NOVEMBERFEST 2009 TALK: COMBINATORIAL GAME CATEGORIES

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1. Introduction

This is joint work with Robin Cockett.

In this talk, I would like to describe how some constructions from combinatorial game theory generalize to category theory, and how category theory may be useful to help analyze combinatorial games. I will begin by briefly describing combinatorial games, then discuss the notion of a "combinatorial game category", as well as give some interesting examples, and some ideas of how they may be useful.

Informally, a combinatorial game has the following properties:

- it is played between two players (usually described as left (L) and right (R)) who alternate taking turns,
- both players have complete information; thus, there is neither random elements nor hidden information,
- each player has only a finite number of moves available to them, and the game ends after a finite number of turns,
- the last player to move wins.

The condition that games may not go on forever is sometimes dropped; we will discuss the resulting "loopy" games later. Also, the winning condition (last player to move wins) is sometimes changed to the last player to move loses; we will also discuss this idea later. For now, however, we will follow the four items above. There are a number of examples of such games, such as:

1.1. Example. A classic example is the game of Nim. In Nim, there are a number of heaps of tokens available. On their turn, a player may take any number of tokens from a single heap. The last player to move wins.

1.2. Example. The game of Domineering is played on an $m \times n$ board. L places 2×1 dominoes on the board, while R places 1×2 dominoes. The dominoes must be placed without overlapping any previous dominoes. As always, the last player with a legal move wins.

Many more examples can be found in [2].

To formulate a mathematical theory of such games, Conway made the following definition:

1.3. Definition. *A* game *is a pair of finite sets of games* $\{(g_i)_I|(h_j)_J\}$

Here, a "game" is really a position; one thinks of the first set $(g_i)_I$ as the positions which L can move to, and the set $(h_j)_J$ as the positions right can move to.

Note that the definition is recursive. All games are generated by building the initial game $0 := \{\emptyset|\emptyset\}$, and then inductively building further games whose options are games already created. So, for example, after 0, we get the games

$$* := \{0|0\}, 1 := \{0|\emptyset\}, -1 := \{\emptyset|0\}$$

and then games whose options are from the set $\{0, *, 1, -1\}$, and so on.

There are two other useful ways to create new games from old ones. The first is by "adding" two games together. To add two games, one creates a copy of each game, and allows players to play in one or the other game for each move. Formally, this is given by the following definition.

1.4. Definition. *Given games* $G = \{(g_i)_I|(h_j)_J\}$ *and* $H = \{(g'_k)_K|(h'_l)_L\}$, *the game* $G+H$ *is given by*

$$G + H := \{(g_i + H)_I, (g'_k + H)_K|(G + h_j)_J, (G + h_l)_L\}$$

Given a game, we can also interchange the roles of left and right.

1.5. Definition. *Given a game* $G = \{(g_i)_I|(h_j)_J\}$, *the game* $-G$ *is given by*

$$-G := \{(-h_j)_J|(-g_i)_I\}$$

Note that both of these definitions are also recursive.

There is also a partial ordering on the set of games:

1.6. Definition. *Say that* $G \leq H$ *if* L *can win the game* $H - G$, *playing second.*

Taking note of this, Joyal [4] discovered that one can extend this idea to give a category whose objects are games. To win a game $H - G$ as L, playing second, one must give a "strategy": a series of responses to each move R can make, until R has no more moves. Joyal takes these as the arrows of the category.

1.7. Definition. *Define a category* **games**, *where:*

- *the objects are games* $\{(g_i)_I | (h_j)_J\}$,
- *an arrow* $G \longrightarrow H$ *is a winning strategy for L playing second in the game* $H - G$.

The identity morphism is the "copycat" strategy: for each move that R makes in $G - G$, L copies the opposite move in the other component.

The composition of $G \xrightarrow{f_1} H$ and $H \xrightarrow{f_2} K$ is slightly more complex. Suppose R makes a move in K. The strategy f_2 then dictates a move in either K or $-H$. If the move is in K, we use that move in $K - G$. If f_2 dictates a move in $-H$, we then copy that move in H, and pretend that R made that move in $H - G$. The strategy f_1 then dictates a move in either H or $-G$. If it is in $-G$, then we take that as our move in $K - G$. If it is in H, then we copy that move over to $-H$, taking that as a R move in $K - H$. This process must terminate, as the game H has only a finite number of moves.

The identity and composition arrows are interesting for a number of reasons. First of all, they describe processes which are used in the practice of combinatorial game theory: the copycat strategy is often described as the "tweedledum-twidledee" strategy and composition the "swivel chair" strategy. In addition, knowing strategies, instead of merely knowing the ordering $\leq$ is of practical importance. After all, when playing a game, it is not enough to know *whether* you win a given game; you need to know *how* to win the game. Thus, including strategies as part of the structure is important both theoretically and practically.

Of course, all this leads one to ask the question: what is a strategy? It is useful to know that combinatorial games form a category; but even more useful would be to understand the nature of the arrows in this particular category, and to see what other categories have arrows that look like strategies.

2. Combinatorial Game Categories

The first thing we need to understand is the role of first-player strategies for L. Again, these will be important both theoretically and practically: we need to know how to win not only when playing second, but also when playing first.

So, the question becomes: what role do first-player strategies form in the category of games? It is easy to see that they do not compose with one another: having a first-player strategy on $-G + H$ and $-H + K$ will *not* give you one on $-G + K$. However, they will compose with second-player strategies. That is, if we have a second-player strategy on $-G + H$ and a first-player strategy on $-H + K$, we can use the same sort of composition as above to get a first-player strategy on $-H + K$. In categorical terms, the fact that we can compose first-player strategies with second-player strategies says nothing more than that the first-player strategies form an endo-module over the category of games.

2.1. Proposition. *On the category* **games***, there is an endomodule* $\mathbf{games} \overset{M}{\twoheadrightarrow} \mathbf{games}$*, where the module arrows* $G \overset{m}{\twoheadrightarrow} H$ *are strategies for* L*, playing first in the game* $-G+H$*.*

Proof. The compositions follow similarly to the compositions in **games**, and the associativity and unit axioms follow similarly as well. ∎

We will denote these strategies with a horizontal arrow with a slash through it, so that $G \twoheadrightarrow H$ represents a first-player strategy for L playing first in $-G+H$.

With both first and second-player strategies available, we can now try and understand the nature of these strategies. What is a second-player strategy? If you, as L, are the second player, you have to have a response to every possible move R could make. That is, you must have a first-player strategy in whatever game R chooses to move to. That is, we have an operation which, given a first-player strategy for any move R could make, produces a second-player strategy. Thus, if $G = \{(g_i)_I|(h_j)_J\}$ and $H = \{(g'_k)_K|(h'_l)_L\}$, we have

$$\forall i \in I, g_i \twoheadrightarrow \{(g'_k)_K|(h'_l)_L\}, \forall l \in L, \{(g_i)_I|(h_j)_J\} \twoheadrightarrow h'_l \Rightarrow G \longrightarrow H$$

This gives in symbols exactly what we said above. R could move to some g_i in the game $-G$, or some h'_l in the game H. Thus, if we have first-player strategies (cross-arrows) for each of the resulting games, we will get a second-player strategy $G \longrightarrow H$.

Conversely, to give a first-player strategy, you need to give a move, and a second-player strategy on the resulting game. This gives two more operations:

$$h_i \longrightarrow H \Rightarrow \{(g_i)_I|(h_j)_J\} \twoheadrightarrow H$$

and

$$G \longrightarrow g'_k \Rightarrow G \twoheadrightarrow \{(g'_k)_K|(h'_l)_L\}$$

In the first one, L specifies a move in $-G$ (h_i), and a second-player strategy on the resulting game. In the second, L specifies a move in H (g'_k) and a second-player strategy on the resulting game. These operations were first described in an unpublished manuscript by Kevin Saff, a student of Robin Cockett's.

We thus have all the ingredients for our definition of a combinatorial game category: second-player strategies, first-player strategies, ways to make new games from old ones, and operations which form new strategies from old ones. This gives:

2.2. Definition. *A* combinatorial game category *(or a* cgc*) consists of*

- *a category* $\mathbf{C}$*,*
- *a module* $\mathbf{C} \overset{M}{\twoheadrightarrow} \mathbf{C}$*,*

- *for each finite set I and J, a functor* $\mathbf{C}^I \times \mathbf{C}^J \xrightarrow{\{-_I|-_J\}} \mathbf{C}$ *("diproduct"), with operations*
- $\forall i \in I, g_i \twoheadrightarrow \{(g'_k)_K|(h'_l)_L\}, \forall l \in L, \{(g_i)_I|(h_j)_J\} \twoheadrightarrow h'_l \Rightarrow \{(g_i)_I|(h_j)_J\} \longrightarrow \{(g'_k)_K|(h'_l)_L\}$ *("ditupling"),*
- $h_i \longrightarrow H \Rightarrow \{(g_i)_I|(h_j)_J\} \twoheadrightarrow H$ *("injection"),*
- $G \longrightarrow g'_k \Rightarrow G \twoheadrightarrow \{(g'_k)_K|(h'_l)_L\}$ *("projection").*
- *and coherence equations.*

There is also a natural notion of *combinatorial game functor* which preserves all operations. We thus have a category of combinatorial game categories and combinatorial game functors, denoted **cgc**.

3. Examples

Naturally, the Joyal category of games will be an example of a combinatorial game category. However, it is more than that: it is essentially the initial object in the category of combinatorial game categories. Suppose we started with nothing, and wished to build a combinatorial game category. By the existence of the diproduct operation, we would immediately get an object $0 := \{\emptyset|\emptyset\}$, just as we did with the original definition of games. However, because of the ditupling operation, we also get an arrow $0 \longrightarrow 0$ (since I and L are empty sets).

We then get more: the objects

$$* := \{0|0\}, 1 := \{0|\emptyset\}, -1 := \{\emptyset|0\}$$

from the diproduct, as well as arrows like

$$* \longrightarrow *, 1 \longrightarrow 1, -1 \longrightarrow -1, -1 \longrightarrow 0 \longrightarrow 1, 1 \twoheadrightarrow 0 \twoheadrightarrow 1$$

from the dituping, injection, and projection operations.

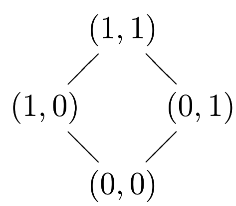
Continuing to build up, we get an enhanced version of Conway's construction. From "nothing", we build all games, but we also build all first and second-player strategies between these games. (Note that there is a slight difference between this initial object and Joyal's category of games: here, the options of a game are a list, whereas in Joyal's category, they were a set; however, this makes no difference when "playing" the game).

The important question then becomes: are there other useful or interesting examples of combinatorial game categories? That is, are there other categories which act like the category of games? Surprisingly, there are are a number of other examples.

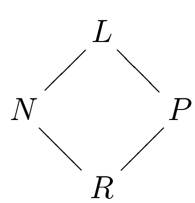
3.1. Example. If $\mathbf{C}$ is a category with products and coproducts, then there is combinatorial game category structure on the product category $\mathbf{C} \times \mathbf{C}$, given by:

- a module arrow $(c, d) \longrightarrow\!\!\!\!\!+ (c', d')$ is an arrow $d \longrightarrow c'$,
- the diproduct is $\{(c_i, d_i)_I | (c'_j, d'_j)_J\} := (\coprod d_i, \prod c'_j)$,
- ditupling is given by tupling and cotuping,
- injection is given by using the injections of $\coprod$,
- projection is givey by using the projections of $\prod$.

This example shows that there are a great number of combinatorial game categories; that is, categories that act like the category of games. Are they useful? It turns out that the simplest non-trivial CGC of this form is very useful for combinatorial game theory. The simplest non-trivial example is when we take $\mathbf{C} = 0 \leq 1$. In this case, $\mathbf{C} \times \mathbf{C}$ is a four-element lattice:



The initiality of the category of games tells us that there is a unique combinatorial game functor from the category of games to this category. Under this functor, each game is assigned one of the above four objects. This assignment gives us nothing more than the "outcome" of a game: the first component tells us who wins if L goes first (1 if L wins, 0 if R wins), and the second component tells us who wins if R goes first (again, 1 if L wins, 0 if R wins). Thus, the lattice above is really the "outcome lattice" from combinatorial game theory:



The fact that the unique functor from games to this lattice is a combinatorial game functor tells us that the operation of assigning an "outcome" to a game preserves diproducts; that is, the outcome of a game is determined by the outcomes of its options. This is also well-known to game theorists, as is the diproduct on this lattice:

$$\{(g_i)_I | (h_j)_J\} = \begin{cases} N & \text{if } \exists i \in I, g_i = L \text{ or } P \text{ and } \exists j \in J, h_j = R \text{ or } P; \\ L & \text{if } \exists i \in I, g_i = L \text{ or } P \text{ and } \forall j \in J, h_j = L \text{ or } N; \\ R & \text{if } \forall i \in I, g_i = R \text{ or } N \text{ and } \exists j \in J, h_j = R \text{ or } P; \\ P & \text{if } \forall i \in I, g_i = R \text{ or } N \text{ and } \forall j \in J, h_j = L \text{ or } N. \end{cases}$$

which tells one how to get the outcome of a game from the outcomes of its options.

We can then think of more general $\mathbf{C}$ as giving more general "outcome game categories". If we take $\mathbf{C} = 0 \leq \frac{1}{2} \leq 1$, we get an outcome category $\mathbf{C} \times \mathbf{C}$ which allows draws. This nine-element lattice is also well known to game theorists (for example, see [5, p. 97]). If we take $\mathbf{C} = [0, 1]$, we get an outcome category which allows "partial wins": outcomes of a game somewhere between 0 and 1. Of course, these are all well-ordered lattices: taking general lattices, or the even more general categories with products and coproducts, gives even more general outcome categories.

In fact, the construction above can be generalized further:

3.2. Example. Any polarized game category $\mathbf{X_o} \overset{M}{\longrightarrow\!\!\!\!\!\!+\;\;} \mathbf{X_p}$ has cgc structure on $\mathbf{X_p} \times \mathbf{X_o}$.

Polarized game categories were described in [3]: they were defined so as to able to abstract the ideas of "polarized game theory" found in various areas of logic. A polarized game is slightly different than a combinatorial game: at each position in the game, only L or R can move from that position; moreoever, if a player moves in a component of a sum, his opponent must respond in the same component. The construction of a cgc from a polarized game category shows how the two notions are related.

4. Loopy Games and Misere Games

(This material was only briefly eluded to at the end of the talk).

As mentioned in the introduction, game theorists often consider more general types of games. In "loopy" games, one is allowed to return to previous board positions, meaning that games could be played forever. In "misere" games, the last player to move *loses* the game. We will briefly consider how to make combinatorial game categories with these more general games.

4.1. Loopy Games. The difficulty with loopy games is with composition. Intuituively, we would like to say that a game is a draw if it goes on infinitely. We would then ask that the arrows of a category of loopy games be strategies for L, playing second, that guarantees either a win or a draw ("survival" strategies). In this way, we guarantee that the copycat strategy is still a strategy. With this formulation, however, composition does not work: when we try to compose a surival strategy on $-G + H$ with one on $-H + K$, we may get infinite moves bouncing back and forth between $-H$ and H, never returning to give a move on $-G$ or K.

There are several ways around this problem; we begin by looking at a way inspired by an idea from proof theory, and then compare it to what is done in combinatorial game theory. From proof theory, the idea is as follows: each loop in a game has a highest point in the game tree to which it returns. For each of these in a game, we assign it either as L or R, and we think of this assignment as part of the data for giving such a game. In a single game, if the play loops infinitely through a L node, then L wins that game; if it loops through an R node, then R wins the game. In a sum $G_1 + \cdots + G_n$, if there are loops in several components, it is a win if all loops go through L or R, and a draw if there is a mixture of L loops and R loops. The negative of a game reverses the assignments of L and R.

With this additional data given for each loopy game, we do get a combinatorial game category of loopy games, where the arrows are survival strategies. That is, the arrows $G \longrightarrow H$ are strategies on $-G+H$ that guarantee either a win or a draw. From the point of view of category theory, these games are interesting: they add initial F-algebras and terminal F-coalgebras to the category of games. For example, consider the loopy game $G = \{G|G\}$, with the loop being a R win. If we take F to be the functor $F(x) = \{x|x\}$, then one can check that G is an initial F-algebra. Conversely, if $G = \{G|G\}$ has its loop being a L win, then G is a terminal F-coalgebra.

From the point of view of combinatorial game theory, these definitions are interesting because they are *different* than how loopy games are normally treated. In combinatorial game theory, one asks that a strategy $G \longrightarrow H$ between loopy games consists of two strategies: a survival strategy on $G^+ \longrightarrow H^+$, and a survival strategy $G^- \longrightarrow H^-$, where $+$ makes all loops L wins, and $-$ makes all loops R wins. Again, this definition of strategy gives a combinatorial game category, but without all initial and terminal F-algebras. It would be interesting to see how the different definitions of "strategy" give rise to different analyses of loopy games.

4.2. Misere Games. In Misere games, one changes the rules by asking that the last player to move *loses*. We cannot use the same notion of arrow as for normal games, as one can show that there is neither a identity $G \longrightarrow G$ nor a composition $G \longrightarrow H \longrightarrow K$ if we take the definition of arrow to be a "misere strategy for L, playing second in $-G+H$" (see Chapter 6 of [1]). However, we can get a more general combinatorial game category which includes normal games, Misere games, and games with other winning conditions.

The idea is to think of a game as a tree, in which the leaves of the tree are either L, R, or have no assignment (as in normal games). If the play of the game ends on an L, L wins, if it ends on an R, R wins. If neither, then the last player to move wins. A misere game is one of these; every final move in the misere game by L is labelled as R, and every final move by R is labelled as L. However, these L/R tree games are more general, and include games where the winner is not determined by who went last, but rather by some other condition (such as the number of points scored).

To define arrows between these games, we follow the example of loopy games. We say that in a sum $G_1 + \cdots + G_n$, the game is a win for L if all G_i's which end in some L or R end in an L node, or R if all end in an R node. The sum is a draw if there is a mixture of L nodes and R nodes. An arrow $G \longrightarrow H$ is then a survival (win or draw) strategy on $-G + H$. This gives another example of a combinatorial game category.

Note, however, that while this category includes all misere games, the arrows between misere games are not winning strategies for misere. Instead, they are strategies which guarantee L a misere win in at least one of the games $-G$ and H. Using these arrows directly, then, may not be useful to analyse misere games.

What may be useful, however, is to consider arrows between a *normal* game and a misere game. Consider an arrow $G \longrightarrow H$, where G is a normal game (no L or R nodes) and H is a misere game (all last moves for L are R nodes, and all last moves for R are L nodes). Such an arrow is a misere strategy on the misere game $-G \oplus H$, where $\oplus$ is a way to sum normal games and misere games: if G is normal and H misere, then $G \oplus H$ is a misere game which is like $G + H$, but if the play ever ends in H, the game is over.

How is this useful? Suppose that we knew that a misere game K was equal to $-G \oplus H$. We could then find a simpler form of $-G$ (say, the canonical form c(G)), and instead analyse the game $K' := -c(G) \oplus H$. Strategies on K correspond to strategies on K'; but because we have simplified G, the game K' is easier to analyse than K. This may give a way to analyse certain misere games, that would be analogous to how one analyses normal games. In a normal game, if a game splits into a sum, then one analyses the game by first simplifying each component of the sum. Here, if we can split a misere game into the $\oplus$ of a normal game and a misere game, one would simplify just the normal game, and analyse the resulting simpler misere game.

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