T-MONOIDS AND
2-DIMENSIONAL CATEGORY THEORY

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1. Introduction

The idea of these notes is two-fold: first of all, to give a brief overview of some advanced topics in category theory, such as enriched categories, internal categories, 2-categories, and double categories, and secondly, to get to a result of myself and Mike Shulman [5] on how a great number of mathematical objects can be constructed in the same way. These mathematical objects include (but are not limited to):

- metric spaces,
- topological spaces,
- closure spaces,
- approach spaces,
- ordered sets,
- rings,
- algebras,
- enriched categories,
- internal categories,
- multicategories.

In fact, the construction will give more than this: it will also give the morphisms between these objects, as well as the "relations" or "bi-modules" between these objects, and the morphisms of these. So, given a certain input (a "double monad" T), we get an output of (structure X, X-morphisms, X-bimodules, X-bimodule morphisms), where X is one of the above, depending on what T we start with.

But, we have quite a ways before we get to that result! To get there, we first need to understand two generalizations of the notion of category: enriched categories, and internal categories.

2. Enriched Categories and Internal Categories

Basic category theory can be generalized in two different ways, depending on what you take as the definition of a category. One definition of a category is the "hom-set" definition: a category $\mathbf{X}$ consists of:

- a set of objects X_0,
- for each pair of objects $x, y \in X_0$, a set of morphisms $\mathbf{X}(x, y)$ (the "hom-sets"),
- a composition operation $\mathbf{X}(x, y) \times \mathbf{Y}(y, z) \xrightarrow{c(x,y,z)} \mathbf{X}(x, z)$,
- identities $1 \xrightarrow{id_x} \mathbf{X}(x, x)$ (where 1 is a 1-element set),
- associativity and unit axioms.

Another definition is the "set of arrows" definition: a category $\mathbf{X}$ consists of:

- a set of objects X_0,
- a set of arrows X_1,
- domain and codomain operations $X_1 \xrightarrow{\text{dom, cod}} X_0$,
- an identity operation $X_0 \xrightarrow{id} X_1$,
- a composition operation $X_2 \xrightarrow{c} X_1$, where X_2 is the set of "composable pairs of arrows",
- assocativity and unit axioms.

Note the difference between the two definitions: in the first, the arrows are broken up into seperate hom-sets $\mathbf{X}(x, y)$, while in the second definition, all the arrows are lumped together in a single set. Generalizing the first definition will give us enriched categories, and generalizing the second definition will give us internal categories.

2.1. Enriched Categories. We begin with some motivation about why we might want to generalize the hom-set definition of a category. In many categories, the hom-"sets" have more structure than merely being sets. For example, consider the category $\mathbf{ab}$ of abelian groups and group homomorphisms. Here, for abelian groups G, H, the hom-set $\mathbf{ab}(G, H)$ has more structure: it is actually itself an abelian group. If we have $f_1, f_2 \in \mathbf{ab}(G, H)$, we can add them by $(f_1 + f_2)(g) = f_1(g) + f_2(x)$, and there is an identity $0(g) = 0$. So, $\mathbf{ab}(G, H)$ is not just a set, it's an abelian group. So, in general, we would like to have a theory of categories in which the hom-"sets" really have more structure: in particular, we would like them to be objects of some *other* category $\mathbf{V}$.

What properties must this category possess? If we would like to have a category where the hom-sets $\mathbf{X}(x, y)$ are objects of some other category $\mathbf{V}$, then composition will be a $\mathbf{V}$-arrow $\mathbf{X}(x, y) \times \mathbf{X}(y, z) \longrightarrow \mathbf{X}(x, z)$. So, one thought is that we will need $\mathbf{V}$ to have products. In fact, this turns out to be too strong. All we need is a category which allows you to "multiply" objects, but neccesarily by taking their categorical product. Such an entity is a monoidal category.

2.2. Definition. *A monoidal category consists of:*

- *a category* $\mathbf{V}$*,*
- *a "multiplication" functor* $\mathbf{V} \times \mathbf{V} \xrightarrow{\otimes} \mathbf{V}$
- *a "unit" object* $I \in \mathbf{V}_0$*,*
- *natural isomorphisms* $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$*,* $X \otimes I \cong X \cong I \otimes X$*,*
- *coherence equations for the natural isomophisms.*

2.3. Example. The category **ab**, with $\otimes$ the tensor product of abelian groups, and $I = \mathcal{Z}$, is a monoidal category.

2.4. Example. For k a field, the category of k-vector spaces, $\mathbf{vec}_k$, is a monoidal category when equipped with $\otimes$ the tensor product of vector spaces, and $I = k$.

2.5. Example. Any category with finite products, where $\otimes$ is the categorical product $\times$, and $I = 1$ (the terminal object). The categories of sets and small categories are examples of these.

2.6. Example. As a smaller example of the above, $\mathbf{V} = 0 \leq 1$, where $\otimes = \wedge$ (infimum), and $I = 1$.

2.7. Example. The set of extended non-negative real numbers $[0, \infty]$, with $\geq$, and $\otimes = +$, $I = 0$.

With this idea of monoidal category, we can consider categories "enriched" in a monoidal category, generalizing the "hom-set" definition.

2.8. Definition. *If* $(\mathbf{V}, \otimes, I)$ *is a monoidal category, then a* $\mathbf{V}$*-enriched category* $\mathbf{X}$ *consists of:*

- *a set of objects* X_0*,*
- *for each pair of objects* $x, y \in X_0$*, a morphism object* $\mathbf{X}(x, y) \in \mathbf{V}$*,*
- *a composition operation* $\mathbf{X}(x, y) \otimes \mathbf{Y}(y, z) \xrightarrow{c(x,y,z)} \mathbf{X}(x, z)$ *(which is now not a function, but a* $\mathbf{V}$*-arrow),*
- *identities* $I \xrightarrow{id_x} \mathbf{X}(x, x)$ *(where 1 is a 1-element set),*

- *associativity and unit axioms.*

Note the similarity to the hom-set definition of a category.

2.9. Example. As discussed above, **ab** is itself an **ab**-category.

2.10. Example. Similarly, $\mathbf{vec}_k$ is both a $\mathbf{vec}_k$ category and an **ab**-category.

2.11. Example. Recall that one-object categories are monoids. One-object **ab**-categories are also interesting: they are rings. Indeed, suppose $\mathbf{X}$ is an **ab**-category with one object $*$. Then it has $\mathbf{X}(*,*)$ some abelian group; call it R. The composition is then an operation $R \otimes R \longrightarrow R$: this provides the ring multiplication. The identity $\mathbf{Z} \longrightarrow R$ supplies the ring identity (recall that giving a group homomophism from the integers to a group is the same as giving an element of that group). The associativity and unit axioms then give the associativity and unit axioms for the ring.

2.12. Example. Similarly, a one-object $\mathbf{vec}_k$-category is a k-algebra.

2.13. Example. For the very small example $\mathbf{V} = (0 \leq 1)$, $\mathbf{V}$-categories are pre-orders (partial orders without anti-symmetry). Indeed, suppose we have a $(0 \leq 1)$-category. That is, we have a set X, and for each $x, y \in X$, $\mathbf{X}(x,y) \in \{0,1\}$. Writing $x \leq y$ if $\mathbf{X}(x,y) = 1$, we can see that the composition arrow $\mathbf{X}(x,y) \wedge \mathbf{X}(y,z) \leq \mathbf{X}(x,z)$ is transitivity of $\leq$. Similarly, the identity arrow $1 \leq \mathbf{X}(x,x)$ expresses reflexivity: $x \leq x$.

2.14. Example. The $([0,\infty], \geq, +, 0)$-categories are also interesting: they are generalized metric spaces [7]. A $[0,\infty]$-category consists of a set X_0, and for each x, y, $\mathbf{X}(x,y) \in [0,\infty]$. Thinking of this as the distance from x to y, we can see that the composition arrow $\mathbf{X}(x,y) + \mathbf{X}(y,z) \geq \mathbf{X}$ is the triangle inequality, while the identity arrow $0 \geq \mathbf{X}(x,x)$ says $\mathbf{X}(x,x) = 0$. These differ from usual metric spaces in that:

- they allow the distance ∞,
- they do not have $\mathbf{X}(x,y) = 0 \Rightarrow x = y$ (though this can be achieved with by modding out),
- they are non-symmetric: $\mathbf{X}(x,y) \neq \mathbf{X}(y,x)$.

This last item, however, can be useful, if one wishes to have a metric to indicate "work done to move around a mountainous region" or the "shortest distance between locations using (possibly 1-way) roads", both of which are non-symmetric. Many general theorems, such as the Banach fixed point theorem, hold in non-symmetric metric spaces (in fact, it would be interesting to find which results of metric space theory rely on symmetry of the metric).

So, enriched categories are useful in a few different ways:

- to add extra structure to "large" categories like **ab** and $\mathbf{vec}_k$,

- to relate structures like ordered sets and metric spaces directly to category theory.

There is also a notion of “enriched functor”; and in many of the examples considered above, these are useful and interesting.

2.15. Definition. *Suppose* $\mathbf{X}$ *and* $\mathbf{Y}$ *are* $\mathbf{V}$*-categories. A* $\mathbf{V}$*-enriched functor* $\mathbf{X}\xrightarrow{F}\mathbf{Y}$ *consists of:*

- *a function* $X_0 \xrightarrow{F} Y_0$*,*
- $\mathbf{V}$*-arrows* $\mathbf{X}(x,y) \xrightarrow{\bar{F}} \mathbf{Y}(Fx,Fy)$ *(think of this as the assignment* $(x \xrightarrow{f} y) \mapsto (Fx \xrightarrow{Ff} Fy)$*),*
- *axioms saying that* $\bar{F}$ *preserves composition and identities.*

2.16. Example. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are $(0 \leq 1)$-categories; that is, pre-orders. Then a $(0 \leq 1)$-functor between them is an order-preserving function, as the existence of $\bar{F}$ says that $x \leq y \Rightarrow Fx \leq Fy$.

2.17. Example. Similarly, for $\mathbf{V} = ([0,\infty], \geq, +, 0)$, a $\mathbf{V}$-functor between metric spaces $\mathbf{X}, \mathbf{Y}$ is a contraction: $\mathbf{X}(x,y) \geq \mathbf{Y}(Fx,Fy)$.

2.18. Example. A **ab**-functor between rings R, S (seen as **ab**-categories) is a ring homomorphism. Indeed, as **ab**-categories, R and S have only one object, so the function on objects does nothing. However, the arrow assignment $R \longrightarrow S$ is an group homomorphism, and the fact that it preserves composition and identities tells one that it is also a ring homomorphism.

2.19. Example. If R is a ring, then an **ab**-functor $R \xrightarrow{F} \mathbf{ab}$ is a (one-sided) R-module. Indeed, on objects, it is a function $\{*\} \longrightarrow \mathbf{ab}$, so that it picks out a single abelian group M. The action on arrows, $\bar{F}$, is then a group homomorphism $R \xrightarrow{\bar{F}} \mathbf{ab}(M,M)$. Thinking of this as the map $r \mapsto (m \mapsto r \cdot m))$, one can check that the axioms state exactly that this gives an action of R on M, so that it is an R-module.

In fact, enriched functors to the base $\mathbf{V}$ (when it itself has a canonical $\mathbf{V}$-category structure) are usually interesting.

2.20. Example. If $\mathbf{X}$ is an ordered set and, then a $(0 \leq 1)$-functor from $\mathbf{X}$ to $(0 \leq 1)$ are the same as up-closed subsets of $\mathbf{X}$ (subsets $A \subseteq X$ such that $a \in A, a \leq x$ implies $x \in A$).

Note that $[0,\infty]$ is itself a $[0,\infty]$-category (ie., metric space), with $d(a,b) = b - a$ (truncated subtraction).

2.21. Example. Suppose that F is a set of bakeries, and we have a way of transporting bread between the bakeries. The cost of such transportation is a metric T on the set F. Suppose further that each bakery f had a certain cost $\phi(f)$ associated to making a loaf of bread there. What does it mean to say that ϕ is a functor $(F,T) \longrightarrow [0,\infty]$? It says that for any two bakeries b_1, b_2, $\phi(b_2) - \phi(b_1) \leq T(b_1,b_2)$, or instead,

$$\phi(b_2) \leq \phi(b_1) + T(b_1,b_2)$$

If ϕ was *not* an enriched functor, then there would be some b_1, b_2 such that

$$\phi(b_2) > \phi(b_1) + T(b_1,b_2)$$

That is, it would be cheaper to make bread at bakery b_1 and transport it to b_2 than it would be to make bread at b_2. Thus, we might as well shut-down bakery b_2. Thus, ϕ is an enriched functor if and only if the production costs at each bakery make sense, given our transportation costs. [1]

So, just as enriched categories capture some useful concepts, so do too enriched functors. Naturally, there is also a notion of "enriched natural transformation" between enriched functors.

2.22. Definition. *Let* $(\mathbf{V},\otimes,I)$ *be a monoidal category, and suppose* $\mathbf{X} \xrightarrow{F,G} \mathbf{Y}$ *are* $\mathbf{V}$*-functors between* $\mathbf{V}$*-categories. A* $\mathbf{V}$*-natural transformation* $F \xrightarrow{\alpha} G$ *consists of:*

- *natural* $\mathbf{V}$*-maps* $I \xrightarrow{\alpha_X} \mathbf{Y}(Fx,Gx)$*,*
- *with an axiom expressing naturality.*

2.23. Example. Recall that in the case $\mathbf{V} = 0 \leq 1$, $\mathbf{V}$-functors $\mathbf{X} \xrightarrow{F,G} \mathbf{Y}$ are order-preserving functions. A $\mathbf{V}$-natural transformation $F \xrightarrow{\alpha} G$ means that we have $1 \leq \mathbf{Y}(Fx,Gx)$; that is, $Fx \leq Gx$ for all x.

2.24. Example. Recall that in the case $\mathbf{V} = [0,\infty]$, $\mathbf{V}$-functors $\mathbf{X} \xrightarrow{F,G} \mathbf{Y}$ are contractions. A $\mathbf{V}$-natural transformation $F \xrightarrow{\alpha} G$ means that we have $0 \geq \mathbf{Y}(Fx,Gx)$; that is, $\mathbf{Y}(Fx,Gx) = 0$ for all x. If the generalized metric space $\mathbf{Y}$ has the usual "identity of indiscerniables" property (that is, $\mathbf{Y}(y_1,y_2) = 0 \Rightarrow y_1 = y_2$), then this means a $\mathbf{V}$-natural transformation exists if only if $F = G$.

2.25. Example. Recall that for $\mathbf{V} = \mathbf{ab}$, and R a ring, $\mathbf{V}$-functors $R \xrightarrow{M,N} \mathbf{ab}$ are R-modules. A $\mathbf{ab}$-natural transformation between them gives a morphism $\mathcal{Z} \longrightarrow \mathbf{ab}(M,N)$, that is, an abelian group morphism $M \xrightarrow{\alpha} N$ which preserves the action of M. Thus, α is a module morphism.

Next time, we'll look at how generalizing the "set of arrows" definition also leads to interesting examples.

[1] Thanks to Simon Willerton for this example.

2.26. Internal Categories. In the previous section, we generalized the hom-set definition by asking that each hom $\mathbf{X}(x, y)$ be an object of another category, rather than a set. On the other hand, the "set of arrows" defintion of a category says that a category is a set of objects, and a set of arrows. An internal category will have the set of objects and set of arrows be objects of some other category. Note the differences: in the first genearlization, the objects are still a set, and each hom $\mathbf{X}(x, y)$ is an object of some other category $\mathbf{V}$. In this generalization, the objects, rather than being a set, are an object of some category $\mathbf{C}$, while the arrows are bundled together into another object of $\mathbf{C}$.

Before we give the full definition, we would like to motivate this idea. Why would want to consider internal categories? To begin with, we will look at a more basic object: groups internal to a category. Often, a group has more structure, such as being a topological group or a Lie group. This idea is captured by considering groups internal to a category.

2.27. Definition. *Let $\mathbf{C}$ be a category with finite products. A* group internal to $\mathbf{C}$ *consists of:*

- *an object G of $\mathbf{C}$,*
- *a multiplication $G \times G \xrightarrow{m} G$,*
- *an identity $1 \xrightarrow{G}$,*
- *axioms to make this a group, such as $(1 \circ e) \circ m = 1$ (ie., $ge = g$).*

2.28. Example. A group internal to **set** is a group.

2.29. Example. A group internal to the category of topological spaces and continous functions, **top**, is a topological group.

2.30. Example. A group internal to the category of smooth manifolds and smooth functions, **man**, is a Lie group.

2.31. Example. A group internal to the category of small categories and functors, **cat**, is a crossed module [6] (for more about this idea, see [2]).

2.32. Example. A group internal to the category of groups and group homomorphisms, **group** is an abelian group. Indeed, let G be an internal group in **group**. Then G has two multiplications: one because it is a group, $\cdot$, and because it is an internal group, $*$. Since $1 \xrightarrow{e} G$ is a group homomorphism, the identity as an internal group must be the same as the identity of G. That $G \times G \xrightarrow{m} G$ is a group homomorphism says that

$$(g_1 * h_1) \cdot (g_2 * h_2) = (g_1 \cdot g_2) * (h_1 \cdot h_2)$$

One can show that this condition, along with the fact that the identities of * and $\cdot$ are the same, shows that, in fact $* = \cdot$, and they are commutative. This is known as the Eckmann-Hilton argument, and is an amusing exercise if you haven't seen it before.

So, groups internal to categories capture useful concepts. To see why categories internal to other categories might be useful, we need to see why *groupoids* are useful.

2.33. Definition. *A* groupoid *is a category in which every arrow is invertible. That is, for each* $A \xrightarrow{f} B$*, there exists a* $B \xrightarrow{g} A$ *such that* $fg = 1_B$*,* $gf = 1_A$*.*

If one thinks of groups as the study of symmetries, then groupoids are the study of "symmetries with many objects". We'll look at a number of examples of groupoids, then talk about internal groupoids and internal categories.

2.34. Example. We will consider two types of puzzles. In the first, the symmetries form a group, in the second, they more naturally form a groupoid [2]. In a Rubik's cube, we can think of sequences of rotations as the elements of a group. By reversing the sequence of rotations, we get inverses, and doing nothing is the identity. One can study properties of the Rubik's cube by studying properties of the group.

Now, consider the "15-puzzle". Here, there are blocks labelled 1-15 set in a 4x4 grid. On each turn, one can exchange a block and the empty spot. Now, consider a sequence of moves. We would like this to also form a group; after all, one can again reverse moves and do nothing. The problem is that one cannot compose two arbitrary moves. For example, we cannot compose "moving the empty block from $(1,1)$ to $(1,2)$" and "moving the empty block from $(4,4)$ to $(3,4)$". The position of the empty block determines whether we can compose two moves. Thus, instead of a group, there is a groupoid with:

- objects of the groupoid being positions (m,n),
- an arrow $(m,n) \longrightarrow (m',n')$ is a sequence of moves of the empty block from (m,n) to (m',n').

Thus, to study this puzzle, we use this groupoid, rather than a group.

2.35. Example. An alternative to the fundamental group of a space is the fundamental groupoid of a space. In the fundamental group, one fixes a basepoint, and considers equivalence classes of loops at that point.

By contrast, the fundamental groupoid considers all points of the space. The fundamental groupoid of a space X has:

- objects the points of the space X,
- an arrow $x \longrightarrow y$ is an equivalence class of paths from x to y,

This can be slightly more convenient, as one doesn't have to choose a base point of the space. In particular, the fundamental groupoid of a space with many components will have information about all the components, whereas with the fundamental group, one only gets information about a single component.

[2]Example from http://cornellmath.wordpress.com/2008/01/27/puzzles-groups-and-groupoids/

2.36. Example. Recall that a category with at most one arrow between any two objects is a pre-ordered set, where $x \leq y$ if there exists an arrow $x \longrightarrow y$. Similarly, a groupoid with at most arrow between any two objects is an equivalence relation: $x \cong y$ if there exists an arrow $x \longrightarrow y$ (the existence of an inverse guarantees the symmetry of the equivalence relation).

2.37. Example. Suppose G acts on a set X. There is a standard equivalence relation on X given by $x \cong y$ if $gx = y$, whose equivalence classes are the orbits. Thus, this forms a groupoid. However, there is a groupoid with more information: the action groupoid, with

- objects the elements of X,
- an arrow $x \longrightarrow y$ is a group element $g \in G$ such that $gx = y$.

Thus, this is an extension of the orbit equivalence relation. However, it also contains more: the subcategory consisting of a single object $x \in X$ is the isotropy subgroup at x: $\{g \in G : gx = x\}$. Thus, this action groupoid contains both the orbit equivalence relation and the isotropy subgroups.

2.38. Example. Recall that for each n, there is a group consisting of the $n \times n$ invertible matrices. These can be combined into a single groupoid, whose objects are the natural numbers, and where an arrow $n \longrightarrow n$ is an invertible $n \times n$ matrix.

As further examples, Grothendieck generalized Galois theory by considering "Galois groupoids": see [3]. Groupoids are also a central idea in Alain Connes' noncommutative geometry: see [4]. Ronnie Brown's survey [**?**] is also a useful reference for further examples and ideas about groupoids.

Since internal groups are useful, and groupoids are useful, it seems important to be able to combine the concepts. We can do this by defining internal groupoids, and more generally, internal categories.

2.39. Definition. *Suppose* $\mathbf{C}$ *is a category with pullbacks. A category internal to* $\mathbf{C}$, $\mathbf{X}$, *consists of:*

- *an object* X_0 *of* $\mathbf{C}$ *(the "object of objects"),*
- *an object* X_1 *of* $\mathbf{C}$ *(the "object of arrows"),*
- *domain and codomain operations* $X_1 \xrightarrow{dom,\ cod} X_0$*,*
- *an identity operation* $X_0 \xrightarrow{id} X_1$*,*
- *a composition operation* $X_2 \xrightarrow{c} X_1$*, where* X_2 *is the set of "composable pairs of arrows",*

- *assocativity and unit axioms.*

X *is an internal groupoid if there is additionally an operation* $X_1 \xrightarrow{()^{-1}} X_1$ *which gives the inverse of an arrow.*

Note that this generalizes the "set of arrows" definition of a category, so that

2.40. Example. A category internal to **set** is a (small) category.

2.41. Example. A groupoid internal to **top** is a topological groupoid.

2.42. Example. A groupoid internal to **man** is a Lie groupoid (which have many applications in differential geometry: see, for example, [8]). As an example, any Lie group G acting on a manifold M gives an "action Lie groupoid", with $X_0 = M$ and $X_1 = G \times M \times M$.

2.43. Example. Somewhat surprisingly, an internal category in **group** is the same as an internal group in **cat**; that is, a crossed module (again, see [6]).

2.44. Example. A category internal to $\mathbf{vec}_k$ is known as a "2-vector space": see [1].

2.45. Example. A category internal to **cat** is a (strict) double category: a category with two types of arrows ("horizontal" and "vertical"), and cells between these arrows. These will be discussed later in detail.

Of course, there is naturally a notion of "internal functor" and "internal natural transformation".

2.46. Definition. *Suppose* **X** *and* **Y** *are internal* **C***-categories. An internal* **C***-functor* $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ *consists of*

- *a* **C***-arrow* $X_0 \xrightarrow{F_0} Y_0$ *(taking "objects to objects"),*
- *a* **C***-arrow* $X_1 \xrightarrow{F_1} Y_1$ *(taking "arrows to arrows"),*
- *axioms saying that* F_1 *preserves domains and codomains,*
- *axioms saying that composition and identities are preserved.*

2.47. Example. If G and H are groups internal to the category of groups (that is, an abelian group), then an internal **group**-functor between them is a group homomorphism.

2.48. Definition. *Suppose we have internal* **C***-categories and internal* **C***-functors* $\mathbf{X} \xrightarrow{F,G} \mathbf{Y}$. *An internal* **C***-natural transformation* $F \xrightarrow{\alpha} G$ *consists of*

- *a* **C***-arrow* $X_0 \xrightarrow{\alpha} Y_1$ *(think of this as the map that sends an object* X *to the arrow* α_X*),*
- *axioms saying that "the domain of* α_X *is* FX*, and the codomain* GX*",*
- *an axiom saying that* α *is "natural".*

Next time, we'll discuss how both enriched categories and internal categories each organize into a "2-category".

3. 2-categories

We have seen in the previous two sections that both enriched and internal categories relate to a number of different concepts in mathematics. But, in what sense are they "categories"? To what extent can we do category theory with them? In this section, we will show that both enriched categories and internal categories naturally organize themselves into 2-categories. In a general 2-category, one can do many of the interesting things one can do what categories: for example, one can talk about adjunctions, monads, Kan extensions, fibrations. By applying this general theory to enriched categories or internal categories, we get notions of enriched or internal adjunctions, enriched or internal monads, etc.

The key element that allows one to "do category theory in a 2-category" is a notion of "arrow between arrow". For ordinary categories, the arrows are functors, and the "arrows-between-arrows" are natural transformations. For enriched or internal categories, these are supplied by the enriched and internal natural transformations. In general, a 2-category is given by:

3.1. Definition. *A 2-category $\mathcal{C}$ consists of:*

- *a class of objects $X, Y \ldots$*
- *between any two objects X, Y a class of arrows F, G, etc.,*
- *between any two arrows F, G, a class of "2-cells" α, beta...*
- *such that the objects and arrows form a category,*
- *there is an identity 2-cell, and one can compose 2-cells "horizontally" and "vertically",*
- *axioms for cell composition.*

(Note: a 2-category is the same as a **cat**-enriched category).

3.2. Example. Categories, functors, and natural transformations form a 2-category.

3.3. Example. Sets, functions, and 2-cells being "equality" (that is, there is a 2-cell $f \xrightarrow{\alpha} g$ if and only $f = g$) form a 2-category. More generally, any category $\mathbf{C}$ can be made into a "discrete" 2-category where the 2-cells are equalities between arrows.

3.4. Example. There can be two different 2-categories on the same objects and arrows. For example, as above, ordered sets, order-preserving functions, and equalities form a 2-category. However, we can replace the 2-cells with inequalities: there exists $f \xrightarrow{\alpha} g$ if $f(x) \leq g(x)$ for all x, and this also forms a 2-category.

3.5. Example. As another example of the above, categories, functors, and natural isomorphisms forms a 2-category.

3.6. EXAMPLE. For any monoidal category $(\mathbf{V}, \otimes, I)$, $\mathbf{V}$-categories, $\mathbf{V}$-functors, and $\mathbf{V}$-natural transformations form a 2-category (note that the ordered set example is one of these, with $\mathbf{V} = 0 \leq 1$).

3.7. EXAMPLE. For any category $\mathbf{C}$ with pullbacks, internal $\mathbf{C}$-categories, internal $\mathbf{C}$-functors, and internal $\mathbf{C}$-natural transformations form a 2-category.

3.8. EXAMPLE. Monoidal categories, monoidal functors (with comparisons $FX \otimes FY \longrightarrow F(X \otimes Y)$ and $I \longrightarrow F(I)$), and monoidal natural transformations (which cohere with the comparisons) form a 2-category.

3.9. EXAMPLE. Categories fibred over a base $\mathbf{E}$, cartesian functors, and cartesian natural transformations form a 2-category.

We will give one example of "doing category theory in a 2-category": adjunctions. Recall that an adjunction $\mathbf{X} \xrightarrow{G} \mathbf{Y}$, $\mathbf{Y} \xrightarrow{F} \mathbf{X}$ is generally defined as natural isomorphisms

$$\mathbf{X}(Fy, x) \cong \mathbf{Y}(y, Gx)$$

However, to define adjunctions in an arbitrary 2-category, we need to use the formulation in which an adjunction consists of natural transformations

$$1_Y \xrightarrow{\eta} GF \text{ and } FG \xrightarrow{\epsilon} 1_X$$

with the "triangle identities"

$$(\epsilon F)(F\eta) = 1_F \text{ and } (G\epsilon)(\eta G) = 1_G$$

We simply replace functor by arrow, natural transformation by 2-cell, and we have the definition of adjunction in a 2-category.

3.10. DEFINITION. *An adjunction in a 2-category consists of arrows* $\mathbf{X} \xrightarrow{G} \mathbf{Y}$, $\mathbf{Y} \xrightarrow{F} \mathbf{X}$ *and 2-cells*

$$1_Y \xrightarrow{\eta} GF \textit{ and } FG \xrightarrow{\epsilon} 1_X$$

such that

$$(\epsilon F)(F\eta) = 1_F \textit{ and } (G\epsilon)(\eta G) = 1_G$$

3.11. EXAMPLE. Of course, an adjuction in the 2-category of categories, functors, and natural transformations is a usual adjunction.

3.12. EXAMPLE. By comparison, an adjunction in the 2-category of categories, functors, and natural isomorphisms is an (adjoint) equivalence of categories. This is the more general notion of when two categories are "the same": instead of asking that we have functors F, G for which $GF(y) = y$ and $FG(x) = x$, we instead ask that they be naturally isomorphic.

3.13. EXAMPLE. An adjunction in sets, functions, and equalities is just an isomorphism.

3.14. Example. An adjunction between ordered sets is a Galois connection.

3.15. Example. An adjunction between $[0, \infty]$-categories (that is, metric spaces) is essentially an isometric isomorphism. Instead of asking that the contractions be inverses of one another, an adjuction asks that, for example, $\mathbf{Y}(y, GFy) = 0$. Thus, if the metric spaces have "identity of indiscernables" (that is, $\mathbf{Y}(y_1, y_2) = 0$ implies $y_1 = y_2$), then an adjoint pair would be an isometric isomorphism.

3.16. Example. An adjunction in the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations is a "monoidal adjunction". Many of the standard adjunctions between monoidal categories are in fact monoidal adjunctions. For example, the free/forgetful adjunction between $(\mathbf{ab}, \otimes, \mathcal{Z})$ and $(\mathbf{set}, \times, 1)$ is actually a monoidal adjunction.

Of course, applying this notion to any 2-category of enriched categories or internal categories gives a notion of enriched adjunction or internal adjunction particular to that setting; the examples above are simply ones for which there is already a name in the literature.

As mentioned above, one can extend other categorical defintions, such as monads, Kan extensions, and fibrations, to an arbitrary 2-category. Applying this to particular 2-categories, such as enriched or internal categories, is an ongoing area of research.

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