
- ***Fibre powers of*** *p, i.e for each object M of $\mathbb{C}$ and each natural number n there is a fibre product[1] of n copies of p_M, written as $T_nM \to M$, and this fibre product is preserved by each iterate T^m of T;*
- ***Additive bundle structure****, i.e natural transformations $+\colon T_2 \to T$ and $0\colon id \to T$ making each projection p_M an additive bundle, i.e., a commutative monoid in the slice category over M in $\mathbb{C}$;*
- *A* ***vertical lift*** *natural transformation $\ell : T \longrightarrow T^2$ such that for each M*

  $$(\ell_M, 0_M) : (p : TM \longrightarrow M, +, 0) \longrightarrow (Tp : T^2M \longrightarrow TM, T(+), T(0))$$

  is an additive bundle morphism[2];
- *A* ***canonical flip*** *natural transformation $c : T^2 \longrightarrow T^2$ such that for each M*

  $$(c_M, 1) : (Tp : T^2M \longrightarrow TM, T(+), T(0)) \longrightarrow (p_T : T^2M \longrightarrow TM, +_T, 0_T)$$

  is an additive bundle morphism;

[1] Recall that a fibre product of a family of morphisms with common codomain M in $\mathbb{C}$ is a product in the slice category $\mathbb{C}/M$. Each fibre product in $\mathbb{C}$ determines an associated diagram in $\mathbb{C}$, called a **fibre product diagram**, consisting of the base object M, the given family of morphisms, and the product projections for the fibre product. Just as with pullbacks, fibre products are equivalently described as certain limits in $\mathbb{C}$.

[2] I.e., the pair $(\ell_M, 0_M)$ is a morphism from p_M to $T(p_M)$ in the arrow category and preserves the given additive bundle structures in the evident sense (see [11, Definition 2.2]).

- ***Coherence of*** ℓ ***and*** c*:* $cc = 1$ *(so* c *is an isomorphism),* $\ell c = \ell$*, and the following diagrams commute:*

$$\begin{tikzcd} T \arrow[r, "\ell"] \arrow[d, "\ell"'] & T^2 \arrow[d, "T\ell"] \\ T^2 \arrow[r, "\ell T"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "Tc"] \arrow[d, "cT"'] & T^3 \arrow[r, "cT"] & T^3 \arrow[d, "Tc"] \\ T^3 \arrow[r, "Tc"'] & T^3 \arrow[r, "cT"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "\ell T"] \arrow[d, "c"'] & T^3 \arrow[r, "Tc"] & T^3 \arrow[d, "cT"] \\ T^2 \arrow[rr, "T\ell"'] & & T^3 \end{tikzcd}$$

- ***Universality of vertical lift****: for each object* M*, if we define* $v : T_2M \longrightarrow T^2M$ *by* $v := \langle \pi_0\ell, \pi_1 0_T\rangle T(+)$*, the following diagram is a pullback that is preserved by each* T^n*:*

$$\begin{tikzcd} T_2(M) \arrow[r, "v"] \arrow[d, "\pi_0 p = \pi_1 p"'] & T^2(M) \arrow[d, "T(p)"] \\ M \arrow[r, "0"'] & T(M) \end{tikzcd}$$

A ***tangent category*** *is a category equipped with a tangent structure. A tangent category is said to have* ***negatives*** *if each of the above commutative monoids* $(p_M, +_M, 0_M)$ *is moreover an abelian group in the slice category over* M*.*

While the axioms for a tangent category at first may appear ad-hoc, recent work of Leung [31] and Garner [21] has shown how tangent categories are related to Weil algebras and how tangent categories are a type of enriched category.

We note that the endofunctor T of a tangent category always has a canonical monad structure (see [11, Proposition 3.4]). This formula was discovered independently by Jubin [26] in the specific case of the tangent category of smooth manifolds.

Proposition 3.2. *Let* $(\mathbb{C}, T)$ *be a tangent category. Then* T *has a monad structure with multiplication and unit given as follows:*

$$T^2M \xrightarrow{<Tp, p_T>} T_2M \xrightarrow{+} TM \qquad\qquad M \xrightarrow{0} TM$$

Before defining a notion of connection in tangent categories, it is helpful to have at hand the following generalization of the notion of vector bundle, given in [13]:

Definition 3.3. *A* ***differential bundle*** *in a tangent category consists of an additive bundle* $(q : E \longrightarrow M, +_{\mathsf{q}} : E_2 \longrightarrow E, 0_{\mathsf{q}} : M \longrightarrow E)$ *with a map* $\lambda : E \longrightarrow T(E)$*, called the* **lift***, such that*

- *finite fibre powers of* q *exist and are preserved by each* T^n*;*
- $(\lambda, 0_M)$ *is an additive bundle morphism from* $(E, q, +_{\mathsf{q}}, 0_{\mathsf{q}})$ *to* $(T(E), T(q), T(+_{\mathsf{q}}), T(0_{\mathsf{q}}))$*;*
- $(\lambda, 0_{\mathsf{q}})$ *is an additive bundle morphism from* $(E, q, +_{\mathsf{q}}, 0_{\mathsf{q}})$ *to* $(T(E), p_E, +_E, 0_E)$*;*

- *the* **universality of the lift** *requires that the following be a pullback:*

$$\begin{tikzcd} E_2 \arrow[r, "\mu:=\langle \pi_0\lambda, \pi_1 0\rangle T(+_{\mathsf{q}})"] \arrow[d, "\pi_0 q = \pi_1 q"'] & T(E) \arrow[d, "T(q)"] \\ M \arrow[r, "0"'] & T(M) \end{tikzcd}$$

where E_2 is the pullback of q along itself;

- *the equation $\lambda\ell_E = \lambda T(\lambda)$ holds.*

We shall write q to denote the entire bundle structure $(q, +_{\mathsf{q}}, 0_{\mathsf{q}}, \lambda)$.

Now let q and q' be differential bundles. A ***bundle morphism*** *between these bundles simply consists of a pair of maps $f_1 : E \longrightarrow E'$, $f_0 : M \longrightarrow M'$ such that $f_1 q' = q f_0$ (first diagram below). A bundle morphism is* ***linear*** *in case, in addition, it preserves the lift, that is $f_1\lambda' = \lambda T(f_1)$ (the second diagram below).*

$$\begin{tikzcd} E \arrow[r, "f_1"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\ M \arrow[r, "f_0"'] & M' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "f_1"] \arrow[d, "\lambda"'] & E' \arrow[d, "\lambda'"] \\ T(E) \arrow[r, "T(f_1)"'] & T(E') \end{tikzcd}$$

Notably, every linear bundle morphism is automatically additive [13, Proposition 2.16].

In the present section, we shall tacitly assume that given differential bundles q satisfy the following additional condition, which is a prerequisite for considering connections on such bundles [14, Def. 2.2], [32, 3.1]:

For all natural numbers n and m, the n-th fibre power $E_n \to M$ of q has a pullback along the projection $T_m M \to M$, and this pullback is preserved by each T^k. (3.i)

Of course, the crucial example of a differential bundle is the tangent bundle

$$\mathsf{p}_M = (p_M, +_M, 0_M, \ell_M)$$

of an object M [13, Example 2.4], and we recommend keeping it in mind in the definitions below. The tangent bundle always satisfies the preceding additional assumption [14, Example 2.3].

Given any differential bundle $\mathsf{q} = (q : E \to M, +_{\mathsf{q}}, 0_{\mathsf{q}}, \lambda)$ we obtain an associated differential bundle

$$T(\mathsf{q}) = (T(q) : TE \to TM, T(+_{\mathsf{q}}), T(0_{\mathsf{q}}), T(\lambda)c_E),$$

defined in [13, §2.3]. Hence TE underlies two differential bundles, namely $T(\mathsf{q})$ and p_E, whose underlying additive bundles appear in Definition 3.3.

We now recall the notion of connection as it is defined in [14] with respect to a tangent category. In this formulation, a connection consists of two parts, a *vertical connection* and a *horizontal connection*, that are suitably compatible. Together, they split the tangent bundle of a given bundle into vertical and horizontal components. A result of Patterson [35, Theorem 1] shows that vertical connections in the category of smooth manifolds correspond to one of the standard formulations of the notion of connection: a covariant (or Koszul) derivative. On the other hand, horizontal connections in smooth manifolds correspond to what are known as linear Ehresmann connections [37, Definition 7.2.1][3]. For smooth manifolds the existence of a covariant derivative/vertical connection is equivalent to the existence of a linear Ehresmann/horizontal connection [37, Proposition 7.5.11] but this is no longer the case in a general tangent category. This led the authors of [14] to employ both notions at once, as seen below.

A covariant derivative (or Koszul connection) is an operation on global sections of certain bundles. Although this notion is one of the most standard formulations of connections, it is inadequate for describing connections in a general tangent category as these are structures internal to the given category and so cannot in general be characterized in terms of global sections. However one can find an appropriate abstract definition based on the work of Patterson [35].

Definition 3.4. *Let q be a differential bundle on E over M. A **vertical connection** on q is a map $K : T(E) \to E$ that is a retraction of $\lambda : E \to T(E)$ and satisfies the following conditions:*

[C.1] $(K, p) : T(\mathsf{q}) \longrightarrow \mathsf{q}$ *is a linear bundle morphism;*

[C.2] $(K, q) : \mathsf{p}_E \longrightarrow \mathsf{q}$ *is a linear bundle morphism.*

Curvature in differential geometry is thought of as a measure of the extent to which a geometric space deviates from being flat n-space [17]. It is typically defined for Riemannian manifolds or, more generally, arbitrary manifolds equipped with a connection. Of particular interest are those connections that have no curvature, that is, they are *flat*.

Definition 3.5. *In a tangent category with a vertical connection K on a differential bundle q, say that the vertical connection is **flat** if $cT(K)K = T(K)K$.*

In the tangent category of smooth manifolds, this is equivalent to the usual definition, by a result of Patterson [35, Theorem 2].

The closely related notion of torsion captures twisting effects that tangent vectors incur when subject to parallel transport. Again, it is of particular interest to see when connections have no torsion; that is, when they are *torsion-free*.

Definition 3.6. *In a tangent category with a vertical connection K on the tangent bundle of an object M (so that $K : T^2(M) \longrightarrow T(M)$), say that the vertical connection is **torsion-free** if $cK = K$.*

[3]Some authors such as Lang simply call these connections [30, pg. 104].

Again, the equivalence of this definition with the standard one follows from a result of Patterson [35, Theorem 3].

Linear Ehresmann connections also generalize to the setting of an arbitrary tangent category, through the notion of *horizontal connection* [14]:

Definition 3.7. *Let* q *be a differential bundle. A* ***horizontal connection*** *on* q *is a map* $H : T(M) \times_M E \longrightarrow T(E)$ *that is a section of* $U = \langle T(q), p\rangle$ *and satisfies the following conditions:*

- $(H, 1_E)$ *is a linear bundle morphism from* $q^*(\mathsf{p_M})$ *to* $\mathsf{p_E}$*;*
- $(H, 1_{T(M)})$ *is a linear bundle morphism from* $p^*(\mathsf{q})$ *to* $T(\mathsf{q})$*.*

Here, we write $q^*(\mathsf{p_M})$ (resp. $p^*(\mathsf{q})$) to denote the pullback of $\mathsf{p_M}$ along q (resp. of q along p_M) [13, Lemma 2.7], which exists as a consequence of our assumption (3.i); see [32, 2.4.7, 2.4.8, §3] for an explicit account of this existence.

As already noted, for smooth manifolds the notions of horizontal and vertical connection are equivalent. In general tangent categories they are not, and this led Cockett and Cruttwell to define a connection to be a pair consisting of one of each satisfying compatibility, as follows.

Definition 3.8 ([14, Def. 5.1])**.** *A* ***connection****,* (K, H)*, on a differential bundle* q *consists of a vertical connection* K *on* q *and a horizontal connection* H *on* q *such that*

- $HK = \pi_1 q 0_{\mathsf{q}}$*;*
- $\langle K, p\rangle\mu + UH = 1_{T(E)}$ *where* μ *is as defined in 3.3 and the addition operation* $+$ *is induced by* $+_E : T_2E \to TE$*.*

These two conditions are called the compatibility conditions *between* H *and* K*.*

Proposition 3.9 ([14, Prop. 3.5])**.** *In a Cartesian tangent category, any differential object* A *(i.e. a differential bundle over* 1*) has a canonical connection* (K, H) *where* $H = \pi_1 0$ *and* K *is the* principal projection $\hat{p} : TA \to A$ *associated to* A *([13, §3], [14, Example 2.3]).*

The third author proved that connections in a tangent category are equivalently described as vertical connections satisfying a certain 'exactness' condition, as follows:

Theorem 3.10 ([32, 8.2(3)])**.** *Let* $\mathsf{q} = (q : E \longrightarrow M, +_{\mathsf{q}}, 0_{\mathsf{q}}, \lambda)$ *be a differential bundle. Then a connection on* q *is equivalently given by a vertical connection* $K : TE \to E$ *such that the following is a fibre product diagram in* $\mathbb{C}$*:*

$$\begin{tikzcd}
& TE \arrow[dl, "T(q)"'] \arrow[d, "p_E"] \arrow[dr, "K"] & \\
TM \arrow[dr, "p_M"'] & E \arrow[d, "q"] & E \arrow[dl, "q"] \\
& M &
\end{tikzcd}$$

Corollary 3.11 ([32, 8.4(3)])**.** *Let M be an object of a tangent category $(\mathbb{C}, T)$. Then a connection on the tangent bundle of M is equivalently given by a vertical connection $K : T^2M \to TM$ on p_M such that the following is a fibre product diagram in $\mathbb{C}$:*

$$\begin{tikzcd}
& T^2M \arrow[dl, "T(p_M)"'] \arrow[d, "p_{TM}"] \arrow[dr, "K"] & \\
TM \arrow[dr, "p_M"'] & TM \arrow[d, "p_M"] & TM \arrow[dl, "p_M"] \\
& M &
\end{tikzcd} \tag{3.ii}$$

Given a morphism K as 3.11, the associated horizontal connection H is characterized by the following:

Theorem 3.12 ([32, 7.8])**.** *Given a vertical connection $K : T^2M \to TM$ on p_M such that* (3.ii) *is a fibre product diagram, there is a unique horizontal connection H such that (K, H) is a connection on the tangent bundle of M in the sense of 3.8. Further, H is the unique morphism $H : TM \times_M TM \to T^2M$ such that*

$$HT(p_M) = \pi_0, \quad Hp_{TM} = \pi_1, \quad HK = p_2 0_M$$

where $p_2 : TM \times_M TM \to M$ is the projection.

In the present paper, we shall represent connections as morphisms K as in 3.10 and 3.11, but we shall also make important use of the associated horizontal connection H. Moreover, throughout the rest of the paper, we shall primarily be concerned with connections on the tangent bundle of an object M. Thus, for brevity, rather than speak of a "connection on the tangent bundle of M", we shall simply say "connection on M". (Connections on tangent bundles are typically referred to as *affine connections*, but, in this paper, this would cause an overload of the term "affine".)

Definition 3.13. *In view of Corollary 3.11 and the discussion above, we will call a morphism $K : T^2M \to TM$ a* ***connection on*** *M if K is a vertical connection on the tangent bundle of M and makes* (3.ii) *a fibre product diagram.*

In section 8 in particular, we will make extensive use of the equational properties that such a connection satisfies:

Proposition 3.14. *If K is a connection on M, then*

(a) (K is a retract of ℓ) $\ell_M K = 1_{TM}$;

(b) (K is a bundle morphism) $Kp_M = p_{TM}p_M = T(p_M)p_M$;

(c) (linearity of K) $K\ell_M = \ell_{TM}T(K)$ and $K\ell_M = T(\ell_M)c_{TM}T(K)$;

(d) (additivity of K) $0_{TM}K = p_M 0_M$, $+_{TM}K = \langle \pi_0 K, \pi_1 K\rangle +_M$, *and* $T(0_M)K = p_M 0_M$, $T(+)K = \langle T(\pi_0)K, T(\pi_1)K\rangle +_M$.

Proof. Properties (a)-(c) follow directly from the definition of a vertical connection on the tangent bundle (see [14, Lemma 3.3], while (d) follows since linear bundle morphisms are additive, as noted above. □

4 Jubin's thesis

We now summarize relevant results of the unpublished Ph.D. thesis of Benoit Jubin [26] that led us to consider the structures defined in this paper. Like Cockett and Cruttwell [11, 12, 13, 14], Jubin was interested in functorial properties of the tangent space construction. He demonstrated that the tangent bundle functor was a monad on the category of smooth manifolds, although he did not identify all of the additional structure that goes into the more abstract definition of *tangent category.* However, he did derive several results specific to the tangent functor on the category of smooth manifolds.

Theorem 4.1 (Jubin [26])**.**

- *The tangent functor on the category of smooth manifolds carries a unique monad structure. Using local coordinates, the multiplication* $\mu\colon T^2M \to TM$ *is given by*

$$\mu\colon T^2M \to TM\colon (x, v, \dot{x}, \dot{v}) \mapsto (x, v + \dot{x}) \ .$$

  The unit $\eta\colon id \to T$ *is given by the zero section.*

- *There are no comonad structures on the tangent functor on the category of smooth manifolds.*

This monad structure exists in any tangent category and indeed appears in [11], as noted above (3.2). But uniqueness depends crucially on the setting of smooth manifolds as does the lack of comonads. These results do not follow from the axioms of tangent category. In particular, we will construct a comonad on a specific tangent category in this paper.

Jubin also studies the category of affine manifolds (see §2). Affine manifolds are the objects of a category Aff in which a morphism is a **locally affine** map, i.e. a smooth map $f : M \to N$ such that for every pair of designated charts $U \cong V \subseteq \mathbb{R}^n$ and $U' \cong V' \subseteq \mathbb{R}^m$ for M and N, respectively, the restriction of f to $U \cap f^{-1}(U')$ has all its second partial derivatives equal to zero.

Theorem 4.2. *The tangent functor on smooth manifolds lifts to an endofunctor on the category of affine manifolds* Aff*.*

Proof. Jubin gives a proof of this result, but it also follows from the more general Theorem 6.1 below. □

Working in the smaller tangent category Aff, we have significantly more freedom to define structures on the tangent functor. Indeed Jubin completely characterizes all monad and comonad structures on the tangent functor on Aff. See Proposition 3.2.1 of [26]. Since this takes place in a category of smooth manifolds, one can define the necessary structural maps using local coordinates.

Theorem 4.3 (Jubin)**.**

- *The only monad structures on the tangent functor on the category* Aff *are indexed by the real numbers, and for a fixed real number a the monad multiplication is given by the following:*

$$\mu^a \colon T^2M \to TM \colon (x, v, w, d) \mapsto (x, v + w + ad)$$

  In this case, the unit map for the monad must be the zero section.

- *The only comonad structures on the tangent functor on the category* Aff *are indexed by the real numbers, and for a fixed real number b the comultiplication is given by the following:*

$$\delta^b \colon TM \to T^2M \colon (x, v) \mapsto (x, v, v, bv)$$

  In this case, the counit map for the comonad is given by the bundle projection.

Jubin furthermore claims that the above monad and comonad structures interact to form bimonads [33]. These are endofunctors with monad and comonad structure as well as a *mixed distributive law* that relates the two structures and satisfies some additional axioms. Since this structure on affine manifolds lifts to the more general setting of tangent categories, we recall the details here.

Definition 4.4. *Let $\mathbb{C}$ be a category and T an endofunctor on $\mathbb{C}$. A* ***bimonad structure*** *on T consists of a monad structure (μ, η) on T, a comonad structure (δ, ϵ) and a mixed distributive law, $\lambda \colon T^2 \to T^2$, from the monad (T, μ, η) to the comonad (T, δ, ϵ) that additionally satisfies the following requirements:*

- *ϵ is a monad morphism [3] from (T, μ, η) to the identity.*
- *η is a comonad morphism from the identity to (T, δ, ϵ).*
- *The following diagram commutes:*

$$\begin{tikzcd} T^2 \arrow[r, "\mu"] \arrow[d, "T\delta"'] & T \arrow[r, "\delta"] & T^2 \\ T^3 \arrow[rr, "\lambda T"'] & & T^3 \arrow[u, "T\mu"'] \end{tikzcd}$$

Theorem 4.5 (Jubin)**.** *The tangent functor T equipped with its a-monad structure and b-comonad structure (Theorem 4.3) is a bimonad. The formula for the mixed distribution is given by (using local coordinates):*

$$\lambda^{a,b} \colon T^2M \to T^2M \colon (x, v, w, d) \mapsto (x, w, v + w + ad, bw - d)$$

Unfortunately, Jubin's proof of this theorem (3.2.2 in [26]) contains some gaps. In particular, the proof given does not show that λ is a distributive law: it only shows that λ satisfies the extra conditions required for a distributive law to be a bimonad. Fortunately, we show in the much more general context of a tangent category that the given λ is indeed a distributive law (8.14) and is part of a bimonad structure (8.15), thus filling in the gap in Jubin's proof while also generalizing it.

5 Geometric and affine structures in tangent categories

A manifold may carry further geometric structure, such as Riemannian structure, and it is only with reference to such additional structure that one can define several important aspects of its geometry, including geodesics, curvature, and parallel transport. The notion of connection captures such structure by means of a formalism that is quite general yet still supports all the latter geometric features, so that there is a sense in which a smooth space carries a fixed geometry once it is equipped with a chosen connection. Thus we are led to define the notion of *geometric space* in a tangent category $\mathbb{C}$, as an object equipped with a connection (5.1). In view of the Auslander-Markus theorem (2.2), the category of affine manifolds has a natural generalization in an arbitrary tangent category, namely the category of geometric spaces whose associated connection is flat and torsion-free. Thus we may pursue certain of the themes of Jubin's thesis within the category of *affine geometric spaces* in $\mathbb{C}$, which we now define:

Definition 5.1. *Let $(\mathbb{C}, T)$ be a tangent category.*

- *A **geometric space** in $\mathbb{C}$ is a pair (M, K) in which M is an object of $\mathbb{C}$ and $K : T^2M \to TM$ is a connection on M (3.13).*

- *A geometric space (M, K) is **flat** (resp. **torsion-free**) if its associated connection K is so (see 3.5 and 3.6).*

- *An **affine geometric space** in $\mathbb{C}$ is a geometric space (M, K) that is both flat and torsion-free.*

- *A map of geometric spaces $f \colon (M, K) \to (M', K')$ is a map $f \colon M \to M'$ in $\mathbb{C}$ such that the following diagram commutes:*

$$\begin{tikzcd} T^2M \arrow[r, "K"] \arrow[d, "T^2f"'] & TM \arrow[d, "Tf"] \\ T^2M' \arrow[r, "K'"'] & TM' \end{tikzcd}$$

- *We write* $\mathsf{Geom}(\mathbb{C}, T)$ *to denote the category of geometric spaces, with the above morphisms. We denote by* $\mathsf{Geom}_{\mathsf{flat}}(\mathbb{C}, T)$ *and* $\mathsf{Geom}_{\mathsf{tf}}(\mathbb{C}, T)$ *the full subcategories of* $\mathsf{Geom}(\mathbb{C}, T)$ *consisting of the flat and torsion-free geometric spaces, respectively.*
- *We write* $\mathsf{Aff}(\mathbb{C}, T)$ *to denote the full subcategory of* $\mathsf{Geom}(\mathbb{C}, T)$ *whose objects are the affine geometric spaces.*

Example 5.2. *By Example 5.7 of [14], any differential object has a canonical choice of connection, given by the formula*

$$K = \langle T(\hat{p})\hat{p}, pp\rangle$$

(its associated horizontal connection is $H = \langle !0_{\mathsf{q}}, \pi_0\hat{p}, \pi_1\hat{p}, \pi_1 p\rangle$*). This connection is flat and torsion-free (see the discussion after Proposition 3.16 and Example 3.21 in [14]). Thus, any differential object has a canonical choice of connection to make it into an affine geometric space.*

Moreover, recall that if $(A, \hat{p}_A)$ *and* $(B, \hat{p}_B)$ *are differential objects, then a* ***linear map*** *between such objects consists of a map* $f : A \longrightarrow B$ *such that* $T(f)\hat{p}_B = \hat{p}_A f$*. It is then easy to show that such a map is also a map between the corresponding affine geometric spaces, i.e., a map in* $\mathsf{Aff}(\mathbb{C}, T)$*.*

Example 5.3. *We note that any Riemannian manifold whose canonical connection (the Levi-Civita connection) is flat is automatically affine, since the Levi-Civita connection is always torsion-free. This gives us access to a wide variety of further examples.*

Recall that the morphisms in the category Aff of affine manifolds are the locally affine maps (§4). The following result shows that these are the same as maps that preserve the associated connections:

Proposition 5.4. *A smooth map* $f : M \to M'$ *between affine manifolds* M *and* M' *is locally affine if and only if* $f : (M, K) \to (M', K')$ *is a morphism of geometric spaces in the tangent category* (Mf, T) *of smooth manifolds, where* K *and* K' *are the associated connections (2.2). Consequently, the category* Aff *of affine manifolds is equivalent to the category* $\mathsf{Aff}(\mathsf{Mf}, T)$ *of affine geometric spaces in* (Mf, T)*.*

Proof. In view of the Auslander-Markus Theorem (2.2), it suffices to prove the first statement above, concerning a given map f. By definition, $f : (M, K) \to (M', K')$ is a morphism of geometric spaces iff $T^2(f)K' = KT(f)$. Since we are dealing with smooth manifolds, it suffices to know that this equality holds in each of the given charts for M. However, by [2, Thm. 1], the Christoffel symbols of the associated connections K and K' are identically zero on each of the given charts, which means that in each of these charts, the connection K (and similarly K') takes the particular form

$$(x, v, w, a) \mapsto (x, a)$$

(see [14, Example 3.6.1] for the relationship of the Christoffel symbols to the vertical connection K).

For the remainder of the proof we will work locally; that is, we consider a pair of charts $U \cong V \subseteq \mathbb{R}^n$ and $U' \cong V' \subseteq \mathbb{R}^m$ in M and M', respectively, and consider the restriction of f to $U \cap f^{-1}(U')$. For simplicity, we will also simply consider the case when $m = 1$. Recall that in a chart $U \subseteq R^n$, for a point $(x, v) \in TU$,

$$T(f)(x, v) = (f(x), D(f)(x, v))$$

where $D(f)(x, v)$ is the directional derivative of f at x in the direction of v, and as a result

$$T^2(f)(x, v, w, a) = (f(x), D(f)(x, v), D(f)(x, w), D(D(f))((x, v), (w, a))).$$

Thus, by the form that K and K' take, we have

$$T(f)(K(x, v, w, a)) = T(f)(x, a) = (f(x), D(f)(x, a))$$

while

$$K'(T^2(f)(x, v, w, a)) = (f(x), D(D(f))((x, v), (w, a))).$$

However, by definition of the directional derivative,

$$D(D(f))((x, v), (w, a)) = \frac{\partial[D(f)(x, v)]}{\partial x} \cdot w + \frac{\partial[D(f)(x, v)]}{\partial v} \cdot a = w^T \cdot H(f)(x) \cdot v + D(f)(x, a)$$

where H is the Hessian of f, ie., the matrix of second partial derivatives of f.

Thus the two terms are equal if and only if

$$w^T \cdot H(f)(x) \cdot v = 0$$

for all x, v, w. But this is true if and only if each second partial derivative of f at x is equal to 0. In other words, the map f is connection-preserving if and only if in each local affine chart, and for each i, $\frac{\partial f}{\partial x_i}$ has each of its partial derivatives equal to 0. □

Every map in $\mathsf{Geom}(\mathbb{C}, T)$ necessarily preserves the associated horizontal connection (5.6). We will prove this by means of the following proposition:

Proposition 5.5. *Suppose* $(f, g) : \mathsf{q} \longrightarrow \mathsf{q}'$ *is a linear map between differential bundles with connections* (K, H) *and* (K', H')*. Then*

$$T(f)K' = Kf \Leftrightarrow (Tg \times f)H' = HT(f).$$

Proof. Suppose $T(f)K' = Kf$. We have

$$\begin{array}{rcl}
\langle K', p\rangle\mu' + U'H' & = & 1 \\
T(f)\langle K', p\rangle\mu' + T(f)U'H' & = & T(f) \\
\langle K, p\rangle(f \times f)\mu' + U(Tg \times f)H' & = & T(f) \text{ (by Lemma 4.2 of [14])} \\
\langle K, p\rangle\mu T(f) + U(Tg \times f)H' & = & T(f) \text{ (by Lemma 2.17 of [13])} \\
H\langle K, p\rangle\mu T(f) + HU(Tg \times f)H' & = & HT(f) \\
0 + (Tg \times f)H' & = & HT(f)
\end{array}$$

as required. For the other direction, suppose $(Tg \times f)H' = HT(f)$. Then we have

$$\begin{array}{rcl}
\langle K, p\rangle\mu + UH & = & 1 \\
\langle K, p\rangle\mu T(f) + UHT(f) & = & T(f) \\
\langle K, p\rangle(f \times f)\mu' + U(Tg \times f)H' & = & T(f) \text{ (by Lemma 4.2 of [14])} \\
\langle K, p\rangle(f \times f)\mu' + T(f)U'H' & = & T(f) \text{ (by Lemma 2.17 of [13])} \\
\langle K, p\rangle(f \times f)\mu'\langle K', p\rangle + T(f)U'H'\langle K', p\rangle & = & T(f)\langle K', p\rangle \\
\langle K, p\rangle(f \times f) & = & T(f)\langle K', p\rangle
\end{array}$$

so that by taking the first projection of both sides, $Kf = T(f)K'$, as required. □

Corollary 5.6. *If M and M' have connections (K, H) and (K', H'), then for any map $f : M \longrightarrow M'$,*

$$T^2(f)K' = KT(f) \Leftrightarrow T_2(f)H' = HT^2(f).$$

Proof. By Proposition 2.4 of [14], the pair $(T(f), f)$ is a linear bundle morphism from the tangent bundle of M to the tangent bundle of M'. Then applying the previous result (5.5) to this linear bundle morphism gives the desired result. □

6 Lifting tangent structure to the geometric categories

The main result we would like to prove in this section is the following:

Theorem 6.1. *Let $(\mathbb{C}, T)$ be a tangent category. There is a functor $T_*\colon \mathsf{Aff}(\mathbb{C}, T) \to \mathsf{Aff}(\mathbb{C}, T)$ given on objects as follows:*

$$(M, K) \mapsto (TM, T(c)cT(K)c)$$

This functor makes $\mathsf{Aff}(\mathbb{C}, T)$ a tangent category.

However, some of the structures that arise in proving this result will also lead us to prove an interesting alternate characterization of flat torsion-free connections (Theorem 6.20).

Before proving the result above, we will pause to consider where the above formula comes from. Given any *strong morphism of tangent categories* $F : \mathbb{C} \to \mathbb{C}'$, in the sense of 6.2 below, we shall show that F sends a connection on M in $\mathbb{C}$ to a connection on FM in $\mathbb{C}'$ (6.5). In particular, the tangent functor $T : \mathbb{C} \to \mathbb{C}$ is a strong morphism of tangent categories when equipped with the transformation c, so by applying T to a connection K on M and composing with a few instances of c we obtain an associated connection $T(c)cT(K)c$ on TM.

Definition 6.2 ([11])**.** *Given tangent categories $(\mathbb{C}, T, p, 0, +, \ell, c)$ and $(\mathbb{C}', T', p', 0', +', \ell', c')$, a **morphism of tangent categories** is a functor $F : \mathbb{C} \to \mathbb{C}'$ equipped with a natural transformation $\alpha = \alpha^F : FT \to T'F$ such that F preserves all the pullbacks that are required to exist as part of the tangent structure on $\mathbb{C}$, and such that*

$$\alpha p'_F = F(p), \quad F(0)\alpha = 0'_F, \quad F(+)\alpha = \alpha_2 +'_F,$$

$$F(\ell)\alpha^{[2]} = \alpha\ell'_F, \quad F(c)\alpha^{[2]} = \alpha^{[2]}c'_F.$$

Here $\alpha^{[2]} = \alpha_T T'(\alpha) : FT^2 \to T'^2F$, and $\alpha_2 : FT_2 \to T'_2F$ is the natural transformation whose components

$$F(TM \times_M TM) = FTM \times_{FM} FTM \longrightarrow T'FM \times_{FM} T'FM$$

*at each object M are induced by α_M on each factor. A morphism of tangent categories (F, α) is said to be **strong** if α is invertible, and **strict** if α is an identity.*

Morphisms of tangent categories are the arrows of a category [11, Def. 2.7], in which the composite of morphisms $(F, \alpha^F) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ and $(G, \alpha^G) : (\mathbb{C}', T') \to (\mathbb{C}'', T'')$ is $(GF, \alpha^F * \alpha^G)$, where we define $\alpha^F * \alpha^G = G(\alpha^F)\alpha^G_F : GFT \to T''GF$.

6.3. A strong morphism of tangent categories $F : (\mathbb{C}, T) \to (\mathbb{C}', T')$ sends each differential bundle $\mathsf{q} = (q : E \to M, +_{\mathsf{q}}, 0_{\mathsf{q}}, \lambda)$ in $\mathbb{C}$ to a differential bundle

$$F(\mathsf{q}) = (F(q), F(+_{\mathsf{q}}), F(0_{\mathsf{q}}), F(\lambda)\alpha^F_E)$$

in $\mathbb{C}'$, and this assignment is functorial with respect to linear morphisms of differential bundles [13, Prop. 4.22].

Lemma 6.4. *Let $(F, \alpha) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be a strong morphism of tangent categories. Then for each object M of $\mathbb{C}$ we have linear isomorphisms of differential bundles as follows:*

1. $(\alpha_M, 1_{FM}) \ : \ F(\mathsf{p}_M) \xrightarrow{\sim} \mathsf{p}'_{FM}$,
2. $(\alpha^{[2]}_M, \alpha_M) \ : \ F(\mathsf{p}_{TM}) \xrightarrow{\sim} \mathsf{p}'_{T'FM}$.
3. $(\alpha^{[2]}_M, \alpha_M) \ : \ FT(\mathsf{p}_M) \xrightarrow{\sim} T'(\mathsf{p}'_{FM})$,

Proof. The fact that 1 is a linear bundle morphism follows immediately from the axioms in 6.2. In particular, $(\alpha_{TM}, 1_{FTM}) : F(\mathsf{p}_{TM}) \to \mathsf{p}'_{FTM}$ is a linear isomorphism, but we also know that the isomorphism $\alpha_M : FTM \to T'FM$ induces a linear isomorphism $(T'(\alpha_M), \alpha_M) : \mathsf{p}'_{FTM} \to \mathsf{p}'_{T'FM}$, and by composition we obtain the linear isomorphism needed in 2. Also, the axioms for a tangent category yield linear isomorphisms $(c_M, 1_{TM}) : T(\mathsf{p}_M) \to \mathsf{p}_{TM}$ and $(c'_{FM}, 1_{T'FM}) : T'(\mathsf{p}'_{FM}) \to \mathsf{p}'_{T'FM}$, so by the functoriality in 6.3, and the fact that c' is an involution, we obtain a linear composite

$$FT(\mathsf{p}_M) \xrightarrow{(F(c_M), 1_{FTM})} F(\mathsf{p}_{TM}) \xrightarrow{(\alpha^{[2]}_M, \alpha_M)} \mathsf{p}'_{T'FM} \xrightarrow{(c'_{FM}, 1_{T'FM})} T'(\mathsf{p}'_{FM}),$$

which can be expressed equally as $(\alpha^{[2]}_M, \alpha_M)$ since $F(c_M)\alpha^{[2]}_M c'_{FM} = \alpha^{[2]}_M c'_{FM} c'_{FM} = \alpha^{[2]}_M$ by 6.2. □

Proposition 6.5. *Let $F : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be a strong morphism of tangent categories, and let $K : T^2M \to TM$ be a connection on M in $\mathbb{C}$. Then the composite*

$$K_F = \left(T'^2FM \xrightarrow{\alpha^{[-2]}_M} FT^2M \xrightarrow{F(K)} FTM \xrightarrow{\alpha_M} T'FM \right)$$

is a connection on FM, where $\alpha = \alpha^F$ and $\alpha^{[-2]} = (\alpha^{[2]})^{-1}$.

Proof. By 3.11 and 3.13, it suffices to show that K_F is a vertical connection and that

$$\begin{tikzcd} & T'^2FM \arrow[dl, "T'(p'_{FM})"'] \arrow[d, "p'_{T'FM}"] \arrow[dr, "K_F"] & \\ T'FM \arrow[dr, "p'_{FM}"'] & T'FM \arrow[d, "p'_{FM}"] & T'FM \arrow[dl, "p'_{FM}"] \\ & FM & \end{tikzcd} \tag{6.i}$$

is a fibre product diagram in $\mathbb{C}'$.

By 3.11, we know that the diagram (3.ii) presents T^2M as a third fibre power of $p_M : TM \to M$. But F preserves finite fibre powers of p_M, so F sends the diagram (3.ii) to a fibre product diagram

$$\begin{tikzcd} & FT^2M \arrow[dl, "FT(p_M)"'] \arrow[d, "F(p_{TM})"] \arrow[dr, "F(K)"] & \\ FTM \arrow[dr, "F(p_M)"'] & FTM \arrow[d, "F(p_M)"] & FTM \arrow[dl, "F(p_M)"] \\ & FM & \end{tikzcd}$$

in $\mathbb{C}'$. By composing with the isomorphism $\alpha_M^{[-2]} : T'^2FM \to FT^2M$, we find that the morphisms

$$\alpha_M^{[-2]}FT(p_M),\ \alpha_M^{[-2]}F(p_{TM}),\ \alpha_M^{[-2]}F(K) \quad : \quad T'^2FM \to FTM$$

present T'^2FM as a third fibre power of $F(p_M)$ in $\mathbb{C}'$. But $F(p_M) = \alpha_M p'_{FM} : FTM \to FM$ since (F, α) is a morphism of tangent categories, so since α is an isomorphism we deduce that the composites

$$f_1 := \alpha_M^{[-2]}FT(p_M)\alpha_M,\ f_2 := \alpha_M^{[-2]}F(p_{TM})\alpha_M,\ f_3 := \alpha_M^{[-2]}F(K)\alpha_M \quad : \quad T'^2FM \to T'FM$$

present T'^2FM as a third fibre power of $p'_{FM} : T'FM \to FM$ in $\mathbb{C}'$.

Hence, in order to show that (6.i) is a fibre product diagram, it suffices to show that $f_1 = T'(p'_{FM}), f_2 = p'_{T'FM}, f_3 = K_F$. The third of these equations holds by the definition of K_F. The first two equations also hold, because

$$\alpha_M^{[2]}T'(p'_{FM}) = \alpha_{TM}T'(\alpha_M)T'(p'_{FM}) = \alpha_{TM}T'F(p_M) = FT(p_M)\alpha_M$$

$$\alpha_M^{[2]}p'_{T'FM} = \alpha_{TM}T'(\alpha_M)p'_{T'FM} = \alpha_{TM}p'_{FTM}\alpha_M = F(p_{TM})\alpha_M$$

since (F, α) is a strong morphism and p' is natural.

Now it suffices to show that K_F is a vertical connection on p'_{FM}. Firstly, K_F is a retraction of $\ell'_{FM} : T'FM \to T'^2FM$ since $\ell'_{FM}K_F = \ell'_{FM}\alpha_M^{[-2]}F(K)\alpha_M = \alpha_M^{-1}F(\ell_M)F(K)\alpha_M = \alpha_M^{-1}F(\ell_M K)\alpha_M = 1_{T'FM}$, because $\ell_M K = 1_{TM}$. Hence it suffices to show that $(K_F, p'_{FM}) : T'(\mathsf{p}'_{FM}) \to \mathsf{p}'_{FM}$ and $(K_F, p'_{FM}) : \mathsf{p}'_{T'FM} \to \mathsf{p}'_{FM}$ are linear morphisms of differential bundles. But since K is a vertical connection on p_M, we know that $(K, p_M) : T(\mathsf{p}_M) \to \mathsf{p}_M$ and

$(K, p_M) : \mathsf{p}_{TM} \to \mathsf{p}_M$ are linear morphisms of differential bundles and so, by 6.3, are sent by F to linear morphisms of differential bundles

$$(F(K), F(p_M)) : FT(\mathsf{p}_M) \to F(\mathsf{p}_M)$$
$$(F(K), F(p_M)) : F(\mathsf{p}_{TM}) \to F(\mathsf{p}_M)\ .$$

Hence by composition with the linear isomorphisms in 6.4 we obtain linear bundle morphisms

$$T'(\mathsf{p}'_{FM}) \xrightarrow{(\alpha_M^{[-2]}, \alpha_M^{-1})} FT(\mathsf{p}_M) \xrightarrow{(F(K), F(p_M))} F(\mathsf{p}_M) \xrightarrow{(\alpha_M, 1_{FM})} \mathsf{p}'_{FM}$$

$$\mathsf{p}'_{T'FM} \xrightarrow{(\alpha_M^{[-2]}, \alpha_M^{-1})} F(\mathsf{p}_{TM}) \xrightarrow{(F(K), F(p_M))} F(\mathsf{p}_M) \xrightarrow{(\alpha_M, 1_{FM})} \mathsf{p}'_{FM}.$$

But the pair (K_F, p'_{FM}) underlies each of these two composites. ☐

Proposition 6.6. *Let $(F, \alpha) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be a strong morphism of tangent categories, and let $K : T^2M \to TM$ be a connection on an object M of $\mathbb{C}$. Then the horizontal connection associated to the connection K_F is the composite*

$$H_F = \left(T'_2FM \xrightarrow{\alpha_2^{-1}} FT_2M \xrightarrow{F(H)} FT^2M \xrightarrow{\alpha_M^{[2]}} T'^2FM \right),$$

recalling that α_2 and $\alpha^{[2]}$ are defined in 6.2.

Proof. By 3.12, it suffices to establish the following equations

$$H_FT'(p'_{FM}) = \pi_0, \quad H_Fp'_{T'FM} = \pi_1, \quad H_FK_F = p'_20'_{FM} \quad : \quad T'_2FM \longrightarrow T'FM,$$

but we know that H satisfies the analogous equations $HT(p_M) = \pi_0$, $Hp_{TM} = \pi_1$, $HK = p_20_M$. Hence we compute that

$$\begin{array}{rcll} H_FT'(p'_{FM}) & = & \alpha_2^{-1}F(H)\alpha_M^{[2]}T'(p'_{FM}) & \\ & = & \alpha_2^{-1}F(H)FT(p_M)\alpha_M & \text{(by 6.4(3))} \\ & = & \alpha_2^{-1}F(\pi_0)\alpha_M & \\ & = & \pi_0\alpha_M^{-1}\alpha_M & \text{(by the definition of } \alpha_2\text{)} \\ & = & \pi_0 & \end{array}$$

$$\begin{array}{rcll} H_Fp'_{T'FM} & = & \alpha_2^{-1}F(H)\alpha_M^{[2]}p'_{T'FM} & \\ & = & \alpha_2^{-1}F(H)F(p_{TM})\alpha_M & \text{(by 6.4(2))} \\ & = & \alpha_2^{-1}F(\pi_1)\alpha_M & \\ & = & \pi_1\alpha_M^{-1}\alpha_M & \text{(by the definition of } \alpha_2\text{)} \\ & = & \pi_1 & \\ H_FK_F & = & \alpha_2^{-1}F(H)\alpha_M^{[2]}\alpha_M^{[-2]}F(K)\alpha_M & \\ & = & \alpha_2^{-1}F(H)F(K)\alpha_M & \\ & = & \alpha_2^{-1}F(p_2)F(0_M)\alpha_M & \\ & = & \alpha_2^{-1}F(p_2)0'_{FM} & \text{(by 6.2)} \\ & = & p'_20'_{FM} & \end{array}$$

since it follows readily from 6.2 that $\alpha_2^{-1}F(p_2) = p'_2$. ☐

Proposition 6.7. *Let $F : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be a strong morphism of tangent categories, and let $K : T^2M \to TM$ be a connection on an object M of $\mathbb{C}$.*

1. *If K is torsion-free, then K_F is a torsion-free connection on FM.*
2. *If K is flat, then K_F is a flat connection on FM.*

Proof. If K is torsion-free, i.e. $c_M K = K$, then

$$c'_{FM}K_F = c'_{FM}\alpha_M^{[-2]}F(K)\alpha_M = \alpha_M^{[-2]}F(c_M)F(K)\alpha_M = \alpha_M^{[-2]}F(K)\alpha_M = K_F$$

by 6.2. Suppose that K is flat, i.e. $c_{TM}T(K)K = T(K)K : T^3M \to TM$. Then

$$\begin{array}{rcl} T'(K_F)K_F & = & T'(\alpha_M^{[-2]})T'F(K)T'(\alpha_M)T'(\alpha_M^{-1})\alpha_{TM}^{-1}F(K)\alpha_M \\ & = & T'(\alpha_M^{[-2]})T'F(K)\alpha_{TM}^{-1}F(K)\alpha_M \\ & = & T'(\alpha_M^{[-2]})\alpha_{T^2M}^{-1}FT(K)F(K)\alpha_M \\ & = & T'(\alpha_M^{[-2]})\alpha_{T^2M}^{-1}F(T(K)K)\alpha_M \\ & = & T'^2(\alpha_M^{-1})T'(\alpha_{TM}^{-1})\alpha_{T^2M}^{-1}F(T(K)K)\alpha_M \\ & = & T'^2(\alpha_M^{-1})\alpha_{TM}^{[-2]}F(T(K)K)\alpha_M \end{array}$$

by the naturality of α^{-1} and the definition of $\alpha^{[-2]}$. Hence

$$\begin{array}{rcl} c'_{T'FM}T'(K_F)K_F & = & c'_{T'FM}T'^2(\alpha_M^{-1})\alpha_{TM}^{[-2]}F(T(K)K)\alpha_M \\ & = & T'^2(\alpha_M^{-1})c'_{FTM}\alpha_{TM}^{[-2]}F(T(K)K)\alpha_M \\ & = & T'^2(\alpha_M^{-1})\alpha_{TM}^{[-2]}F(c_{TM})F(T(K)K)\alpha_M \\ & = & T'^2(\alpha_M^{-1})\alpha_{TM}^{[-2]}F(T(K)K)\alpha_M \\ & = & T'(K_F)K_F. \end{array}$$

by 6.2 and the naturality of c'. ☐

Proposition 6.8. *Every strong morphism of tangent categories $F : (\mathbb{C}, T) \to (\mathbb{C}', T')$ induces a functor*

$$F_* \;:\; \mathsf{Geom}(\mathbb{C}, T) \to \mathsf{Geom}(\mathbb{C}', T')$$

given on objects by $(M, K) \mapsto (FM, K_F)$ and on morphisms by $f \mapsto F(f)$. The analogous claims hold with each of $\mathsf{Geom}_{\mathsf{flat}}$, $\mathsf{Geom}_{\mathsf{tf}}$, and Aff replacing Geom, and in each case we shall denote the resulting functor also by F_*.

Proof. By 6.5 and 6.7, it suffices to show that if $f : (M, K) \to (M', K')$ is a morphism in $\mathsf{Geom}(\mathbb{C}, T)$, then $F(f) : (FM, K_F) \to (FM', K'_F)$ is a morphism in $\mathsf{Geom}(\mathbb{C}', T')$. But this follows immediately from the definitions, using the naturality of α and $\alpha^{[-2]}$. ☐

6.9. Given morphisms of tangent categories $F, G : (\mathbb{C}, T) \to (\mathbb{C}', T')$, a **tangent transformation** $\phi : F \Rightarrow G$ is a natural transformation such that $\alpha^F T'(\phi) = \phi_T \alpha^G$ [13, Def. 4.18]. It is straightforward to show that tangent transformations are closed under vertical composition and are closed under whiskering with morphisms of tangent categories. Hence, in view of 6.2, we obtain a 2-category Tan whose objects are tangent categories, whose 1-cells are strong morphisms, and whose 2-cells are tangent transformations.

Lemma 6.10. *Let $F, G : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be strong morphisms of tangent categories, and let $\phi : F \Rightarrow G$ be a tangent transformation. Then for any connection K on an object M of $\mathbb{C}$, the component ϕ_M underlies a morphism*

$$\phi_M \;:\; (FM, K_F) \longrightarrow (GM, K_G) \tag{6.ii}$$

in $\mathsf{Geom}(\mathbb{C}', T')$. Further, there is a natural transformation

$$\phi_* \;:\; F_* \Rightarrow G_* \;:\; \mathsf{Geom}(\mathbb{C}, T) \to \mathsf{Geom}(\mathbb{C}', T')$$

whose component at each object (M, K) of $\mathsf{Geom}(\mathbb{C}, T)$ is the morphism (6.ii).

Proof. The first claim follows immediately from the naturality of α^F and α^G, and the second is immediate. $\square$

Theorem 6.11. *There are 2-functors*

$$\mathsf{Geom},\ \mathsf{Geom}_{\mathsf{flat}},\ \mathsf{Geom}_{\mathsf{tf}},\ \mathsf{Aff} \;:\; \mathrm{Tan} \to \mathrm{Cat}$$

from the 2-category Tan *of tangent categories (6.9) to the 2-category* Cat *of categories, sending each tangent category $(\mathbb{C}, T)$ to $\mathsf{Geom}(\mathbb{C}, T)$, $\mathsf{Geom}_{\mathsf{flat}}(\mathbb{C}, T)$, $\mathsf{Geom}_{\mathsf{tf}}(\mathbb{C}, T)$, and $\mathsf{Aff}(\mathbb{C}, T)$, respectively. These 2-functors are given on 1-cells by 6.8 and on 2-cells by 6.10.*

Proof. By employing the definitions, as well as the middle-interchange law for Cat, it is straightforward to verify the needed functoriality on 1-cells. Functoriality with respect to vertical composition of 2-cells is immediate, as is the preservation of whiskering by Geom (and hence by the others). $\square$

We now apply this theorem in order to show that $\mathsf{Geom}(\mathbb{C}, T)$ and $\mathsf{Aff}(\mathbb{C}, T)$ are tangent categories, by way of the following general lemma.

Lemma 6.12. *Let $(\mathbb{C}, T, +, 0, \ell, c)$ be a tangent category.*

1. [13] *$(T, c) : (\mathbb{C}, T) \to (\mathbb{C}, T)$ is a strong morphism of tangent categories.*
2. [13] *$(T_n, c_n) : (\mathbb{C}, T) \to (\mathbb{C}, T)$ is a strong morphism of tangent categories for each natural number n, where $c_n : T_nT \to TT_n$ is the unique morphism such that $c_nT(\pi_i) = \pi_i c$ for each $i = 0, ..., n-1$ when we write $\pi_i : T_n \to T$ to denote the projection.*
3. *The following are tangent transformations*

$$p : (T, c) \Longrightarrow (1, 1_T), \quad + : (T_2, c_2) \Longrightarrow (T, c), \quad 0 : (1, 1_T) \Longrightarrow (T, c),$$

$$\ell : (T, c) \Longrightarrow (T, c)^2, \quad c : (T, c)^2 \Longrightarrow (T, c)^2, \quad \pi_i : (T_n, c_n) \Longrightarrow (T, c)$$

for all natural numbers n, i with $i < n$, where $(T, c)^2 = (T, c) \circ (T, c) = (T^2, c * c)$ is the composite 1-cell in Tan, *where $c * c = T(c)c_T : T^3 \to T^3$ (6.2).*

Proof. It suffices to prove 3. Firstly, $p, 0, \ell, c$ are tangent transformations since $cT(p) = p_T = p_T 1_T$, $1_T T(0) = T(0) = 0_T c$, $\ell_T(c * c) = \ell_T T(c) c_T = cT(\ell)$, and $c_T(c * c) = c_T T(c) c_T = T(c) c_T T(c) = (c * c) T(c)$, by the axioms for a tangent category. The definition of c_n immediately entails that each π_i is a tangent transformation. With regard to $+$, one of the axioms for a tangent category entails that $(c, 1_{TM}) : \mathsf{p}_{TM} \to T(\mathsf{p}_M)$ is an additive bundle morphism, so by the definition of c_2 we deduce that

$$\begin{tikzcd} T_2TM \arrow[r, "(c_2)_M"] \arrow[d, "+_{TM}"'] & TT_2M \arrow[d, "T(+_M)"] \\ TTM \arrow[r, "c_M"'] & TTM \end{tikzcd}$$

commutes. $\square$

Corollary 6.13. *Given a tangent category* $(\mathbb{C}, T)$*, we can apply the 2-functor* $\mathsf{Aff} : \mathrm{Tan} \to \mathrm{Cat}$ *to the 1-cells* $T, T_n : (\mathbb{C}, T) \to (\mathbb{C}, T)$ *and 2-cells* $p, +, 0, \ell, c$ *in* Tan *in order to obtain functors*

$$T_*,\ (T_n)_*\ :\ \mathsf{Aff}(\mathbb{C}, T) \to \mathsf{Aff}(\mathbb{C}, T)$$

and natural transformations

$$p_* : T_* \Longrightarrow 1, \quad +_* : (T_2)_* \Longrightarrow T_*, \quad 0_* : 1 \Longrightarrow T_*$$

$$\ell_* : T_* \Longrightarrow T_*^2, \quad c_* : T_*^2 \Longrightarrow T_*^2, \quad (\pi_i)_* : (T_n)_* \Longrightarrow T_*$$

for all natural numbers n, i *with* $i < n$*. We can similarly apply* $\mathsf{Geom}, \mathsf{Geom}_{\mathsf{flat}}, \mathsf{Geom}_{\mathsf{tf}}$ *to the same data in order to obtain endofunctors and natural transformations, for which we employ the same notations.*

In order to show that 6.13 yields a tangent structure on $\mathsf{Aff}(\mathbb{C}, T)$, we shall need certain finite limits in the latter category. To this end we shall employ the following:

Lemma 6.14. *Let* $D : \mathbb{J} \to \mathsf{Geom}(\mathbb{C}, T)$ *be a functor, and let* $\pi = (\pi_j : L \to Dj)_{j \in \mathbb{J}}$ *be a cone on* D*. Writing* $U : \mathsf{Geom}(\mathbb{C}, T) \to \mathbb{C}$ *for the forgetful functor, suppose that* π *is sent by* U *to a limit cone for* UD *that is preserved by* T^k *for each natural number* k*. Then*

(i) π *is a limit cone for* D*,*

(ii) this limit is preserved by each of the endofunctors T_*^k *on* $\mathsf{Geom}(\mathbb{C}, T)$*.*

Proof. Let us write $Dj = (UDj, K_j)$ for each object j of $\mathbb{J}$, and write $L = (L_0, L_1)$. Given any cone $(f_j : (M, K) \to Dj)_{j \in \mathbb{J}}$ on D, we know that $(f_j : M \to UDj)_{j \in \mathbb{J}}$ is a cone on UD and hence induces a morphism $f : M \to L_0$ in $\mathbb{C}$. For each $j \in \mathsf{ob}\,\mathbb{J}$ we compute that

$$KT(f)T(\pi_j) = KT(f_j) = T^2(f_j)K_j = T^2(f)T^2(\pi_j)K_j = T^2(f)L_1T(\pi_j)$$

since f_j and π_j are morphisms in $\mathsf{Geom}(\mathbb{C}, T)$, so since $(T(\pi_j) : TL_0 \to TUDj)_{j \in \mathbb{J}}$ is a limit cone in $\mathbb{C}$ we deduce that $KT(f) = T^2(f)L_1$. Hence $f : (M, K) \to L$ is a morphism in $\mathsf{Geom}(\mathbb{C}, T)$. Thus (i) is proved.

For each natural number k, we know that $T_*^k(\pi) = (T_*^k(\pi_j))_{j \in \mathbb{J}}$ is a cone on the diagram $T_*^k D$ and is sent by U to a limit cone $(T^k(\pi_j) : T^k L_0 \to T^k UDj)_{j \in \mathbb{J}}$ for the diagram $UT_*^k D = T^k UD : \mathbb{J} \to \mathbb{C}$. Further, the latter limit is preserved by $T^{k'}$ for each natural number k', so we can apply (i) to the cone $T_*^k(\pi)$ in order to deduce that $T_*^k(\pi)$ is a limit cone for $T_*^k D$. □

The preceding lemma immediately entails the following:

Lemma 6.15. *Let $(\mathbb{C}, T)$ be a tangent category. Then for each natural number n and each object M of $\mathsf{Aff}(\mathbb{C}, T)$, the morphisms $(\pi_i)_{*M} : (T_n)_* M \to T_* M$ present $(T_n)_* M$ as an n-th fibre power of $p_{*M} : T_* M \to M$ in $\mathsf{Aff}(\mathbb{C}, T)$, and this fibre power is preserved by $T_*^k : \mathsf{Aff}(\mathbb{C}, T) \to \mathsf{Aff}(\mathbb{C}, T)$ for each natural number k. The analogous claims hold with each of $\mathsf{Geom}, \mathsf{Geom}_{\mathsf{flat}}, \mathsf{Geom}_{\mathsf{tf}}$ in place of Aff.*

Theorem 6.16. *Let $(\mathbb{C}, T)$ be a tangent category. Then each of the categories $\mathsf{Geom}(\mathbb{C}, T)$, $\mathsf{Geom}_{\mathsf{flat}}(\mathbb{C}, T)$, $\mathsf{Geom}_{\mathsf{tf}}(\mathbb{C}, T)$, and $\mathsf{Aff}(\mathbb{C}, T)$ is a tangent category when equipped with its endofunctor T_* and natural transformations $p_*, 0_*, +_*, \ell_*, c_*$ as defined in 6.13.*

Proof. All of the needed structure is furnished by 6.13 and 6.15. This structure satisfies the equational axioms for a tangent category, by the 2-functoriality of Geom, $\mathsf{Geom}_{\mathsf{flat}}$, $\mathsf{Geom}_{\mathsf{tf}}$, and $\mathsf{Aff} : \mathrm{Tan} \to \mathrm{Cat}$, so it remains only to verify the *universality of the vertical lift* [13, Def. 2.1]. It suffices to treat the case of $\mathsf{Geom}(\mathbb{C}, T)$, from which the needed property of each of the other categories then follows. For each object M of the category $\mathbb{D} = \mathsf{Geom}(\mathbb{C}, T)$, we must show that a particular commutative square S in $\mathbb{D}$ is a pullback that is preserved by each T_*^n [13, Def. 2.1], where S is defined in terms of the (candidate) tangent structure on $\mathbb{D}$. But the square S is sent by the forgetful functor $U : \mathbb{D} \to \mathbb{C}$ to the similarly defined square in $\mathbb{C}$, which we know is a pullback in $\mathbb{C}$ that is preserved by each T^n. Hence by 6.14 we deduce that S is a pullback square in $\mathbb{D}$ that is preserved by each T_*^n. □

Hence Theorem 6.1 is proved.

Remark 6.17. Given an object (M, K) of $\mathsf{Geom}(\mathbb{C}, T)$, recall that $T_*(M, K) = (TM, K_T)$ (6.8). By 6.5 we obtain the following explicit formula for the connection $K_T : T^3 M \to T^2 M$ on TM:

$$K_T = c_M^{[-2]} T(K) c_M = (c_{TM} T(c_M))^{-1} T(K) c_M = T(c_M) c_{TM} T(K) c_M.$$

For brevity, we will often write this formula as

$$K_T = T(c) c T(K) c.$$

Letting H be the horizontal connection induced by K, we deduce by 6.6 that the associated horizontal connection $H_T : T_2 TM \to T^3 M$ induced by K_T is

$$H_T = (c_M \times c_M) T(H) c_M^{[2]} = (c_M \times c_M) T(H) c_{TM} T(c_M)$$

where $c_M \times c_M : T_2TM = T^2M \times_{TM} T^2M \to T^2M \times_{TM} T^2M = TT_2M$ is induced by c_M on each factor. For brevity, we write this formula also as

$$H_T = (c \times c)T(H)cT(c).$$

The following results will be useful when working with the maps K_T and H_T:

Lemma 6.18. *If $(M, K) \in \mathsf{Geom}(\mathbb{C}, T)$, then:*

(i) $K_T T(p) = T^2(p)K$*;*

(ii) $T(\ell)K_T = K\ell$.

Proof. (i) asserts precisely that $p_M : T_*(M, K) = (TM, K_T) \to (M, K)$ is a morphism in $\mathsf{Geom}(\mathbb{C}, T)$, but this is immediate from 6.13/6.16 since we have a natural transformation $p_* : T_* \Rightarrow 1 : \mathsf{Geom}(\mathbb{C}, T) \to \mathsf{Geom}(\mathbb{C}, T)$ with components $p_{*(M,K)} = p_M$. For (ii) we compute that

$$\begin{aligned} & T(\ell)K_T \\ = \; & T(\ell)T(c)cT(K)c \\ = \; & T(\ell c)cT(K)c \\ = \; & T(\ell)cT(K)c \\ = \; & K\ell c \text{ (by linearity of } K) \\ = \; & K\ell \end{aligned}$$

as required. ☐

Lemma 6.19. *If $(M, K) \in \mathsf{Geom}(\mathbb{C}, T)$ then*

$$H_T T^2(p) = (T(p) \times T(p))H.$$

Proof. We have

$$\begin{aligned} & H_T T^2(p) \\ = \; & (c \times c)T(H)cT(c)T^2(p) \\ = \; & (c \times c)T(H)cT(cT(p)) \\ = \; & (c \times c)T(H)cT(p) \\ = \; & (c \times c)T(H)p \\ = \; & (c \times c)pH \\ = \; & (cp \times cp)H \\ = \; & (T(p) \times T(p))H \end{aligned}$$

as required. ☐

6.1 Alternate characterizations of flat torsion-free connections

The fact that a connection on an object M can be lifted to a connection on TM (and then on T^2M, etc.) leads to two alternate characterizations of when a connection is flat torsion-free. One of these characterizations ((iii) in the result below) effectively says that "a connection is flat torsion-free if and only if it is connection-preserving". If we think of a connection $K : T^2M \longrightarrow TM$ as a 'multiplication', the second characterization says that this operation is 'associative'. These characterizations appear to be new in standard differential geometry.

Theorem 6.20. *Suppose that $(M, K) \in \mathsf{Geom}(\mathbb{C}, T)$. Then the following are equivalent:*

(i) K is flat and torsion-free.

(ii) $K_T K = T(K)K$.

(iii) K is a morphism in $\mathsf{Geom}(\mathbb{C}, T)$ from (T^2M, K_{T^2}) to (TM, K_T).

Proof. We first prove that (i) implies (ii). Assuming that K is flat and torsion-free, consider

$$\begin{aligned} & K_T K \\ = \; & T(c)cT(K)cK \\ = \; & T(c)cT(K)K \text{ (since } K \text{ torsion-free)} \\ = \; & T(c)T(K)K \text{ (since } K \text{ flat)} \\ = \; & T(cK)K \\ = \; & T(K)K \text{ (since } K \text{ torsion-free).} \end{aligned}$$

so that we have (ii).

Next, we prove that (i) implies (iii). Assuming that K satisfies (i), we can apply Theorem 6.16 and Remark 6.17 to deduce that K_T also satisfies (i). Hence, since we have already proved that (i) implies (ii), we deduce that both K and K_T satisfy (ii), a fact that we shall use in the following computations. For K to be a morphism in $\mathsf{Geom}(\mathbb{C}, T)$ between the objects in (iii), we must show that

$$K_{T^2}T(K) = T^2(K)K_T.$$

These are both maps into T^2M. Now by Corollary 3.11, T^2M is the fibre product of three copies of TM, with projections $K, T(p), p$. So, to show the equality of the above maps, it suffices to show their equality when followed by these three projections. For the equality with K, consider

$$\begin{aligned}
& K_{T^2}T(K)K \\
= \;& K_{T^2}K_TK \text{ (by (ii))} \\
= \;& T(K_T)K_TK \text{ (by (ii), applied to } K_T) \\
= \;& T(K_T)T(K)K \text{ (by (ii))} \\
= \;& T(K_TK)K \\
= \;& T(T(K)K)K \text{ (by (ii))} \\
= \;& T^2(K)T(K)K \\
= \;& T^2(K)K_TK \text{ (by (ii))}
\end{aligned}$$

For the equality with p, consider

$$\begin{aligned}
& K_{T^2}T(K)p \\
= \;& K_{T^2}pK \text{ (by naturality of } p) \\
= \;& ppK \text{ (by definition of a connection)} \\
= \;& T^2(K)pp \text{ (by naturality of } p) \\
= \;& T^2(K)K_Tp \text{ (by definition of a connection)}
\end{aligned}$$

Finally, for the equality with $T(p)$, consider

$$\begin{aligned}
& K_{T^2}T(K)T(p) \\
= \;& K_{T^2}T(Kp) \\
= \;& K_{T^2}T(pp) \text{ (by definition of a connection)} \\
= \;& K_{T^2}T(p)T(p) \\
= \;& T^2(p)K_TT(p) \text{ (by 6.18(i))} \\
= \;& T^2(p)T^2(p)K \text{ (by 6.18(i))} \\
= \;& T^2(pp)K \\
= \;& T^2(Kp)K \text{ (by definition of a connection)} \\
= \;& T^2(K)T^2(p)K \\
= \;& T^2(K)K_TT(p) \text{ (by 6.18(i))}
\end{aligned}$$

as required.

We will now prove (iii) implies (ii). Suppose that $T^2(K)K_T = K_{T^2}T(K)$. Then composing both sides of the equation on the left by $T(\ell)$ and on the right by K, we get

$$\begin{aligned}
T(\ell)T^2(K)K_TK &= T(\ell)K_{T^2}T(K)K \\
T(\ell T(K))K_TK &= K_T\ell T(K)K \text{ (using lemma 6.18, applied to } K_T) \\
T(K\ell)K_TK &= K_TK\ell K \text{ (linearity of } K) \\
T(K)T(\ell)K_TK &= K_TK \text{ (definition of connection)} \\
T(K)K\ell K &= K_TK \text{ (by lemma 6.18)} \\
T(K)K &= K_TK \text{ (definition of a connection)}
\end{aligned}$$

so that we have (ii).

Finally, we will show that (ii) implies (i). Suppose that $K_TK = T(K)K$; in other words,

$$T(c)cT(K)cK = T(K)K \ (\star).$$

Composing both sides of this equation on the left by $T(\ell)c$ gives

$$\begin{aligned}
T(\ell)cT(c)cT(K)cK &= T(\ell)cT(K)K \\
c\ell cT(K)cK &= T(\ell)cT(K)K \text{ (by coherence of } \ell \text{ and } c) \\
c\ell T(K)cK &= T(\ell)cT(K)K \\
cK\ell cK &= K\ell K \text{ (by both linearities of } K) \\
cK\ell K &= K \text{ (by definition of a connection)} \\
cK &= K \text{ (by definition of a connection)}
\end{aligned}$$

So we have proven that K is torsion-free. Applying $cK = K$ to $\star$, we get

$$T(c)cT(K)K = T(K)K.$$

Now apply $T(c)$ to both sides of this equation to get

$$\begin{aligned}
T(c)T(c)cT(K)K &= T(c)T(K)K \\
cT(K)K &= T(cK)K \\
cT(K)K &= T(K)K \text{ (since } K \text{ torsion-free)}
\end{aligned}$$

so that K is flat. Thus, we have proven (i).

Altogether, we have proven

$$(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i),$$

and so all three conditions are equivalent. □

This result also allows us to prove several useful results about objects in the affine category.

Corollary 6.21. *If* $(M, K) \in \mathsf{Aff}(\mathbb{C}, T)$ *then:*

(i) $K_T K = T(K)K$*;*

(ii) K *is a morphism in* $\mathsf{Aff}(\mathbb{C}, T)$ *from* (T^2M, K_{T^2}) *to* (TM, K_T) *and provides* (M, K) *with the structure of a flat torsion-free connection in* $\mathsf{Aff}(\mathbb{C}, T)$*.*

(iii) The maps from (ii) form the components of a natural transformation from T_*^2 *to* $T_* : \mathsf{Aff}(\mathbb{C}, T) \longrightarrow \mathsf{Aff}(\mathbb{C}, T)$*.*

Proof. (i) and the fact that K is a morphism in $\mathsf{Aff}(\mathbb{C}, T)$ were proved in the theorem. Moreover, as the tangent structure on $\mathsf{Aff}(\mathbb{C}, T)$ is lifted from $(\mathbb{C}, T)$, this also shows that K is a flat torsion-free connection on (M, K) in the tangent category $\mathsf{Aff}(\mathbb{C}, T)$.

Finally, that these maps form a natural transformation from T_*^2 to T_* on $\mathsf{Aff}(\mathbb{C}, T)$ follows directly from the definition of maps in $\mathsf{Aff}(\mathbb{C}, T)$, namely that such maps preserve the associated connections of the objects. □

7 The 2-comonad of affine geometric spaces

In 6.11 we saw that there are 2-functors $\mathsf{Geom}, \mathsf{Aff} : \mathrm{Tan} \to \mathrm{Cat}$ that send each tangent category $(\mathbb{C}, T)$ to the categories of geometric spaces and affine geometric spaces in $(\mathbb{C}, T)$, respectively. As a consequence we found that $\mathsf{Geom}(\mathbb{C}, T)$ and $\mathsf{Aff}(\mathbb{C}, T)$ are tangent categories (6.16), so it is natural to wonder whether Geom and Aff lift to 2-functors valued in Tan. We now address this question, and we show that Aff underlies a 2-comonad, whose coalgebras are tangent categories whose objects carry affine geometric structure. Here we employ the standard notion of (strict) 2-monad (as employed, for example, in [6]).

Definition 7.1. *Let* $(\mathbb{C}, T)$ *be a tangent category. Each of the following fibre products in* $\mathbb{C}$ *will be called a* ***basic fibre product*** *in* $(\mathbb{C}, T)$*: (1) Each fibre product of the form* T_nM*, and (2) each pullback witnessing the* universality of the vertical lift *[13, Def. 2.1]. A* ***class of endemic fibre products*** *in* $(\mathbb{C}, T)$ *is a class* $\mathcal{F}$ *of finite fibre product diagrams in* $\mathbb{C}$ *that is closed under the application of* T *and contains each basic fibre product. There is clearly a smallest class of endemic fibre products in* $(\mathbb{C}, T)$*, consisting of the fibre product diagrams obtained by repeatedly applying* T *to the basic fibre product diagrams.*

Concretely, we shall represent finite fibre product diagrams in $\mathbb{C}$ as certain functors $D : \mathbb{J}_n \to \mathbb{C}$ on categories $\mathbb{J}_n$ defined as follows. For each natural number n, $\mathbb{J}_n$ is a partially ordered set with $n + 2$ distinct elements $1, 2, ..., n, \perp, \top$, in which $\perp$ is a bottom element, $\top$ is a top element, and the remaining elements $1, 2, ..., n$ are mutually incomparable.

Definition 7.2. *There is a 2-category* Tan_e *whose objects* $(\mathbb{C}, T, \mathcal{F})$ *are tangent categories with a given class of endemic fibre products* $\mathcal{F}$*. A 1-cell* $F : (\mathbb{C}, T, \mathcal{F}) \to (\mathbb{C}', T', \mathcal{F}')$ *in* Tan_e *is a strong morphism of tangent categories that preserves endemic fibre products, i.e. sends fibre product diagrams in* $\mathcal{F}$ *to fibre product diagrams in* $\mathcal{F}'$*. The 2-cells in* Tan_e *are simply tangent transformations.*

Proposition 7.3. *Let $(\mathbb{C}, T, \mathcal{F})$ be a tangent category with endemic fibre products, and let $U = U_{\mathbb{C}} : \mathsf{Geom}(\mathbb{C}, T) \to \mathbb{C}$ denote the forgetful functor. Then $\mathsf{Geom}(\mathbb{C}, T)$ carries a class of endemic fibre products $U^*(\mathcal{F})$, consisting of all diagrams of the form $D : \mathbb{J}_n \to \mathsf{Geom}(\mathbb{C}, T)$ with $UD \in \mathcal{F}$. The functor U underlies a strict morphism of tangent categories, which in turn underlies a 1-cell $U : (\mathsf{Geom}(\mathbb{C}, T), T_*, U^*(\mathcal{F})) \to (\mathbb{C}, T, \mathcal{F})$ in Tan_e. Further, the analogous results hold with each of Aff, $\mathsf{Geom}_{\mathsf{flat}}$, and $\mathsf{Geom}_{\mathsf{tf}}$ in place of Geom.*

Proof. Given any diagram $D : \mathbb{J}_n \to \mathsf{Geom}(\mathbb{C}, T)$ in $U^*(\mathcal{F})$, we know that $UD \in \mathcal{F}$ and hence $T^k UD \in \mathcal{F}$ for every $k \in \mathbb{N}$, by induction on k. Therefore UD is a fibre product diagram in $\mathbb{C}$ that is preserved by each T^k, so by 6.14 we deduce that D is a fibre product diagram in $\mathsf{Geom}(\mathbb{C}, T)$. Note also that $T_*D \in U^*(\mathcal{F})$, since $UT_*D = TUD \in \mathcal{F}$.

Hence $U^*(\mathcal{F})$ is a class of fibre product diagrams in $\mathsf{Geom}(\mathbb{C}, T)$, and $U^*(\mathcal{F})$ is closed under the application of T_*. In view of the construction of the tangent structure on $\mathsf{Geom}(\mathbb{C}, T)$ in 6.16 and 6.15, it is clear that the basic fibre products in $\mathsf{Geom}(\mathbb{C}, T)$ are sent by U to basic fibre products in $(\mathbb{C}, T)$ and hence lie in $U^*(\mathcal{F})$. $\square$

We shall now prove that Geom lifts to a 2-endofunctor on Tan_e. We begin with the following general observation:

Lemma 7.4. *Let $(F, \alpha) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be a morphism of tangent categories. Then $\alpha : FT \Rightarrow T'F$ underlies a tangent transformation*

$$\alpha \ : \ (F, \alpha) \circ (T, c) \Longrightarrow (T', c') \circ (F, \alpha) \ : \ (\mathbb{C}, T) \to (\mathbb{C}', T')$$

where c, c' denote the canonical flips carried by $\mathbb{C}, \mathbb{C}'$, respectively (cf. 6.12).

Proof. Using the definition of composition of morphisms of tangent categories (6.2), we first note that $(F, \alpha) \circ (T, c) = (FT, c*\alpha)$ and $(T', c') \circ (F, \alpha) = (T'F, \alpha * c')$ where $c*\alpha = F(c)\alpha_T : FTT \Rightarrow T'FT$ and $\alpha * c' = T'(\alpha)c'_F : T'FT \Rightarrow T'T'F$. Hence it suffices to show that the diagram

$$\begin{tikzcd} FTT \arrow[r, "c*\alpha"] \arrow[d, "\alpha_T"'] & T'FT \arrow[d, "T'(\alpha)"] \\ T'FT \arrow[r, "\alpha*c'"'] & T'T'F \end{tikzcd}$$

commutes. Indeed,

$$\alpha_T(\alpha * c') = \alpha_T T'(\alpha)c'_F = \alpha^{[2]}c'_F = F(c)\alpha^{[2]} = F(c)\alpha_T T'(\alpha) = (c * \alpha)T'(\alpha)$$

since (F, α) is a morphism of tangent categories (6.2). $\square$

Theorem 7.5. *There is a 2-functor*

$$\mathsf{Geom} \ : \ \mathrm{Tan}_e \to \mathrm{Tan}_e$$

sending each tangent category with endemic fibre products $(\mathbb{C}, T, \mathcal{F})$ to the tangent category $\mathsf{Geom}(\mathbb{C}, T)$ of geometric spaces in $(\mathbb{C}, T)$, equipped with its associated class of endemic fibre products $U^*(\mathcal{F})$ (7.3). Similarly, there are 2-functors $\mathsf{Geom}_{\mathsf{flat}}, \mathsf{Geom}_{\mathsf{tf}}, \mathsf{Aff} : \mathrm{Tan}_e \to \mathrm{Tan}_e$ sending $(\mathbb{C}, T, \mathcal{F})$ to the tangent categories of flat, torsion-free, and affine geometric spaces in $(\mathbb{C}, T)$, respectively.

Proof. We shall treat the case of Geom; the other 2-functors are obtained similarly, using 6.11. Letting $F : (\mathbb{C}, T, \mathcal{F}) \to (\mathbb{C}', T', \mathcal{F}')$ be a 1-cell in Tan_e, we know that the associated isomorphism $\alpha^F : FT \Rightarrow T'F$ is a tangent transformation and so is a 1-cell in Tan. Hence we can apply $\mathsf{Geom} : \mathrm{Tan} \to \mathrm{Cat}$ to α^F in order to obtain an invertible 2-cell $\alpha^F_* : F_*T_* \Rightarrow T'_*F_* : \mathsf{Geom}(\mathbb{C}, T) \to \mathsf{Geom}(\mathbb{C}', T')$ in Cat. We claim that $(F_*, \alpha^F_*) : \mathsf{Geom}(\mathbb{C}, T) \longrightarrow \mathsf{Geom}(\mathbb{C}', T')$ is a 1-cell in Tan_e. Indeed, employing the notation of 7.3, we reason that for each $D \in U^*_{\mathbb{C}}(\mathcal{F})$ the composite F_*D lies in $U^*_{\mathbb{C}'}(\mathcal{F}')$, since $U_{\mathbb{C}'}F_*D = FU_{\mathbb{C}}D \in \mathcal{F}'$ because $U_{\mathbb{C}}D \in \mathcal{F}$. Hence F_* preserves endemic fibre products and so, in particular, sends basic fibre product diagrams to fibre product diagrams. Also, since $\mathsf{Geom} : \mathrm{Tan} \to \mathrm{Cat}$ is 2-functorial and F is a strong morphism of tangent categories, it follows that (F_*, α^F_*) satisfies the equational axioms for a morphism of tangent categories (6.2).

This defines the needed assignment on 1-cells, and the functoriality of this assignment readily follows from the 2-functoriality of $\mathsf{Geom} : \mathrm{Tan} \to \mathrm{Cat}$. Given a 2-cell $\phi : F \Rightarrow G : (\mathbb{C}, T, \mathcal{F}) \to (\mathbb{C}', T', \mathcal{F}')$ in Tan_e, we can apply $\mathsf{Geom} : \mathrm{Tan} \to \mathrm{Cat}$ to obtain a natural transformation $\phi_* : F_* \Rightarrow G_* : \mathsf{Geom}(\mathbb{C}, T) \to \mathsf{Geom}(\mathbb{C}', T')$ in Cat, which is in fact a tangent transformation

$$\phi_* : (F_*, \alpha^F_*) \Rightarrow (G_*, \alpha^G_*) : \mathsf{Geom}(\mathbb{C}, T) \to \mathsf{Geom}(\mathbb{C}', T')$$

since $\phi_{*T_*}\alpha^G_* = (\phi_T\alpha^G)_* = (\alpha^F T'(\phi))_* = \alpha^F_* T'_*(\phi_*)$ by the 2-functoriality of $\mathsf{Geom} : \mathrm{Tan} \to \mathrm{Cat}$. Again using the latter 2-functoriality, the result now follows. $\square$

Theorem 7.6. *There is a 2-comonad $\mathbb{A}\mathsf{ff} = (\mathsf{Aff}, \varepsilon, \delta)$ on Tan_e whose underlying 2-functor*

$$\mathsf{Aff} : \mathrm{Tan}_e \to \mathrm{Tan}_e$$

sends each tangent category with endemic fibre products, $(\mathbb{C}, T, \mathcal{F})$, to the tangent category $\mathsf{Aff}(\mathbb{C}, T)$ of affine geometric spaces in $(\mathbb{C}, T)$. The counit 1-cell

$$\varepsilon_{(\mathbb{C},T,\mathcal{F})} : \mathsf{Aff}(\mathbb{C}, T) \to (\mathbb{C}, T)$$

in Tan_e is the forgetful functor, and the comultiplication 1-cell

$$\delta_{(\mathbb{C},T,\mathcal{F})} : \mathsf{Aff}(\mathbb{C}, T) \to \mathsf{Aff}(\mathsf{Aff}(\mathbb{C}, T)) \tag{7.i}$$

sends each affine geometric space (M, K) in $(\mathbb{C}, T)$ to the affine geometric space $((M, K), K)$ in $\mathsf{Aff}(\mathbb{C}, T)$.

Proof. By 7.3, we know that each forgetful functor $\varepsilon_{(\mathbb{C},T,\mathcal{F})}$ is a strict morphism of tangent categories and is also a 1-cell in Tan_e. Further, it is immediate from the definitions that this defines a 2-natural transformation $\varepsilon : \mathsf{Aff} \to 1_{\mathrm{Tan}_e}$.

With regard to the comultiplication δ, recall that if (M, K) is an affine geometric space in a tangent category $(\mathbb{C}, T)$, then $K : T^2_*(M, K) \to T_*(M, K)$ is a flat torsion-free connection on (M, K) in $\mathsf{Aff}(\mathbb{C}, T)$ (6.21), so $((M, K), K)$ is an affine geometric space in $\mathsf{Aff}(\mathbb{C}, T)$, i.e. an object of $\mathsf{Aff}(\mathsf{Aff}(\mathbb{C}, T))$. For each object $(\mathbb{C}, T, \mathcal{F})$ of Tan_e, this defines $\delta_{(\mathbb{C},T,\mathcal{F})}$

on objects. Given a morphism $f : (M, K) \to (M', K')$ in $\mathsf{Aff}(\mathbb{C}, T)$, it is immediate that $\delta_{(\mathbb{C},T,\mathcal{F})}(f) = f : ((M, K), K) \to ((M', K'), K')$ defines a morphism in $\mathsf{Aff}(\mathsf{Aff}(\mathbb{C}, T))$. Thus we obtain a functor $\delta_{(\mathbb{C},T,\mathcal{F})}$ as in (7.i). The diagram of functors

$$\begin{tikzcd} \mathsf{Aff}(\mathbb{C},T) \arrow[r, "\delta_{(\mathbb{C},T,\mathcal{F})}"] \arrow[dr, equal] & \mathsf{Aff}(\mathsf{Aff}(\mathbb{C},T)) \arrow[d, "\varepsilon_{\mathsf{Aff}(\mathbb{C},T)}"] \\ & \mathsf{Aff}(\mathbb{C},T) \end{tikzcd} \tag{7.ii}$$

clearly commutes, and $\varepsilon_{\mathsf{Aff}(\mathbb{C},T)}$ is a strict morphism of tangent categories and is also a faithful functor, so it follows that $\delta_{(\mathbb{C},T,\mathcal{F})}$ is a strict morphism of tangent categories. Using the commutativity of this diagram, we also find that $\delta_{(\mathbb{C},T,\mathcal{F})}$ preserves endemic fibre products (since $\varepsilon_{\mathsf{Aff}(\mathbb{C},T)}$ *reflects* endemic fibre products). Hence $\delta_{(\mathbb{C},T,\mathcal{F})}$ is a 1-cell in Tan_e.

This defines a natural transformation $\delta : \mathsf{Aff} \to \mathsf{Aff} \circ \mathsf{Aff}$, since if $F : (\mathbb{C}, T, \mathcal{F}) \to (\mathbb{C}', T', \mathcal{F}')$ is a 1-cell in Tan_e then the diagram

$$\begin{tikzcd} \mathsf{Aff}(\mathbb{C},T) \arrow[r, "\delta"] \arrow[d, "F_*"'] & \mathsf{Aff}(\mathsf{Aff}(\mathbb{C},T)) \arrow[d, "(F_*)_*"] \\ \mathsf{Aff}(\mathbb{C}',T') \arrow[r, "\delta"'] & \mathsf{Aff}(\mathsf{Aff}(\mathbb{C}',T')) \end{tikzcd}$$

commutes. Indeed, for each object (M, K) of $\mathsf{Aff}(\mathbb{C}, T)$ we compute that

$$\begin{array}{rclclcl} (F_*)_*(\delta(M,K)) & = & (F_*)_*((M,K),K) & = & (F_*(M,K),K_{F_*}) & = & ((FM,K_F),K_{F_*}) \\ & = & ((FM,K_F),K_F) & = & \delta(FM,K_F) & = & \delta(F_*(M,K)) \end{array}$$

since the definitions of K_F and K_{F_*} readily entail that

$$K_{F_*} = K_F : T'^2_*(FM, K_F) \to T'_*(FM, K_F)$$

by the 2-functoriality of $\mathsf{Aff} = (-)_* : \mathrm{Tan} \to \mathrm{Cat}$. Commutativity on morphisms is immediate. The resulting natural tranformation δ is, moreover, 2-natural, as one readily verifies.

We already know that $(\mathsf{Aff}, \varepsilon, \delta)$ satisfies one of the co-unit laws (7.ii). For the other, we must show that the composite

$$\mathsf{Aff}(\mathbb{C},T) \xrightarrow{\delta_{(\mathbb{C},T,\mathcal{F})}} \mathsf{Aff}(\mathsf{Aff}(\mathbb{C},T)) \xrightarrow{U_*} \mathsf{Aff}(\mathbb{C},T)$$

is the identity, where $U = \varepsilon_{(\mathbb{C},T,\mathcal{F})}$. But this is nearly immediate, since this composite sends each object (M, K) to (M, K_U), while $K_U = K$ since the forgetful functor U is a strict morphism of tangent categories.

For the co-associativity law, we must show that the diagram

$$\begin{tikzcd} \mathsf{Aff}(\mathbb{C},T) \arrow[r, "\delta_{(\mathbb{C},T,\mathcal{F})}"] \arrow[d, "\delta_{(\mathbb{C},T,\mathcal{F})}"'] & \mathsf{Aff}(\mathsf{Aff}(\mathbb{C},T)) \arrow[d, "(\delta_{(\mathbb{C},T,\mathcal{F})})_*"] \\ \mathsf{Aff}(\mathsf{Aff}(\mathbb{C},T)) \arrow[r, "\delta_{\mathsf{Aff}(\mathbb{C},T)}"'] & \mathsf{Aff}(\mathsf{Aff}(\mathsf{Aff}(\mathbb{C},T))) \end{tikzcd}$$

commutes. But the definitions readily entail that both composites send an object (M, K) of $\mathsf{Aff}(\mathbb{C}, T)$ to $(((M, K), K), K)$, and the commutativity on arrows is immediate. $\square$

Definition 7.7. *An* ***affine tangent category*** *is an Eilenberg-Moore $\mathbb{A}\mathsf{ff}$-coalgebra, for the 2-comonad $\mathbb{A}\mathsf{ff} = (\mathsf{Aff}, \varepsilon, \delta)$ defined in 7.6. Hence affine tangent categories are the objects of a 2-category, namely the Eilenberg-Moore 2-category for the 2-comonad $\mathbb{A}\mathsf{ff}$.*

Proposition 7.8. *An affine tangent category is equivalently given by a tangent category with endemic fibre products $(\mathbb{C}, T, \mathcal{F})$ in which each object M is equipped with a flat torsion-free connection K_M such that*

1. *every morphism $f : M \to N$ in $\mathbb{C}$ preserves the given connections K_M and K_N, in the sense that $K_M T(f) = T^2(f) K_N$, and*
2. *for each object M of $\mathbb{C}$, the following diagram commutes:*

$$\begin{tikzcd}
T^3M \arrow[r, "T(c_M)"] \arrow[d, "K_{TM}"'] & T^3M \arrow[r, "c_{TM}"] & T^3M \arrow[d, "T(K_M)"] \\
T^2M \arrow[rr, "c_M"'] & & T^2M
\end{tikzcd}$$

Proof. Suppose that we are given an $\mathbb{A}\mathsf{ff}$-coalgebra $((\mathbb{C}, T, \mathcal{F}), A)$, i.e. an object $(\mathbb{C}, T, \mathcal{F})$ of Tan_e together with a 1-cell $A : (\mathbb{C}, T, \mathcal{F}) \to \mathsf{Aff}(\mathbb{C}, T)$ in Tan_e making the following diagrams commute in Tan:

$$\begin{tikzcd}
(\mathbb{C}, T) \arrow[r, "A"] \arrow[dr, equal] & \mathsf{Aff}(\mathbb{C}, T) \arrow[d, "\varepsilon_{(\mathbb{C},T,\mathcal{F})}"] \\
& (\mathbb{C}, T)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathbb{C}, T) \arrow[r, "A"] \arrow[d, "A"'] & \mathsf{Aff}(\mathbb{C}, T) \arrow[d, "A_*"] \\
\mathsf{Aff}(\mathbb{C}, T) \arrow[r, "\delta_{(\mathbb{C},T,\mathcal{F})}"'] & \mathsf{Aff}(\mathsf{Aff}(\mathbb{C}, T))
\end{tikzcd}
\tag{7.iii}$$

Since $U = \varepsilon_{(\mathbb{C},T,\mathcal{F})}$ is the forgetful functor, the unit law $UA = 1$ entails that A must send each object M of $\mathbb{C}$ to an object of the form (M, K_M) with K_M a flat torsion-free connection on M. For each morphism f in $\mathbb{C}$, as in 1, we have $U(A(f)) = f$, so that $A(f) = f : (M, K_M) \to (N, K_N)$ and 1 holds.

We claim that A is necessarily a *strict* morphism of tangent categories. Indeed, since $UA = 1_{(\mathbb{C},T)}$ in Tan and U is a strict morphism, it follows that the structural isomorphism $\alpha^A : AT \to T^*A$ has $U(\alpha^A) = 1_T : UAT = T \to T = UT^*A$, so that each of its components

$$\alpha^A_M : ATM = (TM, K_{TM}) \to T^*AM = (TM, (K_M)_T)$$

is a morphism in $\mathsf{Aff}(\mathbb{C}, T)$ whose underlying morphism in $\mathbb{C}$ is 1_{TM}. Hence 1_{TM} preserves the connections K_{TM} and $(K_M)_T$, in the sense that $K_{TM}T(1_{TM}) = T^2(1_{TM})(K_M)_T$, i.e. $K_{TM} = (K_M)_T$, so 2 holds since by definition $(K_M)_T = T(c_M)c_{TM}T(K_M)c_M$ and $c_M = c_M^{-1}$. We now deduce also that $AT = T^*A$ as functors and that α^A is the identity transformation on AT.

Conversely, suppose that $(\mathbb{C}, T, \mathcal{F})$ is a tangent category with endemic fibre products and an assignment $M \mapsto K_M$ satisfying 1 and 2. Then we can define a functor $A : \mathbb{C} \to \mathsf{Aff}(\mathbb{C}, T)$ on objects by $AM = (M, K_M)$ and on arrows by $A(f) = f$, whereupon $UA = 1_{\mathbb{C}}$ as functors. Hence it is immediate that A preserves endemic fibre products, since U reflects endemic fibre products. In view of the above, 2 asserts precisely that $K_{TM} = (K_M)_T$ for each object M, i.e. that $ATM = T^*AM$ as objects. But it then follows immediately that $AT = T^*A$ as functors, so since $UA = 1_{\mathbb{C}}$ and U is faithful and is a strict morphism of tangent categories, it follows that A is a strict morphism of tangent categories. Hence A is a 1-cell in Tan_e, and clearly $UA = 1_{(\mathbb{C},T,\mathcal{F})}$ in Tan_e. In the rightmost diagram in (7.iii), each 1-cell is a strict morphism of tangent categories, so the diagram commutes in Tan_e as soon as the underlying diagram in Cat commutes. Indeed, for each object M of $\mathbb{C}$ we compute that

$$A_*(AM) = A_*(M, K_M) = (AM, (K_M)_A) = ((M, K_M), K_M) = \delta_{(\mathbb{C},T,\mathcal{F})}(AM)$$

since it follows readily from the definitions that $(K_M)_A = K_M : T_*^2(M, K_M) \to T_*(M, K_M)$, and the commutativity on arrows is immediate. □

Example 7.9. *Given any tangent category with endemic fibre products,* $(\mathbb{C}, T, \mathcal{F})$*, the category of affine geometric spaces* $\mathsf{Aff}(\mathbb{C}, T)$ *is an affine tangent category, since it is a cofree* $\mathbb{A}\mathsf{ff}$*-coalgebra.*

8 Structures in the affine categories

In the previous sections, we showed how the affine category associated to a tangent category is itself a tangent category, and investigated the resulting 2-functor. In the remainder of the paper we focus on further structure carried by the affine categories themselves. In particular, we generalize many of the results of Jubin [26] regarding the existence of various monads, comonads, and distributive laws in these affine categories.

Proving these results will require extensive calculations. In particular, we will make frequent use of both the axioms of a tangent category (Definition 3.1) and the basic properties of a connection K (Proposition 3.14). As we use many of these axioms frequently and in conjunction with other axioms, we will occasionally not explicitly refer to the specific axioms or properties by name. Moreover, we will also make use of the following notational conventions:

- Throughout, we assume that $(\mathbb{C}, T)$ is a tangent category.
- In the previous sections, we let T_* denote the tangent functor in the affine category; here, we will simply write it as T.
- Similarly, for an object (M, K) in the affine category, objects such as TM, T^2M, etc. will be assumed to be equipped with their canonical choice of connections (eg., TM is equipped with the connection $T(c)cT(K)c$, as per Remark 6.17).

- We will let $+$ denote not only the natural transformation $+ : T_2 \longrightarrow T$, but also the n-fold addition $+_n : T_n \longrightarrow T$.

Related to this last point, in our definitions and calculations, we will often add the same term repeatedly. To faciliate handling such sums, we make the following definitions.

Definition 8.1. *Suppose $f : X \longrightarrow TM$ is a map in $\mathbb{C}$. Define $f \cdot 0 := fp0 : X \longrightarrow TM$, and for any $a \in \mathbb{Z}_{>0}$,*

$$f \cdot a := \langle f, f, \ldots f \rangle + : X \longrightarrow TM$$

(where there are a copies of f). Similarly, if $(\mathbb{C}, T)$ has negatives, with the negation denoted $n : TM \longrightarrow TM$, then define

$$f \cdot (-a) := \langle fn, fn, \ldots fn \rangle + : X \longrightarrow TM$$

(where there are a copies of fn).

Note that for any $g : Y \longrightarrow X$, $g(f \cdot a) = (gf \cdot a)$. More generally, for any collection of maps from a common domain X to TM such that the composites of each of these maps with p_M are equal, one can add such maps, and the resulting addition operation is associative, unital, and commutative (since $p_M : TM \longrightarrow M$ is a commutative monoid in $\mathbb{X}/M$). In our proofs, we will often speak informally of summing such terms.

8.1 Monads

We begin by investigating monad structure on the tangent functor in the affine category. Recall from Section 4 that Jubin showed that for any real number a, one could define a monad on the category of affine manifolds, whose functor was the tangent functor, the unit the projection, and the multiplication $\mu^a : T^2 \longrightarrow T$ defined in local coordinates by

$$(x, v, \dot{x}, \dot{v}) \mapsto (x, v + \dot{x} + a\dot{v}).$$

Our goal is to translate this definition to the affine category of a tangent category.

Now, in any tangent category, given an object M, we have the map $p_{TM} : T^2M \longrightarrow TM$. In local coordinates in the category of smooth manifolds, this is given by

$$(x, v, \dot{x}, \dot{v}) \mapsto (x, v).$$

We also have the map $T(p_M) : T^2M \longrightarrow TM$, which in local coordinates is given by

$$(x, v, \dot{x}, \dot{v}) \mapsto (x, \dot{x}).$$

However, none of the structural maps of a tangent category have the following effect in local coordinates:

$$(x, v, \dot{x}, \dot{v}) \mapsto (x, \dot{v}). \tag{8.i}$$

If we move to the affine category of a tangent category, however, we have more structure available. Specifically, an object in the affine category by definition comes equipped with the connection map $K : T^2M \longrightarrow TM$, which, since the connection is affine, is itself a map in the affine category (see Corollary 6.21). Moreover, for the canonical connection associated to an affine manifold, the effect of K in local coordinates is exactly (8.i) (see the proof of Proposition 5.4).

Now, it is easy to check (as is done in the definition below) that the morphisms $p_{TM}, T(p_M)$, and K are all equal when post-composed by p_M. Thus, in the affine category, we can construct the tuple

$$T^2M \xrightarrow{\langle K,K,K,\ldots K,T(p_M),p_{TM}\rangle} T_{a+2}M$$

(where there are a copies of K). We can then apply $+ : T_{a+2}M \longrightarrow TM$, to get a map $\mu^a : T^2M \longrightarrow TM$. Moreover, in local coordinates $+$ is simply addition of tangent vectors, so that in local coordinates, the map

$$\mu^a := \langle K, K, K, \ldots K, T(p_M), p_{TM}\rangle +$$

is precisely

$$(x, v, \dot{x}, \dot{v}) \mapsto (x, v + \dot{x} + a\dot{v}).$$

(Note that with our notational conventions described above, by associativity of $+$, we can more simply describe μ^a as $\langle K \cdot a, T(p_M), p_{TM}\rangle+$; this is the form we use below).

Thus, the additional structure of the affine category allows us to define a map that generalizes Jubin's monad multiplication in the category of affine manifolds. Our goal in this section is then to show that this map gives rise to monad structure in the more general setting of the affine category of a tangent category.

We begin by showing that this map is well defined, and investigate some of its basic properties.

Proposition 8.2. *For (M, K) an object of $\mathsf{Aff}(\mathbb{C}, T)$, and $a \in \mathbb{Z}_{\geq 0}$, define*

$$\mu^a_{(M,K)} := \langle K \cdot a, T(p_M), p_{TM}\rangle + : T^2M \longrightarrow TM.$$

If $(\mathbb{C}, T)$ has negatives, similarly define $\mu^a_{(M,K)}$ for any $a \in \mathbb{Z}_{<0}$. Then each $\mu^a_{(M,K)}$ is a well-defined map in $\mathsf{Aff}(\mathbb{C}, T)$, and $\mu^a_{(M,K)}p_M = p_{TM}p_M$.

Proof. Using properties of the vertical connection K,

$$Kp_M = p_{TM}p_M = T(p_M)p_M$$

so $\langle K \cdot a, T(p_M), p_{TM}\rangle$ is a well-defined map into the fibre product $T_{a+2}M$, and $\mu^a_{(M,K)}p_M = p_{TM}p_M$. It is in $\mathsf{Aff}(\mathbb{C}, T)$ since K is in $\mathsf{Aff}(\mathbb{C}, T)$ (by Corollary 6.21) and the tangent structure on $\mathbb{C}$ lifts to $\mathsf{Aff}(\mathbb{C}, T)$. $\square$

Note that the case $a = 0$ is the map $\langle Tp, p\rangle+$, which exists in any tangent category and is already known to be the multiplication for a monad (see Proposition 3.2).

To prove that each μ^a provides the multiplication for a monad structure on $\mathsf{Aff}(\mathbb{C}, T)$ will require a number of extensive calculations. To shorten the length of these calculations, we will typically omit the subscript on $\mu^a_{(M,K)}$ and on the tangent category transformations (as we have done in some previous calculations, e.g., in the proof of Lemma 6.18).

Theorem 8.3. *For each $a \in \mathbb{Z}_{\geq 0}$, $\mathbb{T}_a := (T, \mu^a, 0)$ is a monad on $\mathsf{Aff}(\mathbb{C}, T)$. Moreover, if $(\mathbb{C}, T)$ has negatives, then the result also holds for each $a \in \mathbb{Z}$.*

Proof. The naturality of each μ^a follows from Corollary 6.21(iii) and from the naturality of p and $+$.

The unital conditions follow immediately since K preserves both additive structures on T^2, as well as using the tangent category axioms:

$$0\mu^a = 0\langle K \cdot a, T(p), p\rangle + = \langle p0 \cdot a, p0, 1\rangle + = 1,$$

and

$$T(0)\mu^a = T(0)\langle K \cdot a, T(p), p\rangle + = \langle p0 \cdot a, T(1), p0\rangle + = 1.$$

For associativity of the monad, we need to show that

$$T(\mu^a)\mu^a = \mu^a_T\mu^a.$$

First, consider

$$T(\mu^a)\mu^a = T(\mu^a)\langle K \cdot a, T(p), p\rangle + = \langle T(\mu^a)K \cdot a, T(\mu^a)T(p), T(\mu^a)p\rangle + \ (\star)$$

The last two terms in this sum are straightforward to address:

$$T(\mu^a)T(p) = T(\mu^a p) = T(pp)$$

(by Lemma 8.2) and

$$T(\mu^a)p = p\mu^a = p\langle K \cdot a, T(p), p\rangle + = \langle pK \cdot a, pT(p), pp\rangle +.$$

So in total these two terms contribute one $T(pp)$, one $pT(p)$, one pp, and a copies of pK to the sum in $\star$.

The first term of the sum in $\star$ (without the $\cdot a$) is

$$\begin{array}{rl} & T(\mu^a)K \\ = & \langle\langle T(K), T(K), \dots T(K)\rangle T(+), T^2(p), T(p)\rangle T(+)K \\ = & \langle\langle T(K)K, T(K)K, \dots T(K)K\rangle +, T^2(p)K, T(p)K\rangle + \text{ (by additivity of } K) \end{array}$$

so that this term has a copies of $T(K)K$, one of $T^2(p)K$, and one of $T(p)K$. Since there are a of each of these terms, combining with the results above, $\star$ is equal to

$$\langle T(K)K \cdot (a^2), T^2(p)K \cdot a, T(p)K \cdot a, pK \cdot a, T(pp), pT(p), pp\rangle +.$$

We now consider $\mu^a_T \mu^a$. This is equal to

$$\mu^a_T \langle K \cdot a, T(p), p\rangle + = \langle \mu^a_T K \cdot a, \mu^a_T T(p), \mu^a_T p\rangle + \ (\dagger).$$

We consider each of the terms in this sum separately:

$$\begin{array}{cl} & \mu^a_T K \\ = & \langle K_T \cdot a, T(p), p\rangle + K \\ = & \langle K_T K \cdot a, T(p)K, pK\rangle + \text{ (by additivity of } K) \\ = & \langle T(K)K \cdot a, T(p)K, pK\rangle + \text{ (by Corollary 6.21)} \end{array}$$

Since there are a copies of $\mu^a_T K$ in $\dagger$, in total this gives a^2 copies of $T(K)K$, a of $T(p)K$, and a of pK.

The next term in $\dagger$ is

$$\begin{array}{cl} & \mu^a_T T(p) \\ = & \langle K_T \cdot a, T(p), p\rangle + T(p) \\ = & \langle K_T T(p) \cdot a, T(p)T(p), pT(p)\rangle + \text{ (by naturality of } +) \\ = & \langle T^2(p)K \cdot a, T(pp), pT(p)\rangle + \text{ (by Lemma 6.18)} \end{array}$$

The final term in $\dagger$ is $\mu^a_T p = pp$ by Proposition 8.2. Putting all these results together, we get that $\dagger$ is equal to

$$\langle T(K)K \cdot (a^2), T(p)K \cdot a, pK \cdot a, T^2(p)K \cdot a, T(pp), pT(p), pp\rangle +$$

By commutativity of $+$, this is equal to $\star$, and hence we have proven associativity of the monad, as required. $\square$

We can also show that some of Jubin's results on algebras for the monads hold in tangent categories.

Lemma 8.4. *For any object (M, K) of $\mathsf{Aff}(\mathbb{C}, T)$, $((M, K), p_M)$ is an algebra for each monad $\mathbb{T}_a$.*

Proof. This is straightforward, as by Proposition 8.2,

$$\mu^a p = pp = T(p)p$$

and by the tangent category axioms, $0p = 1$. $\square$

Each algebra of $\mathbb{T}_a$ has an associated endomorphism of TM with some interesting properties; in particular, this endomorphism has the property that when applied twice, it gives the sum of a copies of itself (for the case of smooth manifolds, this is on page 30 of Jubin's thesis [26]).

Proposition 8.5. *Suppose that $((M,K),h)$ is an algebra for $\mathbb{T}_a$ in $\mathsf{Aff}(\mathbb{C},T)$. Then the endomorphism*

$$\psi_h := \left(TM \xrightarrow{\ell_M} T^2M \xrightarrow{T(h)} TM \right)$$

has the property that

$$\psi_h h = (1 \cdot a)h$$

and

$$\psi_h \psi_h = \psi_h \cdot a.$$

Proof. For the first claim, consider:

$$\begin{array}{rl} & \psi_h h \\ = & \ell T(h) h \\ = & \ell \mu^a h \text{ (since } h \text{ is an algebra)} \\ = & \ell \langle K \cdot a, T(p), p \rangle + h \\ = & \langle 1 \cdot a, p0, p0 \rangle + h \text{ (properties of } \ell \text{ and } K) \\ = & (1 \cdot a)h \end{array}$$

For the second claim, consider

$$\begin{array}{rl} & \psi_h \psi_h \\ = & \ell T(h) \ell T(h) \\ = & \ell \ell T^2(h) T(h) \text{ (naturality of } \ell) \\ = & \ell \ell T(T(h)h) \\ = & \ell \ell T(\mu^a h) \text{ (since } h \text{ is an algebra structure for } \mu^a) \\ = & \ell \ell T(\mu^a) T(h) \\ = & \ell \ell \langle T(K), \ldots T(K), T^2(p), T(p) \rangle T(+) T(h) \text{ (with } a \text{ copies of } T(K)) \\ = & \langle \ell \ell T(K), \ldots \ell \ell T(K), \ell T(p) \ell, \ell p0 \rangle T(+) T(h) \text{ (using naturality and coherence of } \ell) \\ = & \langle \ell K \ell, \ldots \ell K \ell, p0\ell, p00 \rangle T(+) T(h) \text{ (property of } K \text{, tan. cat. axiom)} \\ = & \langle \ell, \ldots \ell, p0T(0), p0T(0) \rangle T(+) T(h) \text{ (property of } K \text{, tan. cat. axiom)} \\ = & \langle \ell, \ldots \ell \rangle T(+) T(h) \text{ (} T(0) \text{ the unit for } T(+)) \end{array}$$

Now, note that by coherence of ℓ, $\ell T(p) T(0) = p0T(0) = p00 = \ell p0$, so that by [13, Lemma 2.6(ii)], the above becomes

$$\langle \ell, \ldots \ell \rangle + T(h)$$

which, by naturality of $+$, becomes

$$\langle \ell T(h), \ldots \ell T(h) \rangle + = a \cdot (\ell T(h)) = a \cdot \psi_h,$$

as required. $\square$

Note that for the algebra p_M, the associated endomorphism ψ_{p_M} is the 0 morphism, as

$$\psi_{p_M} = \ell T(p) = p0.$$

8.2 Comonads

The goal of this section is to generalize some of Jubin's results on comonads in the affine category. In section 4 we saw that for any real number b, Jubin defined $\delta^b : T \longrightarrow T^2$ with local coordinate description

$$(x, w) \mapsto (x, w, w, b \cdot w) \ ,$$

which was the comultiplication for a comonad structure on T. We would like to define an analogue of this map in the affine category of a tangent category.

Any tangent category has a natural transformation that bears some similarities to the transformation above: the map

$$v := \left(T_2 \xrightarrow{\langle \pi_0 \ell, \pi_1 0 \rangle T(+)} T^2 \right)$$

(defined in the universality of the vertical lift axiom) has local coordinate description

$$(x, w_1, w_2) \mapsto (x, w_2, 0, w_1)$$

We can modify this map to instead use b copies of ℓ:

$$v^b := \left(T_2 \xrightarrow{\langle \pi_0 \ell \cdot b, \pi_1 0 \rangle T(+)} T^2 \right);$$

its effect in local coordinates is

$$(x, w_1, w_2) \mapsto (x, w_2, 0, b \cdot w_1) \ .$$

This is not yet the desired map δ^b. However, we will use it in the definition of δ^b. Thus, we first describe its properties; note that this map exists in any tangent category, not just the affine category of a tangent category.

Proposition 8.6. *For each object M of $(\mathbb{C}, T)$, and each $b \in \mathbb{Z}_{\geq 0}$, define*

$$v^b_M := \langle \pi_0 \ell \cdot b, \pi_1 0 \rangle T(+) : T_2 M \longrightarrow T^2 M,$$

with a similar definition for any $b \in \mathbb{Z}_{<0}$ if $(\mathbb{C}, T)$ has negatives. Then v^b_M is well defined, and (omitting the subscripts):

(i) $v^b p = \pi_1$;

(ii) $v^b T(p) = \pi_0 p 0 = \pi_1 p 0$;

(iii) If K is a connection on M, then $v^b K = \pi_0 \cdot b$.

Proof. Note that $\pi_0\ell T(p) = \pi_0 p0 = \pi_1 p0 = \pi_1 0T(p)$, so $\langle \pi_0\ell \cdot b, \pi_1 0\rangle$ defines a map into the pullback $T(T_2M)$; thus v^b_M is well-defined. This calculation also proves (ii).

For (i), using naturality and additivity of p,

$$v^b p = \langle \pi_0\ell \cdot b, \pi_1 0\rangle T(+)p = \langle \pi_0\ell p \cdot b, \pi_1 0p\rangle + = \langle \pi_0 p0 \cdot b, \pi_1\rangle + = \pi_1.$$

For (iii), using the fact that K is additive and a retract of ℓ,

$$v^b K = \langle \pi_0\ell \cdot b, \pi_1 0\rangle T(+)K = \langle \pi_0\ell K \cdot b, \pi_1 0K\rangle + = \langle \pi_0 \cdot b, \pi_1 p0\rangle + = \pi_0 \cdot b.$$

$\square$

Now, while we also have the maps $0_{TM} : TM \longrightarrow T^2M$ and $T(0_M) : TM \longrightarrow T^2M$, which have the effects

$$(x, w) \mapsto (x, w, 0, 0) \text{ and } (x, w) \mapsto (x, 0, w, 0),$$

and the addition maps $+_{TM}$ and $T(+_M)$, no combination of these additions or zeros with v^b gives the desired map. What is required is a map from TM to T^2M that has local coordinate effect

$$(x, w) \mapsto (x, w, w, 0).$$

While we do not have such a map from the general structure of a tangent category, we do have such a map in the affine category of a tangent category. In particular, if H is the associated horizontal connection of an object (M, K) (see Theorem 3.12) in the affine category, one can readily check that $\langle 1, 1\rangle H : TM \longrightarrow T^2M$ has precisely the local coordinate effect above. If we then consider $\langle 1, 1\rangle\langle v^b, H\rangle+$, it has the local effect

$$(x, w) \mapsto (x, w, w) \mapsto ((x, w, 0, b \cdot w), (x, w, w, 0)) \mapsto (x, w, w, b \cdot w)\ ,$$

which is exactly what we want.

Proposition 8.7. *For any $(M, K) \in \mathsf{Aff}(\mathbb{C}, T)$, and each $b \in \mathbb{Z}_{\geq 0}$, define*

$$\delta^b_{(M,K)} := \langle 1, 1\rangle\langle v^b, H\rangle + : TM \longrightarrow T^2M$$

(where H is the corresponding horizontal connection associated to K—Theorem 3.12) with a similar definition for any $b \in \mathbb{Z}_{<0}$ if $(\mathbb{C}, T)$ has negatives. Then $\delta^b_{(M,K)}$ is well-defined, a map in $\mathsf{Aff}(\mathbb{C}, T)$, and (dropping the subscripts)

(i) $\delta^b p = 1$;

(ii) $\delta^b T(p) = 1$;

(iii) $\delta^b K = 1 \cdot b$.

Proof. By Proposition 8.6(i), $v^b p = \pi_1$, and since H is a horizontal connection, $Hp = \pi_1$. Thus $\langle v^b, H\rangle$ defines a map into the pullback $T_2(TM)$ and so $\delta^b_{M,K}$ is well defined. Moreover,

$$\delta^b p = \langle 1,1\rangle \pi_1 = 1$$

as required for (i). By Corollary 6.21, since K is a connection on (M,K) in $\mathsf{Aff}(\mathbb{C},T)$, by Theorem 3.12, it has a unique compatible horizontal connection $\mathsf{Aff}(\mathbb{C},T)$; since this would also be a horizontal connection in $(\mathbb{C},T)$, it must be H itself. Thus H is a map in $\mathsf{Aff}(\mathbb{C},T)$, and as a result δ^b is as well.

For (ii), using Proposition 8.6(ii) and the fact that H is a horizontal connection,

$$\delta^b T(p) = \langle 1,1\rangle\langle v^b, H\rangle{+}T(p) = \langle 1,1\rangle\langle v^b T(p), HT(p)\rangle{+} = \langle 1,1\rangle\langle \pi_0 p0, \pi_0\rangle{+} = \langle p0, 1\rangle{+} = 1.$$

Finally, for (iii), using Proposition 8.6(iii) and the fact that H is a horizontal connection,

$$\delta^b K = \langle 1,1\rangle\langle v^b, H\rangle{+}K = \langle 1,1\rangle\langle v^b K, HK\rangle{+} = \langle 1,1\rangle\langle \pi_0\cdot b, \pi_1 p0\rangle{+} = \langle 1\cdot b, p0\rangle{+} = 1\cdot b,$$

as required.

□

For ease of notation, we will usually omit the subscripts on these maps. With the definition in hand, we can now prove it gives comonad structure.

Theorem 8.8. *For each $b \in \mathbb{Z}_{\geq 0}$, $\mathbb{T}^b := (T, \delta^b, p)$ is a comonad on $\mathsf{Aff}(\mathbb{C},T)$; if $(\mathbb{C},T)$ has negatives then the result also holds for any $b \in \mathbb{Z}$.*

Proof. By Corollary 5.6, the horizontal connections H associated to the objects (M,K) in $\mathsf{Aff}(\mathbb{C},T)$ form the components of a natural transformation in $\mathsf{Aff}(\mathbb{C},T)$ from T_2 to T^2. Since the other components of δ^b are elements of the tangent structure (which lifts to $\mathsf{Aff}(\mathbb{C},T)$), each δ^b is thus a natural transformation from T to T^2 in $\mathsf{Aff}(\mathbb{C},T)$.

The co-unit equations follow immediately from Proposition 8.7 (i and ii).

For co-associativity, we need to show that $\delta^b T(\delta^b)$ equals $\delta^b \delta^b_T$; these are maps into $T^3 M$. Recall from Corollary 3.11 that since TM has a connection (K,H), T^2M is a fibre product of three copies of TM with projections $K, T(p), p$ (thus, this is a jointly monic triple). That is, $T^2 M$ is isomorphic to $T_3 M$ with the above projections. But then since T preserves this fibre product, $T^3 M$ is the fibre product of three copies of $T^2 M$ with projections $T(K), T^2(p), T(p)$:

$$\begin{tikzcd}
 & T^3M \arrow[dl, "T(K)"'] \arrow[d, "T^2(p)"] \arrow[dr, "T(p)"] & \\
T^2M \arrow[dr, "T(p)"'] & T^2M \arrow[d, "T(p)"] & T^2M \arrow[dl, "T(p)"] \\
 & TM &
\end{tikzcd}$$

But, as above, each of these copies of T^2M is a fibre product with projections $K, T(p), p$. Moreover, since $Kp = T(p)p = pp$,

$$T(K)T(p) = T^2(p)T(p) = T(p)T(p),$$

and so T^3M is isomorphic to T_7M, with the seven projections

$$T(K)K, T(K)p, T^2(p)K, T^2(p)p, T(p)K, T(p)p, T(K)T(p) = T^2(p)T(p) = T(p)T(p).$$

Thus, the above is a jointly monic 7-tuple, and so to check the equality of $\delta^b T(\delta^b)$ and $\delta^b \delta^b_T$, it suffices to check that the two are equal when post-composed by each of the above 7 maps.

We will first show that these two terms are equal when post-composed by p. Indeed, using Proposition 8.7(i),

$$\delta^b T(\delta^b)p = \delta^b p \delta^b = \delta^b \delta^b_T p.$$

Thus, we know the terms are equal when post-composed by $T(K)p = pK$, $T^2(p)T(p) = pT(p)$, and $T(p)p = pp$.

We can also show that the terms are equal when post-composed by $T(p)$, using Proposition 8.7 (i and ii):

$$\delta^b T(\delta^b)T(p) = \delta^b T(\delta^b p) = \delta^b = \delta^b \delta^b_T T(p).$$

Thus, the terms are equal when post-composed by $T(p)T(p)$ and $T(p)K$.

For equality when post-composing with $T(K)K$, consider:

$$\begin{aligned}
& \delta^b T(\delta^b)T(K)K \\
= \;& \delta^b T(\delta^b K)K \\
= \;& \delta^b T(1 \cdot b)K \text{ (by Proposition 8.7(iii))} \\
= \;& \delta^b \langle 1, 1, \ldots 1\rangle T(+)K \\
= \;& \delta^b \langle K, K, \ldots K\rangle + \text{ (since } K \text{ additive)} \\
= \;& \delta^b (K \cdot b)
\end{aligned}$$

while

$$\begin{aligned}
& \delta^b \delta^b_T T(K)K \\
= \;& \delta^b \delta^b_T K_T K \text{ (by Corollary 6.21)} \\
= \;& \delta^b (1 \cdot b)K \text{ (by Proposition 8.7(iii))} \\
= \;& \delta^b (K \cdot b) \text{ (since } K \text{ additive)}
\end{aligned}$$

so that the terms are equal when post-composed by $T(K)K$.

Finally, we need to check equality when post-composed by $T^2(p)K$. First, using Proposition 8.7(ii and iii), we have

$$\delta^b T(\delta^b)T^2(p)K = \delta^b T(\delta^b T(p))K = \delta^b K = 1 \cdot b.$$

On the other hand,

$$\begin{array}{rl} & \delta^b \delta^b_T T^2(p) K \\ = & \delta^b \delta^b_T K_T T(p) \text{ (by naturality of } K) \\ = & \delta^b (1 \cdot b) T(p) \text{ (by Proposition 8.7(iii))} \\ = & \delta^b (T(p) \cdot b) \text{ (by naturality of } +) \\ = & 1 \cdot b \text{ (by Proposition 8.7(ii))} \end{array}$$

Thus $\delta^b T(\delta^b)$ and $\delta^b \delta^b_T$ are equal when post-composed by each of the maps in the jointly monic 7-tuple, and so are equal, as required for co-associativity.

□

In section 8.1, we found examples of algebras for each of the monads $\mathbb{T}_a$, and found some structure associated to these algebras. As we shall see here, related results hold for the coalgebras of the comonads $\mathbb{T}^b$.

Lemma 8.9. *For any object (M, K) of $\mathsf{Aff}(\mathbb{C}, T)$, $((M, K), 0_M)$ is a coalgebra for each of the comonads $\mathbb{T}^b$.*

Proof. First, note that using additivity of ℓ,

$$\langle 0, 0 \rangle v^b = \langle 0\ell \cdot b, 00 \rangle T(+) = 00$$

and then using this and additivity of H,

$$0\delta^b = \langle 0, 0 \rangle \langle v^b, H \rangle + = 00 = 0T(0)$$

Moreover, by a tangent category axiom, $0p = 1$. So 0_M is a coalgebra for $(\delta^b, 0)$. □

Note that in general, a coalgebra for $\mathbb{T}^b$ will be a vector field, since the co-unit of the comonad is p. Moreover, we also have the following results (for smooth manifolds, these are found on page 30 of [26]).

Proposition 8.10. *Suppose that $((M, K), j)$ is a coalgebra for $\mathbb{T}^b$ in $\mathsf{Aff}(\mathbb{C}, T)$. Then the endomorphism*

$$\phi_j := \left(TM \xrightarrow{T(j)} T^2 M \xrightarrow{K} TM \right)$$

has the property that

$$j \phi_j = j \cdot b$$

and

$$\phi_j \phi_j = \phi_j \cdot b.$$

Proof. For the first claim,

$$\begin{aligned}
& j\phi_j \\
= \;& jT(j)K \\
= \;& j\delta^b K \text{ (since } j \text{ a coalgebra)} \\
= \;& j(1 \cdot b) \text{ (by Proposition 8.7(iii))} \\
= \;& (j \cdot b)
\end{aligned}$$

For the second claim, consider

$$\begin{aligned}
& \phi_j\phi_j \\
= \;& T(j)KT(j)K \\
= \;& T(j)T^2(j)K_TK \text{ (since } j \text{ is a map in } \mathsf{Aff}(\mathbb{C}, T) \text{ from } (M,K) \text{ to } (TM, K_T)) \\
= \;& T(j)T^2(j)T(K)K \text{ (by Corollary 6.21(i))} \\
= \;& T(jT(j)K)K \\
= \;& T(j\delta^b K)K \text{ (since } j \text{ is a coalgebra)} \\
= \;& T(j(1 \cdot b))K \text{ (by Proposition 8.7(iii))} \\
= \;& \langle T(j), \ldots T(j)\rangle T(+)K \text{ (where there are } b \text{ instances of } T(K)) \\
= \;& \langle T(j)K, \ldots T(j)K\rangle + \text{ (by additivity of } K) \\
= \;& T(j)K \cdot b \\
= \;& \phi_j \cdot b
\end{aligned}$$

as required. ☐

Note that for the coalgebra $j = 0_M$, the associated map ϕ_{0_M} is simply the zero vector at each point, as by additivity of K,

$$\phi_{0_M} = T(0_M)K = p0.$$

8.3 Distributive laws

We now consider distributive laws between the various monads and comonads defined in the previous sections. Part of what we do here fills in a gap in Jubin's work. In Jubin's thesis [26], he claimed that for each $a, b \in \mathbb{Z}$, there was a natural transformation $\lambda^{a,b} : T^2 \longrightarrow T^2$ on the category of affine manifolds so that the triple $(\mathbb{T}_a, \mathbb{T}^b, \lambda^{a,b})$ was a bimonad. Now, part of the definition of a bimonad is that the map $\lambda^{a,b}$ should be a distributive law from the monad $\mathbb{T}_a$ to the comonad $\mathbb{T}^b$. However, Jubin does not prove this in his thesis: he instead proves the additional conditions for a distributive law to be a bimonad (see page 31 of [26]). Fortunately, this is merely an omission and not an error: in this section, we define a generalization of Jubin's $\lambda^{a,b}$ maps for tangent categories with negatives, and indeed show that they are distributive laws.

However, we also prove the existence of other distributive laws. The canonical flip $c : T^2 \longrightarrow T^2$, an element of any tangent category, is highly compatible with the monads and comonads defined above: we show in this section that it is a distributive law *from any* of the possible monads or comonads defined above *to any of the other* possible monads or comonads.

We begin with the results on c being a distributive law in many different ways, as this will require no additional assumptions on the tangent category. To define Jubin's $\lambda^{a,b}$ maps, we will need to assume the tangent category has negatives. As far as we are aware, all results in this section are new, even for smooth manifolds.

Theorem 8.11. *For any $a_1, a_2, b_1, b_2 \in \mathbb{Z}_{\geq 0}$, $c : T^2 \longrightarrow T^2$ is a distributive law from $\mathbb{T}_{a_1}$ to $\mathbb{T}_{a_2}$, $\mathbb{T}_{a_1}$ to $\mathbb{T}^{b_1}$, $\mathbb{T}^{b_1}$ to $\mathbb{T}_{a_1}$, and from $\mathbb{T}^{b_1}$ to $\mathbb{T}^{b_2}$. If $(\mathbb{C}, T)$ has negatives, a similar result holds for any $a_1, a_2, b_1, b_2 \in \mathbb{Z}$.*

Proof. The equations to prove all of these various results have some commonalities; to prove all of these requires proving that for appropriate a, b,

$$(1)\ T(0)c = 0 \text{ and } T(0) = 0c,$$

$$(2)\ cT(c)\mu^a_T = T(\mu^a)c \text{ and } \mu^a_T c = T(c)cT(\mu^a),$$

$$(3)\ cT(p) = p \text{ and } T(p) = cp,$$

$$(4)\ \delta^b_T T(c)c = cT(\delta^b) \text{ and } c\delta^b_T = T(\delta^b)c_T T(c).$$

However, note that since $c^2 = 1$, each of the pairs of equations in (1)-(4) are equivalent to each other. Thus, we only need to prove one of each pair. Moreover, (1) and (3) are automatic from the equations of a tangent category. Thus, it suffices to prove one of the equations in (2) and one of the equations in (4). Here is the calculation for the first equation in (2) (using a variety of tangent category axioms):

$$\begin{aligned}
& cT(c)\mu^a_T \\
= \;& cT(c)\langle K_T \cdot a, T(p), p\rangle + \\
= \;& cT(c)\langle T(c)cT(K)c \cdot a, T(p), p\rangle + \\
= \;& \langle cT(c)T(c)cT(K)c \cdot a, cT(cp), cT(c)p\rangle + \\
= \;& \langle T(K)c \cdot a, cT(T(p)), cpc\rangle + \\
= \;& \langle T(K)c \cdot a, T^2(p)c, T(p)c\rangle + \\
= \;& \langle T(K) \cdot a, T^2(p), T(p)\rangle T(+)c \text{ (since } c \text{ is additive)} \\
= \;& T(\langle K \cdot a, T(p), p\rangle +)c \\
= \;& T(\mu^a)c
\end{aligned}$$

To conclude, we prove the first equation in (4): $\delta^b_T T(c)c = cT(\delta^b)$. For this, we will first show that $v^b_T T(c)c = (c \times c)T(v^b)$:

$$\begin{array}{rl} & v^b_T T(c)c \\ = & \langle \pi_0\ell \cdot b, \pi_1 0\rangle T(+)T(c)c \\ = & \langle \pi_0\ell \cdot b, \pi_1 0\rangle T(+c)c \\ = & \langle \pi_0\ell \cdot b, \pi_1 0\rangle T(\langle \pi_0 c, \pi_1 c\rangle T(+))c \text{ (additivity of } c) \\ = & \langle \pi_0\ell \cdot bT(c), \pi_1 0T(c)\rangle T^2(+)c \text{ (} T \text{ preserves the relevant pullback)} \\ = & \langle \langle \pi_0\ell, \ldots \pi_0\ell\rangle + T(c)c, \pi_1 0T(c)c\rangle T^2(+) \text{ (naturality of } c) \\ = & \langle \langle \pi_0\ell T(c), \ldots \pi_0\ell T(c)\rangle + c, \pi_1 c0c\rangle T^2(+) \text{ (naturality of } + \text{ and } 0) \\ = & \langle \langle \pi_0\ell T(c)c, \ldots \pi_0\ell T(c)c\rangle T(+), \pi_1 cT(0)\rangle T^2(+) \text{ (additivity of } c) \\ = & \langle \langle \pi_0 cT(\ell), \ldots \pi_0 cT(\ell)\rangle T(+), \pi_1 cT(0)\rangle T^2(+) \text{ (by a tangent category axiom)} \end{array}$$

while

$$\begin{array}{rl} & (c \times c)T(v^b) \\ = & (c \times c)T(\langle \pi_0\ell \cdot b, \pi_1 0\rangle T(+)) \\ = & (c \times c)\langle \langle \pi_0 T(\ell), \ldots \pi_0 T(\ell)\rangle T(+), \pi_1 T(0)\rangle T^2(+) \text{ (} T \text{ preserves the pullback)} \\ = & \langle \langle \pi_0 cT(\ell), \ldots \pi_0 cT(\ell)\rangle T(+), \pi_1 cT(0)\rangle T^2(+) \end{array}$$

so that indeed $v^b_T T(c)c = (c \times c)T(v^b)$. Using various tangent category axioms, we also have that

$$H_T T(c)c = (c \times c)T(H)cT(c)T(c)c = (c \times c)T(H)cc = (c \times c)T(H).$$

Thus

$$\begin{array}{rl} & \delta^b_T T(c)c \\ = & \langle 1, 1\rangle \langle v^b_T, H_T\rangle + T(c)c \\ = & \langle 1, 1\rangle \langle v^b_T T(c)c, H_T T(c)c\rangle T(+) \text{ (naturality of } + \text{ and additivity of } c) \\ = & \langle 1, 1\rangle \langle (c \times c)T(v^b), (c \times c)T(H)\rangle T(+) \text{ (by the results above)} \\ = & \langle c, c\rangle \langle T(v^b), T(H)\rangle T(+) \\ = & c\langle 1, 1\rangle T(\langle v^b, H\rangle +) \\ = & c\langle T(1), T(1)\rangle T(\langle v^b, H\rangle +) \\ = & cT(\langle 1, 1\rangle \langle v^b, H\rangle +) \\ = & cT(\delta^b) \end{array}$$

as required. □

Now we would like to define an analogue of Jubin's distributive laws. For each $a, b \in \mathbb{Z}$, he wished to define a distributive law of $\mathbb{T}_a$ over $\mathbb{T}^b$. Since his definition involved negatives (even for a, b positive), an analogous definition in a tangent category will require that the tangent category have negatives. The formula for the distributive law in the general setting was arrived at through similar reasoning to that described for finding the monad and comonad structures: investigating Jubin's local coordinate definition and finding the appropriate analogues in a tangent category. Specifically, we first define for each $b \in \mathbb{Z}$ a map $\alpha^b : T^2M \longrightarrow TM$, whose effect in local coordinates is

$$(x, v, w, d) \mapsto (x, bw - d).$$

Using this map, we then define, for each $a, b \in \mathbb{Z}$, $\lambda^{a,b} : T^2M \longrightarrow T^2M$, whose effect in local coordinates is

$$(x, v, w, d) \mapsto (x, w, v + w + ad, bw - d).$$

This is how Jubin defines his distributive laws. We first check that we can define these maps in tangent categories, then show that they satisfy the axioms required to be distributive laws.

Proposition 8.12. *Suppose that $(\mathbb{C}, T)$ has negatives, and (M, K) is an object of $\mathsf{Aff}(\mathbb{C}, T)$. For each $b \in \mathbb{Z}$, define*

$$\alpha^b_{(M,K)} := \langle Kn, T(p) \cdot b\rangle + : T^2M \longrightarrow TM,$$

and for each $a, b \in \mathbb{Z}$, define

$$\lambda^{a,b}_{(M,K)} = \langle\langle \alpha^b \ell, T(p)0\rangle T(+), \langle \mu^a, T(p)\rangle H\rangle + : T^2M \longrightarrow T^2M.$$

Then α^b and $\lambda^{a,b}$ are well-defined morphisms in $\mathsf{Aff}(\mathbb{C}, T)$ and give rise to natural transformations. Further,

(i) $\lambda^{a,b}p = T(p)$;

(ii) $\lambda^{a,b}T(p) = \mu^a$;

(iii) $\lambda^{a,b}K = \alpha^b$.

Proof. By Corollary 6.21, K is a morphism in $\mathsf{Aff}(\mathbb{C}, T)$, and since the equations $Knp = Kp = T(p)p$ hold in $\mathsf{Aff}(\mathbb{C}, T)$ we find that $\langle Kn, T(p) \cdot b\rangle$ defines a map into the pullback $T_2(M)$ in $\mathsf{Aff}(\mathbb{C}, T)$, so α^b is well defined.

To show that $\lambda^{a,b}$ is well-defined, first consider

$$\alpha^b \ell T(p) = \alpha^b p0 = T(p)p0 = T(p)0T(p),$$

so that $\langle \alpha^b \ell, T(p)0\rangle$ defines a map into the pullback $T(T_2M) \cong T^2M \times_{TM} T^2M$ in $\mathsf{Aff}(\mathbb{C}, T)$. Also,

$$\mu^a p = pp = T(p)p,$$

so $\langle \mu^a, T(p)\rangle$ defines a map into the pullback $T_2(M)$. Finally, by various tangent category axioms,

$$\langle \alpha^b \ell, T(p)0\rangle T(+)p = \langle \alpha^b \ell p, T(p)0p\rangle + = \langle \alpha^b p0, T(p)\rangle + = T(p)$$

and using that fact that H is a horizontal connection,

$$\langle \mu^a, T(p)\rangle Hp = \langle \mu^a, T(p)\rangle \pi_1 = T(p),$$

so $\langle\langle \alpha^b \ell, T(p)0\rangle T(+), \langle \mu^a, T(p)\rangle H\rangle$ defines a map into $T_2(TM)$; thus $\lambda^{a,b}$ is itself well defined. These results also prove (i). We then have natural transformations $\alpha^b : T^2 \longrightarrow T$ and $\lambda^{a,b} : T^2 \longrightarrow T^2$ since they are defined as composites and pairings of other natural transformations.

For (ii), consider

$$\begin{aligned}
& \lambda^{a,b} T(p) \\
= \; & \langle\langle \alpha^b \ell, T(p)0\rangle T(+), \langle \mu^a, T(p)\rangle H\rangle + T(p) \\
= \; & \langle\langle \alpha^b \ell, T(p)0\rangle T(+p), \langle \mu^a, T(p)\rangle HT(p)\rangle + \text{ (naturality of } +) \\
= \; & \langle\langle \alpha^b \ell, T(p)0\rangle T(\pi_1 p), \langle \mu^a, T(p)\rangle \pi_0\rangle + \text{ (properties of } H \text{ and } +) \\
= \; & \langle T(p)0T(p), \mu^a\rangle + \\
= \; & \langle T(p)p0, \mu^a\rangle + \text{ (by naturality of 0)} \\
= \; & \mu^a
\end{aligned}$$

For (iii), consider

$$\begin{aligned}
& \lambda^{a,b} K \\
= \; & \langle\langle \alpha^b \ell, T(p)0\rangle T(+), \langle \mu^a, T(p)\rangle H\rangle + K \\
= \; & \langle\langle \alpha^b \ell K, T(p)0K\rangle +, \langle \mu^a, T(p)\rangle HK\rangle + \text{ (by additivity of } K) \\
= \; & \langle\langle \alpha^b, T(p)p0\rangle +, \langle \mu^a, T(p)\rangle \pi_1 p0\rangle + \text{ (by properties of } K) \\
= \; & \alpha^b \text{ (0 the unit of } +)
\end{aligned}$$

as required. $\square$

Before proving that $\lambda^{a,b}$ is a distributive law, it will be helpful to record a few other results related to these maps.

Lemma 8.13. *For each $a, b \in \mathbb{Z}$,*

(i) $\lambda^{a,b}_T T(K)K = \alpha^b_T K = \langle T(K)Kn, T(p)K \cdot b\rangle +$;

(ii) $\lambda^{a,b}_T T^2(p)K = \alpha^b_T T(p) = \langle T^2(p)Kn, T(pp) \cdot b\rangle +$;

(iii) $\lambda^{a,b}_T T(p)K = \mu^a_T K = \langle T(K)K \cdot a, T(p)K, pK\rangle +$;

(iv) $T(\delta^b)\lambda^{a,b}_T T(K)K = pp0$;

(v) $T(\delta^b)\lambda^{a,b}_T T^2(p)K = \alpha^b$;

(vi) $T(\delta^b)\lambda^{a,b}_T T(p)K = \langle K \cdot (ab), K, p \cdot b\rangle +$.

Proof. (i)

$$\begin{aligned}
& \lambda_T^{a,b} T(K)K \\
= \; & \lambda_T^{a,b} K_T K \text{ (by Corollary 6.21(i))} \\
= \; & \alpha_T^b K \text{ (by Prop. 8.12(iii))} \\
= \; & \langle K_T n, T(p)\cdot b\rangle + K \\
= \; & \langle K_T K n, T(p)K\cdot b\rangle + \text{ (additivity of } K) \\
= \; & \langle T(K)Kn, T(p)K\cdot b\rangle + \text{ (Corollary 6.21)}
\end{aligned}$$

(ii)

$$\begin{aligned}
& \lambda_T^{a,b} T^2(p)K \\
= \; & \lambda_T^{a,b} K_T T(p) \text{ (naturality of } K) \\
= \; & \alpha_T^b T(p) \text{ (by Prop. 8.12(iii))} \\
= \; & \langle K_T n, T(p)\cdot b\rangle + T(p) \\
= \; & \langle K_T T(p) n, T(p)T(p)\cdot b\rangle + \text{ (naturality of } + \text{ and } n) \\
= \; & \langle T^2(p)Kn, T(p)T(p)\cdot b\rangle + \text{ (by Lemma 6.18)}
\end{aligned}$$

(iii)

$$\begin{aligned}
& \lambda_T^{a,b} T(p)K \\
= \; & \mu^a K \text{ (by Prop. 8.12(iii))} \\
= \; & \langle K_T\cdot a, T(p), p\rangle + K \\
= \; & \langle T(K)K\cdot a, T(p)K, pK\rangle + \text{ (additivity of } K \text{ and Corollary 6.21(i))}
\end{aligned}$$

(iv)

$$\begin{aligned}
& T(\delta^b)\lambda_T^{a,b} T(K)K \\
= \; & T(\delta^b)\langle T(K)Kn, T(p)K\cdot b\rangle + \text{ (by (i))} \\
= \; & \langle T(\delta^b K)Kn, T(\delta^b p)K\cdot b\rangle + \\
= \; & \langle T(1\cdot b)Kn, K\cdot b\rangle + \text{ (by Proposition 8.7)} \\
= \; & \langle Kn\cdot b, K\cdot b\rangle + \text{ (additivity of } K) \\
= \; & pp0 \text{ (cancellation of negative terms)}
\end{aligned}$$

(v)

$$\begin{aligned}
& T(\delta^b)\lambda_T^{a,b} T^2(p)K \\
= \; & T(\delta^b)\langle T^2(p)Kn, T(p)T(p)\cdot b\rangle + \text{ (by (ii))} \\
= \; & \langle T(\delta^b T(p))Kn, T(\delta^b p)T(p)\cdot b\rangle + \\
= \; & \langle Kn, T(p)\cdot b\rangle + \text{ (by Proposition 8.7)} \\
= \; & \alpha^b
\end{aligned}$$

(vi)

$$\begin{array}{rl} & T(\delta^b)\lambda_T^{a,b}T(p)K \\ = & T(\delta^b)\mu_T^a K \text{ (by Prop. 8.12(ii))} \\ = & T(\delta^b)\langle K_T \cdot a, T(p), p\rangle{+}K \\ = & T(\delta^b)\langle K_T K \cdot a, T(p)K, pK\rangle{+} \text{ (by additivity of } K) \\ = & T(\delta^b)\langle T(K)K \cdot a, T(p)K, pK\rangle{+} \text{ (by Corollary 6.21(i))} \\ = & \langle T(1\cdot b)K \cdot a, T(1)K, p\delta^b K\rangle{+} \text{ (by Proposition 8.7 and naturality)} \\ = & \langle K \cdot (ab), K, p \cdot b\rangle{+} \text{ (by additivity of K and Proposition 8.7(iii))} \end{array}$$

□

We can now prove that the transformations $\lambda^{a,b}$ are mixed distributive laws.

Proposition 8.14. *For each $a, b \in \mathbb{Z}, \lambda^{a,b}$ is a mixed distributive law from the monad $\mathbb{T}_a$ to the comonad $\mathbb{T}^b$ in $\mathsf{Aff}(\mathbb{C}, T)$.*

Proof. As per definition 2.1 in [33], we need to establish four equations:

$$0_T\lambda^{a,b} = T(0),$$

$$\lambda^{a,b}p_T = T(p),$$
$$T(\delta^b)\lambda_T^{a,b}T(\lambda^{a,b}) = \lambda^{a,b}\delta_T^b,$$
$$T(\lambda^{a,b})\lambda_T^{a,b}T(\mu^a) = \mu_T^a\lambda^{a,b}.$$

For the first equation, using additivity of K and naturality of 0,

$$0\alpha^b = 0\langle Kn, T(p)\cdot b\rangle{+} = \langle p0n, p0\cdot m\rangle{+} = p0$$

and

$$0\mu^a = 0\langle K\cdot a, T(p)p\rangle{+} = \langle 0\cdot a, p0, 1\rangle{+} = 1,$$

so then

$$\begin{array}{rl} & 0\lambda^{a,b} \\ = & 0\langle\langle\alpha^b\ell, T(p)0\rangle T(+), \langle\mu^a, T(p)\rangle H\rangle{+} \\ = & \langle p0\ell, p00\rangle T(+), \langle 1, p0\rangle H\rangle{+} \\ = & \langle p0T(0), p0T(0)\rangle T(+), T(0)\rangle{+} \text{ (additivity of } H \text{ in the second component)} \\ = & \langle p0T(0), T(0)\rangle{+} \\ = & \langle p00, T(0)\rangle{+} \text{ (by naturality of 0)} \\ = & T(0) \end{array}$$

as required.

The second equation is Proposition 8.12(i).

The third equation asks for equality of two arrows with codomain T^3. Thus, as in the proof of the coassociativity of δ, it suffices to check the equality of these maps when post-composed by pK, $pT(p)$, pp, $T(p)K$, $T(p)T(p)$, $T(K)K$, and $T^2(p)K$. The first three of these are covered by checking equality when followed by p:

$$\begin{aligned}
& T(\delta^b)\lambda_T^{a,b}T(\lambda^{a,b})p \\
= \;& T(\delta^b)\lambda_T^{a,b}p\lambda^{a,b} \\
= \;& T(\delta^b)T(p)\lambda^{a,b} \text{ (by Prop. 8.12(i))} \\
= \;& \lambda^{a,b} \text{ (by Prop. 8.7(i))} \\
= \;& \lambda^{a,b}\delta_T^b p \text{ (again by Prop. 8.7(i))}
\end{aligned}$$

as required. For the equality when followed by $T(p)K$, consider

$$\begin{aligned}
& T(\delta^b)\lambda_T^{a,b}T(\lambda^{a,b})T(p)K \\
= \;& T(\delta^b)\lambda_T^{a,b}T(T(p))K \text{ (by Prop. 8.12(i))} \\
= \;& T(\delta^b)\lambda_T^{a,b}T^2(p)K \\
= \;& \alpha^b \text{ (by Prop. 8.13(v))}
\end{aligned}$$

while

$$\lambda^{a,b}\delta_T^b T(p)K = \lambda^{a,b}K = \alpha^b$$

by Propositions 8.7(ii) and 8.12(iii).

For equality when followed by $T(p)T(p)$, consider

$$\begin{aligned}
& T(\delta^b)\lambda_T^{a,b}T(\lambda^{a,b})T(p)T(p) \\
= \;& T(\delta^b)\lambda_T^{a,b}T(T(p))T(p) \text{ (by Prop.8.12(i))} \\
= \;& T(\delta^b)\lambda_T^{a,b}T^2(p)T(p) \\
= \;& T(\delta^b)\lambda_T^{a,b}T(p)T(p) \text{ (naturality of } p) \\
= \;& T(\delta^b)\mu_T^a T(p) \text{ (by Prop. 8.12(ii))} \\
= \;& T(\delta^b)\langle K_T \cdot a, T(p), p\rangle + T(p) \\
= \;& T(\delta^b)\langle K_T T(p) \cdot a, T(p)T(p), pT(p)\rangle + \text{ (by naturality of } +) \\
= \;& T(\delta^b)\langle T^2(p)K \cdot a, T(pp), pT(p)\rangle + \text{ (by Lemma 6.18(i))} \\
= \;& \langle T(\delta^b T(p))K \cdot a, T(\delta^b pp), p\delta^b T(p)\rangle + \\
= \;& \langle K \cdot a, T(p), p\rangle + \text{ (by Proposition 8.7)} \\
= \;& \mu^a
\end{aligned}$$

while

$$\lambda^{a,b}\delta_T^b T(p)T(p) = \lambda^{a,b}T(p) = \mu^a$$

by Propositions 8.7(ii) and 8.12(ii).

For the equality when followed by $T(K)K$, consider

$$\begin{aligned}
& T(\delta^b)\lambda_T^{a,b}T(\lambda^{a,b})T(K)K \\
= \; & T(\delta^b)\lambda_T^{a,b}T(\alpha^b)K \text{ (by Prop. 8.12(iii))} \\
= \; & T(\delta^b)\lambda_T^{a,b}\langle T(K)T(n), T^2(p), T^2(p), \ldots, T^2(p)\rangle T(+)K \text{ (b copies of $T^2(p)$)} \\
= \; & T(\delta^b)\lambda_T^{a,b}\langle T(K)Kn, T^2(p)K \cdot b\rangle + \text{ (additivity of K)} \\
= \; & \langle pp0, \alpha^b \cdot b\rangle + \text{ (by 8.13(iv and v))} \\
= \; & \alpha^b \cdot b
\end{aligned}$$

while

$$\begin{aligned}
& \lambda^{a,b}\delta_T^b T(K)K \\
= \; & \lambda^{a,b}\delta_T^b K_T K \text{ (by Corollary 6.21(i))} \\
= \; & \lambda^{a,b}(1 \cdot b)K \text{ (by Proposition 8.7(iii))} \\
= \; & \lambda^{a,b}(K \cdot b) \text{ (since K additive)} \\
= \; & \alpha^b \cdot b \text{ (by Prop. 8.12(iii))}
\end{aligned}$$

Hence, for the third equation it now suffices to check equality when post-composed by $T^2(p)K$. For this, consider

$$\begin{aligned}
& T(\delta^b)\lambda_T^{a,b}T(\lambda^{a,b})T^2(p)K \\
= \; & T(\delta^b)\lambda_T^{a,b}T(\mu^a)K \text{ (by Prop. 8.12(ii))} \\
= \; & T(\delta^b)\lambda_T^{a,b}\langle T(K)K \cdot a, T^2(p)K, T(p)K\rangle + \text{ (definition of μ^a, additivity of K)} \\
= \; & \langle pp0 \cdot a, \alpha^b, \langle K \cdot (ab), K, p \cdot b\rangle +\rangle + \text{ (by Prop. 8.12)} \\
= \; & \langle Kn, T(p) \cdot b, K \cdot (ab), K, p \cdot b\rangle + \text{ (by definition of α)} \\
= \; & (\langle K \cdot a, T(p), p\rangle +) \cdot b \\
= \; & \mu^a \cdot b
\end{aligned}$$

while

$$\begin{aligned}
& \lambda^{a,b}\delta_T^b T^2(p)K \\
= \; & \lambda^{a,b}\delta_T^b K_T T(p) \text{ (by naturality of K)} \\
= \; & \delta^b(1 \cdot b)T(p) \text{ (by Proposition 8.7(iii))} \\
= \; & \delta^b(T(p) \cdot b) \text{ (by naturality of $+$)} \\
= \; & \mu^a \cdot b \text{ (by Prop. 8.12(iii))}
\end{aligned}$$

as required.

For the fourth equation, we need to show

$$T(\lambda^{a,b})\lambda_T^{a,b}T(\mu^a) = \mu_T^a\lambda^{a,b}.$$

These are both maps into T^2, so it suffices to check their equality when post-composed by K, $T(p)$, and p.

We will start with the equality when followed by p. Consider

$$\begin{aligned}
& T(\lambda^{a,b})\lambda_T^{a,b}T(\mu^a)p \\
= \ & T(\lambda^{a,b})\lambda_T^{a,b}p\mu^a \\
= \ & T(\lambda^{a,b})T(p)\mu^a \text{ (by Prop. 8.12(i))} \\
= \ & T^2(p)\mu^a \text{ (by Prop. 8.12(i))} \\
= \ & \mu_T^a T(p) \text{ (naturality of } \mu^a) \\
= \ & \mu_T^a \lambda^{a,b} p \text{ (by Prop. 8.12(i))}
\end{aligned}$$

Next, we consider the equality when followed by $T(p)$. Consider

$$\begin{aligned}
& T(\lambda^{a,b})\lambda_T^{a,b}T(\mu^a)T(p) \\
= \ & T(\lambda^{a,b})\lambda_T^{a,b}T(pp) \text{ (by Proposition 8.2)} \\
= \ & T(\lambda^{a,b})\lambda_T^{a,b}T(p)T(p) \\
= \ & T(\lambda^{a,b})\mu_T^a T(p) \text{ (by Prop. 8.12(ii))} \\
= \ & T(\lambda^{a,b})T^2(p)\mu^a \text{ (naturality of } \mu^a) \\
= \ & T(\mu^a)\mu^a \text{ (by Prop. 8.12(ii))} \\
= \ & \mu_T^a \mu^a \text{ (since } \mu^a \text{ is associative)} \\
= \ & \mu_T^a \lambda^{a,b} T(p) \text{ (by Prop. 8.12(ii))}
\end{aligned}$$

Finally, we need to check the equality when followed by K. For this, first consider $\mu_T^a\lambda^{a,b}K$. By Proposition 8.12(iii), this equals $\mu_T^a\alpha^b$, which equals

$$\mu_T^a\langle Kn, T(p)\cdot b\rangle + = \langle \mu_T^a Kn, \mu_T^a T(p)\cdot b\rangle + \ (\star).$$

Now by additivity of K and Corollary 6.21(i),

$$\mu_T^a K = \langle K_T\cdot a, T(p), p\rangle + K = \langle K_T K\cdot a, T(p)K, pK\rangle + = \langle T(K)K\cdot a, T(p)K, pK\rangle +$$

while by naturality of $+$ and Lemma 6.18(i),

$$\mu_T T(p) = \langle K_T\cdot a, T(p), p\rangle + T(p) = \langle K_T T(p)\cdot a, T(pp), pT(P)\rangle + = \langle T^2(p)K\cdot a, T(pp), pT(p)\rangle +$$

Thus $\star$ consists of the sum of $-a$ copies of $T(K)K$, -1 of $T(p)K$, -1 of pK, ab copies of $T^2(p)K$, b copies of $T(pp)$, and b copies of $pT(p)$.

We now need to compare this to $T(\lambda^{a,b})\lambda_T^{a,b}T(\mu^a)K$. First, by additivity of K,

$$T(\mu^a)K = T(\langle K\cdot a, T(p), p\rangle +)K = \langle T(K)K\cdot a, T^2(p)K, T(p)K\rangle +$$

Then $\lambda_T^{a,b}T(\mu^a)K$ is the sum of the components

$$\lambda_T^{a,b}T(K)K\cdot a = \langle T(K)Kn\cdot a, T(p)K\cdot(ab)\rangle + \text{ (by Lemma 8.13(i))};$$

$$\lambda^{a,b}_T T^2(p)K = \langle T^2(p)Kn, T(p)T(p)\cdot b\rangle + \text{ (by Lemma 8.13(ii))};$$
$$\lambda^{a,b}_T T(p)K = \langle T(K)K\cdot a, T(p)K, pK\rangle + \text{ (by Lemma 8.13(iii))}.$$

However, note that in this sum, there are a copies of $T(K)K$ and of its negative, $T(K)Kn$. Thus, these terms cancel, giving $ab+1$ copies of $T(p)K$, -1 of $T^2(p)K$, b of $T(p)T(p) = T(pp)$, and one of pK ($\star\star$). We now need to pre-compose each of these terms with $T(\lambda^{a,b})$.

Using Proposition 8.12(i),

$$T(\lambda^{a,b})T(p)K = T(\lambda^{a,b}p)K = T(T(p))K = T^2(p)K,$$

and using that same result and naturality of p,

$$T(\lambda^{a,b})T(p)T(p) = T(T(p))T(p) = T(T(p)p) = T(pp).$$

Using Proposition 8.12(ii) and additivity of K,

$$T(\lambda^{a,b})T^2(p)K = T(\mu^a)K = \langle T(K)K\cdot a, T^2(p)K, T(p)K\rangle +$$

Moreover, using naturality and Proposition 8.12(iii),

$$T(\lambda^{a,b})pK = p\lambda^{a,b}K = p\alpha^b = \langle pKn, pT(p)\cdot b\rangle +$$

Combining each of these with the numbers of the terms found in $\star\star$, we now have a total of $ab+1$ copies of $T^2(p)K$, $-a$ of $T(K)K$, -1 of $T^2(p)K$, -1 of $T(p)K$, b of $T(pp)$, -1 of pK, and b of $pT(p)$. After summing the $T^2(p)K$ terms, this gives us the same result as $\star$, as required.

□

8.4 Bimonad and Hopf structure

In this final section, we generalize Jubin's results on bimonad and Hopf monad structure in the category of affine manifolds to the affine category of a tangent category, showing that in particular each $\lambda^{a,b}$ is not just a distributive law but in fact a bimonad; in some cases it is also a Hopf monad.

However, before we get to this, it is worth mentioning another point of interest. Since Theorem 8.11 has shown that c is a mixed distributive law of $\mathbb{T}_a$ over $\mathbb{T}^b$, one may wonder whether it also provides bimonad or Hopf monad structure. Unfortunately, in general this is not the case. For c to be a bimonad from $\mathbb{T}_a$ to $\mathbb{T}^b$ it would need to satisfy the equation

$$T(\delta^b)cT(\mu^a) = \mu^a\delta^b.$$

One can check that in local coordinates of affine manifolds this equation is not satisfied. More generally, in any tangent category, for these to be equal they would need to be equal when post-composed by $T(p)$; however, by Proposition 8.7(ii),

$$\mu^a\delta^b T(p) = \mu^a$$

while

$$\begin{aligned} & T(\delta^b)cT(\mu^a)T(p) \\ = \; & T(\delta^b)cT(pp) \text{ (by Proposition 8.2)} \\ = \; & T(\delta^b)pT(p) \text{ (tangent category axiom for } c\text{)} \\ = \; & p\delta^b T(p) \text{ (naturality of } p\text{)} \\ = \; & p \text{ (by Proposition 8.7(ii))} \end{aligned}$$

Even for $a = 0$, in most tangent categories $\mu^a \neq p$, and thus c will not give a bimonad structure. However, the morphisms $\lambda^{a,b}$ will, as Jubin found in the category of smooth manifolds.

As in the previous section, all these results require that the tangent category have negatives.

Theorem 8.15. *For any tangent category* $(\mathbb{C}, T)$ *that has negatives, for each* $a, b \in \mathbb{Z}$,

$$(\mathbb{T}_a, \mathbb{T}^b, \lambda^{a,b})$$

is a bimonad on $\mathsf{Aff}(\mathbb{C}, T)$.

Proof. We have already taken care of many of the parts of the definition (Definition 4.4). By Theorem 8.3, each $\mathbb{T}_a = (T, \mu^a, 0)$ is a comonad, by Theorem 8.8, each $\mathbb{T}^b = (T, \delta^b, p)$ is a comonad, and by Proposition 8.14, $\lambda^{a,b}$ provides a mixed distributive law between each such monad and comonad pair. The fact that p and 0 are monad and comonad morphisms also follows relatively easily: this requires precisely that

$$T(p)p = \mu^a p, \; 0\delta^b = 00, \text{ and } 0p = 1.$$

The first follows from naturality of p and Proposition 8.2, and the third is part of the definition of a tangent category. For the second, we first calculate (using various tangent category axioms) that

$$\langle 0, 0\rangle v^b = \langle 0, 0\rangle\langle \pi_0\ell \cdot b, \pi_1 0\rangle T(+) = \langle 0\ell \cdot b, 00\rangle T(+) = \langle 0T(0) \cdot b, 00\rangle T(+) = 00$$

so that since H is a horizontal connection we deduce that

$$0\delta^b = \langle\langle 0, 0\rangle v^b, \langle 0, 0\rangle H\rangle + = \langle 00, 00\rangle + = 00$$

as required.

The part of the proof that will take the most work is to show that

$$\mu^a\delta^b = T(\delta^b)\lambda_T^{a,b}T(\mu^a).$$

Now, these are maps with codomain T^2M. Recall from Corollary 3.11 that T^2M is a fibre product of three copies of TM, with projections $K, T(p), p$. Thus, to check the equality of the above maps, it suffices to check their equality when post-composed by $K, T(p)$, and p.

We begin with p. By Proposition 8.7(i),

$$\mu^a\delta^b p = \mu^a$$

while using naturality and Propositions 8.7(i) and 8.12(i),

$$T(\delta^b)\lambda_T^{a,b}T(\mu^a)p = T(\delta^b)\lambda_T^{a,b}p\mu^a = T(\delta)T(p)\mu^a = T(\delta p)\mu^a = \mu^a.$$

So the terms are equal when post-composed by p.

For equality with $T(p)$, using Proposition 8.7(ii),

$$\mu^a\delta^b T(p) = \mu^a.$$

On the other hand,

$$\begin{array}{rl} & T(\delta^b)\lambda_T^{a,b}T(\mu^a)T(p) \\ = & T(\delta^b)\lambda_T^{a,b}T(pp) \text{ (by Proposition 8.2)} \\ = & T(\delta^b)\lambda_T^{a,b}T(p)T(p) \\ = & T(\delta^b)\mu_T^a T(p) \text{ (by Prop. 8.12(ii))} \\ = & T(\delta^b)\langle K_T\cdot a, T(p), p\rangle + T(p) \\ = & T(\delta^b)\langle K_T T(p)\cdot a, T(p)T(p), pT(p)\rangle + \text{ (by naturality of } +) \\ = & T(\delta^b)\langle T^2(p)K\cdot a, T(p)T(p), pT(p)\rangle + \text{ (by Lemma 6.18(i))} \\ = & \langle T(\delta^b T(p))K\cdot a, T(\delta^b p)T(p), p\delta^b T(p)\rangle + \\ = & \langle K\cdot a, T(p), p\rangle + \text{ (by Proposition 8.7)} \\ = & \mu^a \end{array}$$

Thus the terms are equal when post-composed by $T(p)$.

The longest part of the proof is the equality when followed by K. First, consider $\mu^a\delta^b K$:

$$\begin{array}{rl} & \mu^a\delta^b K \\ = & \mu^a(1\cdot b) \text{ (by Proposition 8.7)} \\ = & \langle K\cdot a, T(p), p\rangle(1\cdot b) \\ = & \langle K\cdot(ab), T(p)\cdot b, p\cdot b\rangle + \text{ (by associativity of addition)} \end{array}$$

That is, $\mu^a\delta^b K$ consists of the sum of ab copies of K, b copies of $T(p)$, and b copies of p.

We now consider $T(\delta^b)\lambda_T^{a,b}T(\mu^a)K$; we need to show it also consists of the sum of ab copies of K, b copies of $T(p)$, and b copies of p. First, consider $T(\mu^a)K$:

$$T(\mu^a)K = \langle T(K), T(K), \ldots T(K), T^2(p), T(p)\rangle T(+)K = \langle T(K)K\cdot a, T^2(p)K, T(p)K\rangle +$$

by additivity of K. Thus, to consider

$$T(\delta^b)\lambda_T^{a,b}T(\mu^a)K, (\star)$$

we will consider each of $T(\delta^b)\lambda_T^{a,b}T(K)K$ (1), $T(\delta^b)\lambda_T^{a,b}T^2(p)K$ (2), and $T(\delta^b)\lambda_T^{a,b}T(p)K$ (3) separately; $\star$ is then a sum of these terms.

To find $T(\delta^b)\lambda_T^{a,b}T(K)K$, we first calculate $\lambda_T^{a,b}T(K)K$:

$$\begin{aligned} & \lambda_T^{a,b}T(K)K \\ = \ & \lambda_T^{a,b}K_TK \text{ (by Corollary 6.21(i))} \\ = \ & \alpha_T^bK \text{ (by Prop. 8.12(iii))} \\ = \ & \langle K_Tn, T(p)\cdot b\rangle + K \\ = \ & \langle K_TKn, T(p)K\cdot b\rangle + \text{ (additivity of } K) \\ = \ & \langle T(K)Kn, T(p)K\cdot b\rangle + \text{ (Corollary 6.21(i))} \end{aligned}$$

So then (1) becomes

$$\begin{aligned} & T(\delta^b)\langle T(K)Kn, T(p)K\cdot b\rangle + \\ = \ & \langle T(\delta^bK)Kn, T(\delta^bp)K\cdot b\rangle + \\ = \ & \langle T(1\cdot b)Kn, K\cdot b\rangle + \text{ (by Proposition 8.7)} \\ = \ & \langle Kn\cdot b, K\cdot b\rangle + \text{ (additivity of } K) \end{aligned}$$

That is, (1) sums b copies of negative K and b copies of K. Thus (1) is simply a zero term.

We now turn to (2); that is, we consider $T(\delta^b)\lambda_T^{a,b}T^2(p)K$:

$$\begin{aligned} & T(\delta^b)\lambda_T^{a,b}T^2(p)K \\ = \ & T(\delta^b)T^2(p)\lambda^{a,b}K \text{ (by naturality of } \lambda^{a,b}) \\ = \ & T(\delta^bT(p))\alpha^b \text{ (using Prop. 8.12(iii))} \\ = \ & \alpha^b \text{ (by Proposition 8.7(ii))} \\ = \ & \langle Kn, T(p)\cdot b\rangle + \end{aligned}$$

Thus, (2) contributes one negative copy of K and b copies of $T(p)$ to $\star$.

We now consider (3):

$$\begin{aligned} & T(\delta^b)\lambda_T^{a,b}T(p)K \\ = \ & T(\delta^b)\mu_T^aK \text{ (by Prop. 8.12(ii))} \\ = \ & T(\delta^b)\langle K_T\cdot a, T(p), p\rangle + K \\ = \ & T(\delta^b)\langle K_TK\cdot a, T(p)K, pK\rangle + \text{ (additivity of } K) \\ = \ & T(\delta^b)\langle T(K)K\cdot a, T(p)K, pK\rangle + \text{ (by Corollary 6.21(i))} \\ = \ & \langle T(\delta^bK)K\cdot a, T(\delta^bp)K, p\delta^bK\rangle + \text{ (using naturality of } p) \\ = \ & \langle T(1\cdot b)K\cdot a, K, p(1\cdot b)\rangle + \text{ (by Proposition 8.7)} \\ = \ & \langle K\cdot(ba), K, p\cdot b\rangle + \text{ (by additivity of } K) \end{aligned}$$

Thus, (3) contributes $ba = ab$ copies of K, one additional copy of K, and b copies of p.

Thus, since $\star$ is the sum of (1), (2), and (3), in total, $\star$ contains $ab+1-1=ab$ copies of K, b copies of $T(p)$, and b copies of p. This is the same as $\mu^a\delta^b K$, and thus we have the desired equality of the terms when post-composed by K.

We have shown that $\mu^a\delta^b$ and $T(\delta^b)\lambda_T^{a,b}T(\mu^a)$ are equal when post-composed by K, $T(p)$, and p, and thus, by Corollary 3.11, $\mu^a\delta^b = T(\delta^b)\lambda_T^{a,b}T(\mu^a)$, as required. $\square$

Finally, we consider Hopf monad structure (see [33, Definition 5.2]). For this, we need an antipode: a natural transformation $S : T \longrightarrow T$ with certain equational requirements. In [26, Theorem 3.2.2], antipodes are provided for any a, b such that $1+ab \neq 0$. However, the antipode is defined by scalar multiplying a tangent vector by $-\frac{1}{1+ab}$. In a general tangent category (even with negatives) we do not have structure that corresponds to "division by scalars". However, if $a=0$ or $b=0$, the above is simply negation of the tangent vector. In this case, we do indeed get Hopf monad structure in the general setting of a tangent category with negatives.

Theorem 8.16. *If $(\mathbb{C}, T)$ has negatives, then for $a, b \in \mathbb{Z}$, if either $a=0$ or $b=0$ then*

$$(\mathbb{T}_a, \mathbb{T}^b, \lambda^{a,b}, n)$$

is a Hopf monad on $\mathsf{Aff}(\mathbb{C}, T)$.

Proof. By the previous result, all we need to show is that n is an antipode, i.e., that we have

$$\delta^b n\mu^a = \delta^b T(n)\mu^a = p0 : TM \longrightarrow TM$$

(see [33, Definition 5.2]). We first calculate $\delta^b T(n)\mu^a$:

$$\begin{array}{cl} & \delta^b T(n)\mu^a \\ = & \delta^b T(n)\langle K\cdot a, T(p), p\rangle + \\ = & \delta^b\langle Kn\cdot a, T(p), pn\rangle + \text{ (by additivity of } K\text{, naturality of } p\text{, and the equation } np=p) \\ = & \langle (1\cdot b)n\cdot a, 1, n\rangle + \text{ (by Proposition 8.7)} \\ = & \langle 1, n\rangle + \text{ (since } a=0 \text{ or } b=0) \\ = & p0. \end{array}$$

Similarly,

$$\begin{array}{cl} & \delta^b n\mu^a \\ = & \delta^b n\langle K\cdot a, T(p), p\rangle + \\ = & \delta^b\langle Kn\cdot a, T(p)n, p\rangle + \text{ (by additivity of } K \text{ and the equation } np=p) \\ = & \langle (1\cdot b)n\cdot a, n, 1\rangle + \text{ (by Proposition 8.7)} \\ = & \langle n, 1\rangle + \text{ (since } a=0 \text{ or } b=0) \\ = & p0 \end{array}$$

as required. $\square$

References

[1] L. Auslander. The structure of complete locally affine manifolds. *Topology* 3, pp. 131-139, (1964).

[2] L. Auslander, L. Markus. Holonomy of flat affinely connected manifolds. *Annals of Mathematics*, 62, (1955).

[3] M. Barr, C. Wells. *Toposes, Triples and Theories*, Springer-Verlag, (1984).

[4] K. Bauer, B. Johson, C. Osborne, E. Riehl, and A. Tebbe, Directional derivatives and higher order chain rules for abelian functor calculus, available at http://arxiv.org/abs/1610.01930 (2017).

[5] F. Bergeron, G. Labelle, P. Leroux. *Combinatorial Species and Tree-like Structures*. Cambridge University Press, (1997).

[6] R. Blackwell, G. M. Kelly, A. J. Power. Two-dimensional monad theory, *Journal of Pure and Applied Algebra* 59, pp. 1–41, (1989).

[7] R. Blute, J.R.B. Cockett, R.A.G. Seely. Differential categories. *Mathematical Structures in Computer Science* 16, pp. 1049-1083, (2006).

[8] R. Blute, J.R.B. Cockett, R.A.G. Seely. Cartesian differential categories. *Theory and Applications of Categories* 22, pp. 622–672, (2009).

[9] R. Blute, T. Ehrhard, C. Tasson. A Convenient Differential Category, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 53, pp. 211-232, (2012).

[10] M. Bunge and P. Sawyer. On connections, geodesics and sprays in synthetic differential geometry. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 25, pp 221–258, 1984.

[11] J.R.B. Cockett, G. Cruttwell. Differential structure, tangent structure, and SDG. *Applied Categorical Structures* 22, pp. 331-417, (2014).

[12] J.R.B. Cockett, G. Cruttwell. The Jacobi identity for tangent categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 56, pp. 301-316, (2015).

[13] J.R.B. Cockett, G. Cruttwell. Differential bundles and fibrations for tangent categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 59 (1), pg. 10–92, 2018.

[14] J.R.B. Cockett, G. Cruttwell. Connections in tangent categories. *Theory and Applications of Categories* 32, pp. 835-888 (2017).

[15] G. Cruttwell. Forms and exterior differentiation in Cartesian differential categories. *Theory and Applications of Categories* 28, pp. 981-1001, (2013).

[16] G. S. H. Cruttwell, R. B. B. Lucyshyn-Wright. A simplicial foundation for differential and sector forms in tangent categories. *Journal of Homotopy and Related Structures*, 2018.

[17] C. Dodson, T. Poston. *Tensor Geometry: The Geometric Viewpoint and its Uses.* Springer-Verlag, (2009).

[18] T. Ehrhard, L. Regnier The differential λ-calculus. *Theoretical Computer Science* 309 pp. 1–41, (2003).

[19] T. Ehrhard, L. Regnier. Differential interaction nets. Workshop on Logic, Language, Information and Computation (WoLLIC), invited paper. *Electronic Notes in Theoretical Computer Science* 123, March 2005, Elsevier.

[20] A.Frölicher, A. Kriegl. *Linear Spaces and Differentiation Theory*, Wiley, (1988).

[21] R. Garner, An embedding theorem for tangent categories. *Advances in Mathematics* 323 (2018), 668–687.

[22] J.-Y. Girard, Linear logic. *Theoretical Computer Science* 50 (1987) 1–102.

[23] W. Goldman. Two examples of affine manifolds. *Pacific Journal of Mathematics* 94, pp. 327–330, (1981).

[24] W. Goldman. Complete affine manifolds: A survey. Preprint found at http://www.math.umd.edu/∼wmg, (2013).

[25] N. Hicks. Connexion preserving, conformal, and parallel maps. Michigan Mathematical Journal 10, pp. 295-302, (1963).

[26] B. Jubin. *The Tangent Functor Monad and Foliations.* Ph.D. Thesis, University of California, Berkeley (2012). Available at arXiv:1401.0940.

[27] A. Kock. *Synthetic Differential Geometry, Second Edition*, Cambridge University Press, (2006).

[28] I. Kolar, P. Michor, and J. Slovak. *Natural Operations in Differential Geometry.* Springer-Verlag, (1993).

[29] A. Kriegl, P. Michor. *The Convenient Setting of Global Analysis.* American Mathematical Society, (1997).

[30] S. Lang. *Fundamentals of differential geometry.* Springer-Verlag, (1999).

[31] P. Leung, *Classifying tangent structures using Weil algebras*, *Theory and Applications of Categories* **32** (2017), no. 9, 286–337.

[32] R. B. B. Lucyshyn-Wright. On the geometric notion of connection and its expression in tangent categories. *Theory and Applications of Categories* (2018, to appear). Available as arXiv:1705.10857.

[33] B. Mesablishvili, R. Wisbauer. Bimonads and Hopf monads on categories. *Journal of K-Theory* 7, pp. 349–388, (2011).

[34] K. Nomizu, T. Sasaki. *Affine Differential Geometry*. Cambridge University Press, (1994).

[35] L. Patterson. Connexions and prolongations. *Canadian Journal of Mathematics* 27, pp766-791, (1975).

[36] J. Rosický. Abstract tangent functors. *Diagrammes* 12, Exp. No. 3, (1984).

[37] J. Szilasi, R. Lovas, and D. Kertész. *Connections, sprays, and Finsler structures*. World Scientific, (2014).