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# Combinatorial Game Categories

J.R.B. Cockett, G.S.H. Cruttwell and K. Saff †

*Department of Computer Science, University of Calgary, Calgary AB, Canada.*

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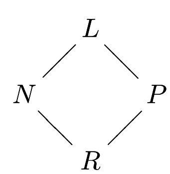
We develop a theory of combinatorial game categories. These generalize Joyal's category of combinatorial games, and include many other examples, such as loopy games, outcome lattices, and polarized game categories.

## 1. Introduction

In 1977, André Joyal observed that combinatorial games organized themselves into a compact closed category (Joyal 1977). This observation was taken up by the logic community, and various models for logic were based around modified versions of the category of combinatorial games. However, what was not apparent was whether other categories acted like Joyal's category of combinatorial games. That is, there was no answer to the question: "when is a category a combinatorial game category"?

The current paper answers that question. Joyal's category is (with a small modification) the initial category of combinatorial games. Significantly, however, there are other examples, many of which already occur in the combinatorial game literature. Examples of this include the "outcome lattice"




the "games born by day n", "consecutive move-ban games" and a variant of "loopy" games. Other examples occur outside of combinatorial game theory, such as the polarized games of (Cockett and Seely 2007). Interestingly, not all of these combinatorial game categories have a natural compact closed structure as the initial one does. Most free combinatorial game categories are not naturally compact (see remarks at the end of Section 4), and examples such as the loopy game category do not even have a natural monoidal structure (see section 7). Thus, the analysis of combinatorial games does not rely on the addition or subtraction of games. The existence of compact monoidal structure for the

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initial category of combinatorial games is merely a happy coincidence.

To develop a theory of combinatorial game categories, we work as in (Cockett and Seely 2007). That is, we begin by developing a proof theory for combinatorial games, then describe the categorical semantics for this proof theory. This approach then brings together three disciplines: combinatorial game theory, proof theory, and category theory. The advantange of this multi-faceted approach is that each subject gives a different perspective. The combinatorial game theory literature helps us understand how to work in detail with combinatorial games. Proof theory helps us understand the tree-like interplay between the two players of a combinatorial game. The category theory gives us alternate models of combinatorial game theory, as well as allows us to describe universal constructions.

The first three sections of the paper describe these three different, but related, approaches, to combinatorial game theory. In the first section, we give a brief overview of combinatorial games. In the next section, we describe the syntax for combinatorial games. In the third section, we give a categorical semantics for the proof theory syntax, and show that as mentioned above, there are a number of interesting examples.

Following this, we relate the polarized game categories of Cockett and Seely (Cockett and Seely 2007) and combinatorial game categories by showing that each polarized game category gives rise to a combinatorial game category. (Unfortunately, this construction is not universal.) In particular, the polarized game category of finite Abramsky-Jagadeesan games (Abramsky 1997) gets sent to the combinatorial game category of "consecutive move-ban games", which have found applications in misère game theory (Ottaway 2009). This provides a potential source of further research, as misère game theory, in which the last player to move loses, is generally considered much more difficult than "normal play" game theory, in which the last player to move wins (Plambeck and Siegel 2008).

Next, we provide an application of the theory, by describing idempotents and splittings in categories of combinatorial games. This relates to the notions of "dominated" and "reversible" options in combinatorial game theory, and provides an alternative approach to the "canonical" form of a game.

Finally, we investigate the idea of "loopy" games. In a loopy game, one is allowed to return to a previous board position. Thus, there is a potential for infinite play in a loopy game. Naturally, this can lead to a number of problems. In particular, the question of who wins such a game and how one can compose strategies between such games. We investigate three different approaches to this problem, showing that one solution in particular has well-behaved categorical properties. In essence, adding loopy games to regular games is the same as adding initial and terminal algebras for various functors.

We hope that this paper will be the starting point for further interaction between combinatorial game theory and other areas of mathematics.

## 2. Brief Game Theory Overview

Informally, a combinatorial game has the following properties: (Albert *et al.* 2007, p. xi)

- it is played between two players (usually described as Left and Right) who alternate taking turns,
- it has a clearly defined ruleset, stating what moves players can make,
- both players have complete information, and there are no are sources of randomness,
- from each position, only a finite number of moves is available for each player, and the game ends after a finite number of moves.

For determining who wins or loses the game, one of two criteria is generally used: either the last player to move wins ("normal" play) or the last player to move loses ("misère" play). Misère games are generally much harder to analyze than normal-play games (Plambeck and Siegel 2008). One reason for this is that equivalence between games (defined below) greatly reduces the number of games one has to consider in normal play. However, in misère play, the equivalence classes are often very small, making the analysis more difficult.

In this paper, we will restrict our attention to the normal play convention. There have been recent advances in misère play; notably the "indistinguishability quotient" construction of (Plambeck and Siegel 2008), and it would be interesting to try and understand their construction as it relates to the ideas in this paper, but we leave this for future work.

**Example 2.1.** A classic example is the game of Nim. In Nim, there are a number tokens, arranged in heaps. On each turn, a player may take any number of tokens from a single heap. The last player to move wins.

**Example 2.2.** The game of Domineering is played on an $m \times n$ board. Left places $2 \times 1$ dominoes on the board, while Right places $1 \times 2$ dominoes. The dominoes must be placed without overlapping any previous dominoes. Again, the last player with a legal move wins.

Many more examples can be found in (Berlekamp *et al.* 2001).

To formulate a mathematical theory of such games, Conway made the following definition:

**Definition 2.3.** A *game* is a pair of finite sets of games $\{(g_i)_I | (h_j)_J\}$

One thinks of the first set $(g_i)_I$ as the games which Left can move to, and the set $(h_j)_J$ as the games Right can move to.

Note that the definition is recursive. All games are generated by building the initial game $0 := \{\emptyset | \emptyset\}$ (in which neither player has a move available) and then inductively building further games whose options are games already created. So, for example, after

0, we get the games

$$* := \{0|0\}, 1 := \{0|\emptyset\}, -1 := \{\emptyset|0\}$$

and then games whose options are from the set $\{0, *, 1, -1\}$, and so on.

For example, the position in Nim with no tokens would be represented by 0. The position with a single token available would be represented by $* = \{0|0\}$. An empty 2x2 board in Domineering is represented by the game $\{1| -1\}$, as Left can move to a game with another move available for her, and Right similarly.

There are two other useful ways to create new games from old ones. The first is by "adding" two games together. To add two games, one essentially creates a copy of each game, and allows players to play in one or the other game for each move. Formally, this is given by the following definition.

**Definition 2.4.** Given games $G = \{(g_i)_I|(h_j)_J\}$ and $H = \{(g'_k)_K|(h'_l)_L\}$, the disjunctive sum $G + H$ is a game given by

$$G + H := \{(g_i + H)_I, (g'_k + H)_K|(G + h_j)_J, (G + h_l)_L\}$$

For example, if $G$ is the Nim game with a single heap of 5 tokens, and $H$ the Nim game with a single heap of 7 tokens, then a Nim game with one heap of 5 tokens and one of 7 is the game $G + H$. Similarly, if $G$ is a 2x6 board of Domineering with a single play by Left of a 2x1 domino in the 3rd column, then $G = H + K$, where $H$ is the game of Domineering with an empty 2x2 board, and $K$ the game of Domineering with an empty 2x3 board.

Given a game, we can also interchange the roles of Left and Right:

**Definition 2.5.** Given a game $G = \{(g_i)_I|(h_j)_J\}$, the game $-G$ is given by

$$-G := \{(-h_j)_J|(-g_i)_I\}$$

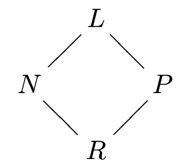
Note that the definitions of sum and negative are recursive.

Assuming perfect play on the part of each player, we assign each game $G$ an outcome $o(G)$:

— $o(G) = L$ if Left wins going first or second,
— $o(G) = R$ if Right wins going first or second,
— $o(G) = N$ if the first (or "next") player wins
— $o(G) = P$ if the second (or "previous") player wins.

For example, if $G$ is a game of Domineering on an empty 2x2 board, then $o(G) = N$, since after the first player makes their move, the second player has no moves left. If $G$ is a game of Nim with two separate heaps of one token each, then $o(G) = P$, since no matter which token the first player takes, the second player can take the other one, winning the game. The *fundamental theorem of game theory* (Albert *et al.* 2007, p. 35) states that each game has exactly one outcome.

If we prefer the Left player to the Right player[†], then the outcomes are ordered as follows:




Using the notion of outcome, and the ordering given above, one can define a partial order and an equivalence relation on the set of all games.

**Definition 2.6.** For games $G$ and $H$, write $G \leq H$ if $o(G+X) \leq o(H+X)$ for all games $X$. Similarly, say that $G$ is equivalent to $H$, and write $G \cong H$, if $o(G+X) = o(H+X)$ for all games $X$.

Notice that the partial ordering and equivalence relation are with respect to addition by any other game $X$. The reason for this is that many games $G$ (such as a position of Domineering) naturally break down into a number of disjoint board positions; that is, $G = \sum_i H_i$. To analyze $G$, we would like to be able to replace each $H_i$ with some simpler element of its equivalence class. With the above definition of equivalence, we know that replacing each element in this way will not affect the outcome of $G$; if we merely asked that $G$ and $H$ are equivalent if $o(G) = o(H)$, we would not have this.

Of course, to carry out this type of analysis, it would be helpful if each equivalence class had a particularly simple representative. Then, to analyze a sum $G + H$, we would first reduce both $G$ and $H$ to their simpler forms, then analyze the game. Fortunately, such a simpler form exists.

Suppose we are given a game and, as Left, we have two options, $g_i$ and $g_j$. Suppose we also know that $g_i \geq g_j$. If this case, playing perfectly, we would never choose $g_j$. We say that $g_i$ *dominates* $g_j$, and we may as well remove that option from the game. One can then check that the game with $g_j$ removed is equivalent (in the above sense) to the original game.

There is another way to simplify a game $G$. Suppose Right moves to some game $h_i$, and Left has some particular option of $h_i$, $h_i^L$ that is so good that she always moves to it, regardless of what else is going on in the game. In other words, $h_i^L \geq G$. In this case, we say $h_i$ is *reversible*, and we may as well replace $h_i$ with the Right options of $h_i^L$, since Left will automatically move to $h_i^L$ if Right moves to $h_i$.

The formal definitions of dominated and reversible options are as follows.

**Definition 2.7.** For a game $G = \{(g_i)_I|(h_j)_J\}$, say that $g_i$ is a *dominated Left option* if there exists some $g_{i'}$ such that $g_{i'} \geq g_i$. Say that $h_j$ is a *dominated Right option* if there

[†] This is the standard convention in game theory.

exists some $h_{j'}$ such that $h_{j'} \leq h_j$.

Say that $h_j$ is a *reversible Right option* if it has some Left option $h_j^L$ such that $h_j^L \geq G$. Say that $g_i$ is a *reversible Left option* if it has some Right option $g_i^R$ such that $g_i^R \leq G$.

By removing dominated options, and "reversing" reversible options, we arrive at the simplest, "canonical form" of a game.

**Definition 2.8.** Say that $G$ is *in canonical form* if it has no dominated or reversible options, and each of its options are in canonical form.

Each game has a unique canonical form to which it is equivalent (Albert *et al.* 2007, p. 81-82).

Before discussing our approach to combinatorial games, we need to discuss one further bit of structure, discovered by André Joyal (Joyal 1977). Joyal found that the poset of games actually extends to a full category of games. An alternate form of the $\leq$ relation is the following: $G \leq H$ if and only if Left can win the game $H - G$, playing second (Albert *et al.* 2007, p. 74). Taking note of this, Joyal discovered that one can extend this idea to give a category whose objects are games. To win a game $H - G$ as Left, playing second, one must give a "strategy": a series of responses to each move Right can make, until Right has no more moves.

As an example, imagine a game of Nim with two heaps of two. If Right takes an entire heap, Left takes the other, winning the game. If Right takes a single token from a heap, Left takes a single token from the other. Right must then finish a heap, and Left finishes the other heap, winning the game. Thus, Left has a strategy on this game, playing second.

Joyal takes these "strategies" as the arrows of the category.

**Definition 2.9.** Define a category **games**, where:
— the objects are games $\{(g_i)_I | (h_j)_J\}$,
— an arrow $G \longrightarrow H$ is a winning strategy for Left playing second in the game $H - G$.

The identity morphism is the "copycat" or "Tweedledum and Tweedledee" strategy: for each move that Right makes in $G - G$, Left copies the opposite move in the other component (this is the strategy Left follows in the above example).

The composite of $G \xrightarrow{f_1} H$ and $H \xrightarrow{f_2} K$ is slightly more complex, it is sometimes referred to as the "swivel chair" strategy in game theory. Suppose Right makes a move in $K$. The strategy $f_2$ then dictates a move in either $K$ or $-H$. If the move is in $K$, we use that move in $K - G$. If $f_2$ dictates a move in $-H$, we then copy that move in $H$, and pretend that Right made that move in $H - G$. The strategy $f_1$ then dictates a move in either $H$ or $-G$. If it is in $-G$, then we take that as our move in $K - G$. If it is in $H$, then we copy that move over to $-H$, taking that as a Right move in $K - H$. This process must terminate, as the game $H$ has only a finite number of moves.

The identity and composition arrows are interesting in that not only do they give a category, but also because they describe processes which game theorists use themselves. In addition, knowing about strategies instead of just the ordering $\leq$ is useful for actual playing these games. After all, when playing a game, it is not enough to know whether you win a given game; you need to know how to win the game, and this is given by having a winning strategy.

Of course, all this leads one to ask the question: what is a strategy? It is useful to know that combinatorial games form a category. However, even more useful would be to understand the nature of the arrows in this particular category, and to see what other categories have arrows that look like strategies. In the next section, we begin to formalize these ideas, by describing a proof theory that describes the interaction between second-player strategies and first-player strategies. This then leads us to a notion of "combinatorial game category", of which the category **games** is one model, but of which, surprisingly, there are also many other models.

## 3. Proof Theory for Combinatorial Games

We have found that one can describe a category of games whose arrows are winning strategies (from now on, "strategy" will always mean "winning strategy"). However, we would like an explicit description of these strategies. With such a description in hand, we can then see if there are other settings which act like combinatorial games. To do this, we will begin by developing a proof theory for combinatorial games which abstractly describes these strategies. That is, in this section, we think of a game $G$ as a formula, and an arrow $G \longrightarrow H$, (ie., a strategy on the game $-G+H$) as a proof of the assertion "$G$ implies $H$".

The formulae of our theory will either be atoms (basic "games" for which we know no further structure) or compound formulae of the form $\{(g_i)_I|(h_j)_J\}$. We would like terms in our theory that will allow us to give proofs of the proposition $G \longrightarrow H$. To do this, we must answer the question: what is a second-player strategy for Left in the game $-G+H$? For each move that Right could make (either in $-G$ or $H$), it consists of a good response. What does a "response" consist of? A response is something different: a response is given by a first-player strategy for Left, as after Right makes their move, you, as Left, are the first player. Thus, for any move Right could make, you need a first-player strategy on the resulting game.

Thus, to construct a proof theory for strategies, we will need another type of proof, which corresponds to first-player strategies for Left. As we have seen above, each second-player strategy corresponds to a number of first-player strategies, one for each possible move for Right. What does a first-player strategy consist of? As the first player, you need to choose a move from which you can win. What does this mean? After your move, you

$$\frac{}{a \xrightarrow{1_a} a}\ \text{atomic identity}$$

$$\frac{G \xrightarrow{f} g_i}{G \overset{\sigma_i \cdot f}{\nrightarrow} \{(g_i)_{i\in I}|(h_j)_{j\in J}\}}\ \text{inject} \qquad \frac{h_j \xrightarrow{f} H}{\{(g_i)_{i\in I}|(h_j)_{j\in J}\} \overset{\pi_j \cdot f}{\nrightarrow} H}\ \text{project}$$

$$\frac{(g_i \overset{s_i}{\nrightarrow} \{(g'_k)_{k\in K}|(h'_l)_{l\in L}\})_{i\in I}, (\{(g_i)_{i\in I}|(h_j)_{j\in J}\} \overset{r_l}{\nrightarrow} h'_l)_{l\in L}}{\{(g_i)_{i\in I}|(h_j)_{j\in J}\} \xrightarrow{[(s_i)_{i\in I},(r_l)_{l\in L}]} \{(g'_k)_{k\in K}|(h'_l)_{l\in L}\}}\ \text{dituple}$$

Table 1. *Terms for the combinatorial game logic.*

are the second player, so a first-player strategy must consist of a move, together with a second-player strategy on the resulting game.

Thus, our proof theory will have two types of proof (corresponding to first and second player strategies), as well as several terms. One of these terms will, from a number of first-player strategies ("good responses") produce a second-player strategy. We call this "ditupling" by analogy with the categorical operations of tupling and cotupling. Suppose we want to get a second-player strategy

$$\{(g_i)_I|(h_j)_J\} \longrightarrow \{(g'_k)_K|(h'_l)_L\}.$$

The possible moves that Right could make are the $g_i$ or the $h'_l$, as we are dealing with the negative of the first game. Thus, if we have a first-player strategy on each of the games that results after those moves, we will have a second-player strategy. This is what the ditupling term produces.

Two other terms will, given a second-player strategy, produce a first-player strategy ("move to the game with that second-player strategy"), and these we call injection and projection. If we want a first-player strategy

$$\{(g_i)_I|(h_j)_J\} \nrightarrow \{(g'_k)_K|(h'_l)_L\},$$

we can do one of two things. We can either make some move $h_j$, and have a second-player strategy on the resulting game (this is projection), or we could make a move $g'_k$, and have a second-player strategy on that game (this is injection).

Finally, we will need a term which produces the copycat strategy - that is, for any $a$, we need a second-player strategy $a \longrightarrow a$; these are simply identities. Together, our terms are given in Table 1 (note that the sets $I$ and $J$ are all finite sets).

Of course, the logic should have cuts - given a proof $A \longrightarrow B$ and a proof $B \longrightarrow C$,

$$\frac{a \xrightarrow{f} b,\; b \overset{p}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} c}{a \overset{f;p}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} c}\ \text{m-cut} \qquad \frac{a \overset{p}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} b,\; b \xrightarrow{g} c}{a \overset{p;g}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} c}\ \text{m-cut}$$

$$\frac{a \xrightarrow{g} b,\; b \xrightarrow{g} c}{a \xrightarrow{f;g} c}\ \text{cut}$$

Table 2. *Basic cuts for the combinatorial game logic.*

$f; 1 \Rightarrow f$
$1; f \Rightarrow f$
$f'; (\sigma_i \cdot f) \Rightarrow \sigma_i \cdot (f'; f)$
$(\pi_j \cdot f); f' \Rightarrow \pi_j \cdot (f; f')$
$(\sigma_i \cdot f); [(s_i)_{i\in I}, (r_l)_{l\in L}] \Rightarrow f; s_i$
$[(s_i)_{i\in I}, (r_l)_{l\in L}]; (\pi_l \cdot f) \Rightarrow r_l; f$
$[(s_i)_{i\in I}, (r_l)_{l\in L}]; [(s'_k)_{k\in K}, (r'_n)_{n\in N}] \Rightarrow [(s_i; [(s'_k)_{k\in K}, (r'_n)_{n\in N}])_{i\in I}, ([(s_i)_{i\in I}, r_l)_{l\in L}]; r'_n)_{n\in N}]$

Table 3. *Rewrite rules for the combinatorial game logic.*

we must have a proof $A \longrightarrow C$. But this only deals with the strategies where Left is the second player - do we have any cuts where Left is the first player? It is easy to see that if we have first-player strategies $A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ and $B \longrightarrow\!\!\!\!\!\!\!+\;\; C$, then, in general, we will not get a first-player strategy $A \longrightarrow\!\!\!\!\!\!\!+\;\; C$. As an example, take $A$ and $C$ to be the empty game, and $C$ to be the game of Nim with exactly one token. There are obviously first-player strategies on $-A + B$ and $-B + C$ (simply take the single token in the game $B$). But there is no first-player strategy on $-A + C$, as it is the empty game. Thus, we cannot ask for cuts of the form $A \longrightarrow\!\!\!\!\!\!\!+\;\; B$, $B \longrightarrow\!\!\!\!\!\!\!+\;\; C \Rightarrow A \longrightarrow\!\!\!\!\!\!\!+\;\; C$.

What we do have, however, is mixed cuts. If we have a first-player strategy $A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ and a second-player strategy $B \longrightarrow C$, we can form a first-player strategy $A \longrightarrow\!\!\!\!\!\!\!+\;\; C$: simply play the first move one is given in $A \longrightarrow\!\!\!\!\!\!\!+\;\; B$, then follow the usual composite of second-player strategies. Similarly, $A \longrightarrow B$ and $B \longrightarrow\!\!\!\!\!\!\!+\;\; C$ will give $A \longrightarrow\!\!\!\!\!\!\!+\;\; C$. Thus, we have three types of cuts for our combinatorial game logic: see Table 2.

Of course, we will also need rewrite rules which describe the interaction of the cuts with the terms of the logic: see Table 3. With the rewrites, it is easy to see that the logic satisfies cut-elimination: every proof that uses a cut can be rewritten to a proof that does not use cuts. Indeed, if we define the "height" of a cut to be the number of occurrences of injection, projection, or ditupling in that cut, then any rewrite involving cuts gives cuts with height strictly less. Thus, continually applying the rewriting rules to the terms in a proof will always reduce to a proof without cuts. Moreover, it is also easy to see that the logic satisfies the Church-Rosser property: if a proof has two possible rewrites, then those themselves can be rewritten to a common proof. That is, we have:

**Proposition 3.1.** The combinatorial game logic satisfies cut-elimination and the Church-Rosser property.

Thus, the logic for combinatorial games is well-behaved.

## 4. Combinatorial Game Categories

With the logic of combinatorial games in hand, we can now give axioms for a combinatorial game category. The first issue that we must deal with is the nature of the second type of inference in the logic we described above. That is, we must have a categorical interpretation of the first-player strategies for Left (the second-player strategies will be the arrows in the category). The cuts we described above show that the first-player strategies for Left give a module (otherwise known as a profunctor, or distributor) from the category of games to itself.

**Proposition 4.1.** On the category **games**, there is a module $\mathbf{games} \overset{M}{-\!\!\!+\!\!\!\rightarrow} \mathbf{games}$, where the module arrows $G \overset{m}{-\!\!\!+\!\!\!\rightarrow} H$ are strategies for Left, playing first.

Before we define a combinatorial game category, we describe a 2-category of "module categories", where a module category consists of a category equipped with an endo-module.

**Definition 4.2.** Define a 2-category **modcat**, where

— an object is a category $\mathbf{C}$ equipped with an endo-module $\mathbf{C} \overset{M}{-\!\!\!+\!\!\!\rightarrow} \mathbf{C}$,

— an arrow between $(\mathbf{C}, M)$ and $(\mathbf{D}, N)$ consists of a functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$ as well as a module morphism

$$\begin{array}{ccc} \mathbf{C} & \overset{M}{-\!\!\!+\!\!\!\longrightarrow} & \mathbf{C} \\ {\scriptstyle F}\downarrow & \widetilde{F} \Downarrow & \downarrow{\scriptstyle F} \\ \mathbf{D} & \underset{N}{-\!\!\!+\!\!\!\longrightarrow} & \mathbf{D} \end{array}$$

— a 2-cell between $(F, \widetilde{F})$ and $(G, \widetilde{G})$ consists of a natural transformation $F \xrightarrow{\alpha} G$ such that for any $M_X$-arrow $c \overset{m}{-\!\!\!+\!\!\!\rightarrow} c'$, we have that

$$\begin{array}{ccc} Fc & \overset{\widetilde{F}m}{-\!\!\!+\!\!\!\longrightarrow} & Fc' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Gx & \underset{\widetilde{G}m}{-\!\!\!+\!\!\!\longrightarrow} & Gx' \end{array}$$

commutes. The composition and identities are the natural ones.

We now give the definition of a combinatorial game category. It is precisely the proof theory for combinatorial games, translated into categorical terms.

**Definition 4.3.** A *combinatorial game category* (or a *cgc*) consists of

— a module category $(\mathbf{C}, M)$,
— for each finite set $I$ and $J$, a functor $\mathbf{C}^I \times \mathbf{C}^J \xrightarrow{\{-_I|-_J\}} \mathbf{C}$ ("diproduct"), with operations
— $\forall i \in I, g_i \not\longrightarrow \{(g'_k)_K|(h'_l)_L\}, \forall l \in L, \{(g_i)_I|(h_j)_J\} \not\longrightarrow h'_l \Rightarrow \{(g_i)_I|(h_j)_J\} \longrightarrow \{(g'_k)_K|(h'_l)_L\}$ ("ditupling"),
— $h_i \longrightarrow H \Rightarrow \{(g_i)_I|(h_j)_J\} \not\longrightarrow H$ ("injection"),
— $G \longrightarrow g'_k \Rightarrow G \not\longrightarrow \{(g'_k)_K|(h'_l)_L\}$ ("projection").
— and coherence equations, which are the last five rewrite rules for the combinatorial game logic.

Say that a combinatorial game category is a *combinatorial game lattice* if, between any two objects, there is at most one $\mathbf{C}$-arrow, and at most one $M$-arrow.

An alternative form of the coherence equations is given by asking that the ditupling be a natural equivalence

$$\frac{(g_i \xrightarrow{s_i} \{(g'_k)_{k\in K}|(h'_l)_{l\in L}\})_{i\in I}, \{(g_i)_{i\in I}|(h_j)_{j\in J}\} \xrightarrow{r_l} h'_l)_{l\in L}}{\{(g_i)_{i\in I}|(h_j)_{j\in J}\} \xrightarrow{[(s_i)_{i\in I},(r_l)_{l\in L}]} \{(g'_k)_{k\in K}|(h'_l)_{l\in L}\}} \text{ dituple}$$

and that the final rewrite rule is satisfied.

**Definition 4.4.** Define a 2-category **cgc**, where

— an object is a combinatorial game category,
— an arrow (combinatorial game functor) is a module morphism which preserves diproducts up to isomorphism, and preserves projections, injections, and ditupling exactly,
— a 2-cell is a module natural transformation.

Obviously, the category **games** will be an example of a combinatorial game category. Before we get into other examples, we show how to build a free combinatorial game category out of an arbitrary module category.

**Proposition 4.5.** There is a forgetful 2-functor $\mathbf{modcat} \xrightarrow{U} \mathbf{cgc}$ which has a left 2-adjoint $F$, which constructs the free combinatorial game category based on a module category.

*Proof.* Given the definitions of **modcat** and **cgc**, it is obvious that forgetting the combinatorial game structure gives a 2-functor.

Given a module category $(\mathbf{C}, M)$, we inductively define a cgc $F(\mathbf{C}, M)$ as follows:

— the objects are those of $M$, together with objects $\{(g_i)_{i\in I}|(h_j)_{j\in J}\}$, where the $g_i$ and $h_j$ are objects of $F(\mathbf{C}, M)$,
— the arrows are those of $\mathbf{C}$, together with arrows

$$\{(g_i)_{i\in I}|(h_j)_{j\in J}\} \xrightarrow{[(s_i)_{i\in I},(r_l)_{l\in L}]} \{(g'_k)_{k\in K}|(h'_l)_{l\in L}\}$$

for cross-arrows

$$(g_i \overset{s_i}{\relbar\!\!\!+\!\!\!\rightarrow} \{(g'_k)_{k\in K}|(h'_l)_{l\in L}\})_{i\in I}, \{(g_i)_{i\in I}|(h_j)_{j\in J}\} \overset{r_l}{\relbar\!\!\!+\!\!\!\rightarrow} h'_l)_{l\in L},$$

— the cross-arrows are those of $M$, together with cross-arrows

$$a \overset{\sigma_i \cdot f}{\relbar\!\!\!+\!\!\!\rightarrow} \{(g_i)_{i\in I}|(h_j)_{j\in J}\}$$

for an arrow $a \xrightarrow{f} g_i$, and

$$\{(g_i)_{i\in I}|(h_j)_{j\in J}\} \overset{\pi_j \cdot f}{\relbar\!\!\!+\!\!\!\rightarrow} a$$

for an arrow $h_j \xrightarrow{f} a$.

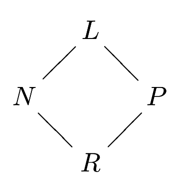
In addition, we quotient out the arrows and cross-arrows by the equivalence relations generated by the conversion relations for the type theory. This is well-defined as the term language has cut-elimination and satisfies Church-Rosser.

The unit of the adjunction is the inclusion from $(\mathbf{C}, M)$ to $F(\mathbf{C}, M)$. If we have a module functor $\mathbf{C} \xrightarrow{f} U(\mathbf{D})$, we can define a pcg functor $F(\mathbf{C}) \xrightarrow{\bar{f}} \mathbf{D}$ which acts as $f$ does on the objects of $\mathbf{C}$, and takes the constructed diproducts in $F(\mathbf{C})$ to the chosen diproducts in $\mathbf{D}$. We can similarly extend module natural transformations. It is then easy to check that we have a 2-adjunction. □

**Example 4.6.** The free game category on the empty module category is essentially **games**, as described above. The only difference is that in this category, the options of the game are a list, rather than a set, so that, for example, the objects $\{g, g|h\}$ and $\{g|h\}$ are distinct. This free game category will be the initial object in the category **cgc**, and hence for any other combinatorial game category $\mathbf{C}$, there will be a unique combinatorial game category $\mathbf{games} \longrightarrow \mathbf{C}$.

While the category **games** is a free combinatorial game category, most of the interesting examples will not be free.

**Example 4.7.** The outcome lattice




is a combinatorial game lattice, with the following additional structure: there are cross-arrows between every pair except $(L, P)$, $(L, R)$, $(P, P)$, and $(P, R)$ (that is, there is a cross-arrow $A \relbar\!\!\!+\!\!\!\rightarrow B$ if it possible to win a game $-G + H$ as Left, playing first, with

$o(G) = A$, $o(H) = B$), and

$$\{(g_i)_I|(h_j)_J\} = \begin{cases} N & \text{if } \exists i \in I, g_i = L \text{ or } P \text{ and } \exists j \in J, h_j = R \text{ or } P; \\ L & \text{if } \exists i \in I, g_i = L \text{ or } P \text{ and } \forall j \in J, h_j = L \text{ or } N; \\ R & \text{if } \forall i \in I, g_i = R \text{ or } N \text{ and } \exists j \in J, h_j = R \text{ or } P; \\ P & \text{if } \forall i \in I, g_i = R \text{ or } N \text{ and } \forall j \in J, h_j = L \text{ or } N. \end{cases}$$

Combinatorial game theorists will recognize this as the table one uses to compute the outcome of a game from the outcomes of its options. The unique combinatorial game functor from **games** to **O** gives the outcome of a game.

In fact, as we shall see later, this combinatorial game category is an example of a more general construction, which allows us to build other "outcome" combinatorial game lattices.

**Example 4.8.** Any category with a 0 object has a (rather trivial) cgc structure, with the module being the identity, and $\{G|H\} = 0$ for all $G, H$. In particular, a one-object one-arrow category equipped with this cgc structure is the terminal object in **cgc**.

**Example 4.9.** It is easily checked that a product of two cgc's has canonical cgc structure.

**Example 4.10.** If **C** is a combinatorial game category, then pos(**C**) is also a combinatorial game category, where pos "makes **C** into a module poset"; that is, it reduces all arrows between objects to a single arrow, and similarly with cross-arrows. In particular, pos(**games**) is the usual lattice of combinatorial games.

We can also show that the "games born by day n" form a combinatorial game lattice. Recall that the day of a game is defined recursively: if a game $G$ has options all of day $n$ or less (and at least one option born by day $n$), then $G$ is born by day $n+1$. In (Calistrate *et al.* 2002) and (Fraser et al. 1987), the authors show that the equivalence classes of games born by day $N$ form a lattice. We would like to show that these lattices are in fact combinatorial game lattices. To do this, we first show that the games born by day $n$ are equivalent to "quotienting out by games born by day $n-1$"[‡]. Then, to form the diproduct of games born by day $n$, we form the diproduct in **games**, then quotient out by games born by day $n-1$ to get us back to games born by day $n$. Thus, we need to prove a few results first.

**Lemma 4.11.** If $G \not\leq H$, and both are born on day $n+1$ or less, then there exists a game born by day $n$ or less such that $o(G+X) \not\leq o(H+X)$.

*Proof.* If $G \not\leq H$, then Left can win $G-H$, playing first. Thus there exists a $G^L$ such that $H \leq G^L$ or there exists a $H^R$ such that $H^R \leq G$. Suppose the first case is true. Since $H \leq G^L, o(H-G^L) \in R \cup P$. Now, Left can also win playing first in $G-G^L$, by taking the $G^L$ move in $G$, then following the copycat strategy. Thus $o(G-G^L) \in L \cup N$. Thus, taking $X = -G^L$ (which is born by day $n$ or less), we get $o(G+X) \not\leq o(H+X)$. The second case is similar, as we can take $X = -H^R$. □

[‡] The authors are grateful to Aaron Siegel for providing assistance with this proof.

**Definition 4.12.** Say that $G$ is equivalent to $H$ up to day $n$, and write $G \cong_n H$, if $o(G+X) = o(H+X)$ for all games $X$ born by day $n$.

**Proposition 4.13.** The games born by day $n+1$ are equivalent to taking the set of all games and modding out by $\cong_n$.

*Proof.*
Suppose we have $G, H$ born by day $n+1$, with $G \neq H$, and assume without loss of generality that $G \not\leq H$. Then the previous lemma implies that $G$ can be distinguished from $H$ by a game born by day $n$.

Conversely, suppose that we have an arbitrary game $G$. We need to show that under $\cong_n$, it is equivalent to some game born by day $n+1$. Then define

$$L = \{X \in G_n : X \not\leq G\}$$

and

$$R = \{X \in G_n : X \not\geq G\}$$

Putting $G' = \{L|R\}$, we get that $G$ is born by day $n+1$, and it is easy to check that $G \cong_n G'$. □

**Proposition 4.14.** The lattice of games born by day $n$ form a combinatorial game lattice, where

— $\{(g_i)_I|(h_j)_J\} := \{(g_i)_I|(h_j)_J\}/\cong_n$
— $G \leq H$ if there exists a pair of games $G', H'$ such that $G \cong_n G'$, $H \cong_n H'$, and an arrow $G' \longrightarrow H'$,
— $G \twoheadrightarrow H$ if there exists a pair of games $G', H'$ such that $G \cong_n G'$, $H \cong_n H'$, and a cross-arrow $G' \twoheadrightarrow H'$,

*Proof.* The previous proposition gives that the diproduct is well-defined. Moreover, the lemma ensures that the lattice structure is the same as the lattice structure for the games born by day n. The axioms for a combinatorial game category are a direct result of the definitions. □

Note, however, that while the definition of $\leq$ does not introduce any new inequalities, the definition of $\twoheadrightarrow$ introduces new cross-inequalities. For example, in the lattice of games born by day 1, there is no first-player win from 1 to 1, but there is a cross-inequality $1 \twoheadrightarrow 1$ in the combinatorial game category.

Finally, since the initial cgc is compact monoidal, one might be curious when the free cgc construction produces a compact monoidal category. In particular, suppose we start with a compact monoidal category **C**. We form the chaotic module category **C**$'$ on this category, adding a single module arrow between any two objects of **C**. If we apply the free cgc functor $F$ to **C**$'$, do we get a compact monoidal category? By constructing the monoidal structure similarly to that of **games**, it is easy to show the following:

**Proposition 4.15.** If $\mathbf{C}$ has monoidal structure, then the underlying category of $F(C')$ has a natural monoidal structure, where the unit is $I$, and:

$$C \otimes D := C \otimes D,$$

$$A \otimes \{(C_i)_I | (D_j)_J\} := \{(A \otimes C_i)_I | (A \otimes D_j)_J\},$$

$$\{(C_i)_I | (D_j)_J\} \otimes A := \{(C_i \otimes A)_I | (D_j \otimes A)_J\},$$

$$\{(A_i)_I | (A'_j)_J\} \otimes \{(B_k)_K | (B'_l)_L\} := \{(A_I \otimes H)_I, (G \otimes B_k)_K | (A'_j \otimes H)_J, (G \otimes b'_l)_L\}$$

(where, for the last definition, $G = \{(a_i)_I | (a'_j)_J\}$, $H = \{(b_k)_K | (b'_l)_L\}$).

Unfortunately, the same is not true of compact closed structure. If $\mathbf{C}$ has compact structure $*$, one could define

$$\{(g_i)_I | (h_j)_J\}^* := \{(h^*_j)_J | (g^*_i)_I\}$$

This gives an involutive functor on the underlying category of $F(C')$. However, in the free game category, there are no arrows from objects of $\mathbf{C}$ to diproducts, so if $G$ is a diproduct, then there are no arrows

$$I \longrightarrow G^* \otimes G, G \otimes G^* \longrightarrow I$$

and so $F(\mathbf{C}')$ cannot be compact closed.

## 5. Combinatorial Game Categories and Polarized Categories

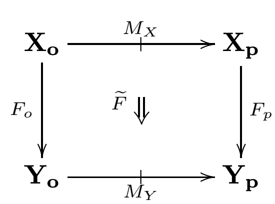
In this section, we describe the relationship between polarized categories, polarized game categories, module categories, and combinatorial game categories. In particular, we will show that any polarized game category gives a combinatorial game category. An intriguing aspect of this is that the outcome cgc is an example of this construction. Another example of the construction shows that games with a consecutive move-ban (Ottaway 2009) form a combinatorial game category.

From (Cockett and Seely 2007), we recall the definition of the 2-category of polarized categories:

**Definition 5.1.** The 2-category **polcat** consists of:

— an object is a pair of categories $(\mathbf{X_o}, \mathbf{X_p})$, with a module $\mathbf{X_o} \xrightarrow{M} \mathbf{X_p}$,

— an arrow between $(\mathbf{X_o} \xrightarrow{M_X} \mathbf{X_p})$ and $(\mathbf{Y_o} \xrightarrow{M_Y} \mathbf{Y_p})$ consists of functors $\mathbf{X_o} \xrightarrow{F_o} \mathbf{Y_0}$, $\mathbf{X_p} \xrightarrow{F_p} \mathbf{Y_p}$, as well as a module morphism



— a 2-cell between $(F_o, F_p, \widetilde{F})$ and $(G, \widetilde{G})$ consists of natural transformations $F_o \xrightarrow{\alpha_o} G_o$, $G_p \xrightarrow{\alpha_p} G_P$ such that for any $M_X$-arrow $h \xrightarrow{m} g$, we have that

$$\begin{array}{ccc} F_o h & \xrightarrow{\widetilde{F}m} & F_p g \\ {\scriptstyle \alpha_o}\downarrow & & \downarrow{\scriptstyle \alpha_p} \\ G_o h & \xrightarrow[\widetilde{G}m]{} & G_p g' \end{array}$$

commutes. The composition and identities are the natural ones.

Polarized categories and module categories relate as follows:

**Proposition 5.2.** The inclusion $\mathcal{I}$ of **modcat** into **polcat** has a right 2-adjoint $\mathcal{P}$ and a left 2-adjoint $\mathcal{S}$:

$$\begin{array}{c} \textbf{polcat} \\ \dashv \ \mathcal{I}\uparrow \ \dashv \\ \textbf{modcat} \end{array}$$

*Proof.* Suppose $(\mathbf{X_o} \xrightarrow{M_X} \mathbf{X_p})$ is a polarized category. We need to describe $\mathcal{P}$ of it, which should be a module category. We take the category to be $\mathbf{X_p} \times \mathbf{X_o}$. The module structure on this category, $M$, is given by

$$M[(g_1, h_1), (g_2, h_2)] = M_X(h_1, g_2)$$

We need to define the composition of module arrows with $\mathbf{X}$-arrows on either side. If we have arrows

$$(g_1', h_1') \xrightarrow{(f_1, f_2)} (g_1, h_1) \xrightarrow{m} (g_2, h_2)$$

then define $(f_1, f_2); m := f_2; m$. Similarly, if we have arrows

$$(g_1, h_1) \xrightarrow{m} (g_2, h_2) \xrightarrow{(f_1, f_2)} (g_2', h_2')$$

then define $m; (f_1, f_2) := m; f_1$.

The associativity and unit axioms for $M$ follow directly from the associativity and unit axioms for $\mathbf{M_X}$. This thus defines a module category.

Suppose now that we have a polarized functor $(\mathbf{X_o} \xrightarrow{M_X} \mathbf{X_p}) \xrightarrow{(F_o, F_p, \widetilde{F})} (\mathbf{Y_o} \xrightarrow{M_Y} \mathbf{Y_p})$. We define $\mathcal{P}(F_o, F_p, \widetilde{F})$ to be $F_p \times F_o$. The module map is a restricted version of $\widetilde{F}$, and all axioms are easily checked.

If we have a polarized natural transformation $(\alpha_o, \alpha_p)$, we can define a module natural transformation by $\alpha_p \times \alpha_o$. The axioms directly follow. Finally, it is easily checked that

$\mathcal{P}$ is a 2-functor.

Suppose we have a polarized functor $\mathcal{I}(\mathbf{X} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+} \mathbf{X}) \longrightarrow (\mathbf{X_o} \overset{M_X}{\longrightarrow\!\!\!\!\!\!\!\!+} \mathbf{X_p})$. This consists of functors $\mathbf{X} \xrightarrow{F_o} \mathbf{X_o}$, $\mathbf{X} \xrightarrow{F_p} \mathbf{X_p}$, as well as a module morphism which assigns to each $x \twoheadrightarrow x'$ a $F_o x \twoheadrightarrow F_p x'$.

Conversely, a module functor $(\mathbf{X} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+} \mathbf{X}) \longrightarrow \mathcal{P}(\mathbf{X_o} \overset{M_X}{\longrightarrow\!\!\!\!\!\!\!\!+} \mathbf{X_p})$ consists of a functor $\mathbf{X} \xrightarrow{F} \mathbf{X_p} \times \mathbf{X_o}$, and a module morphism which assigns to each $x \twoheadrightarrow x'$ a $F_1 x \twoheadrightarrow F_2 x'$. Thus, it is easily seen that these two categories are naturally isomorphic, and we have a 2-adjunction.

To define $\mathcal{S}$ of a polarized category $(\mathbf{X_o} \overset{M_X}{\longrightarrow\!\!\!\!\!\!\!\!+} \mathbf{X_p})$, we take the coproduct of $X_p$ and $X_o$ as the category; the module arrows are those given by $M_X$ and no others. Checking that this provides a left 2-adjoint is then similar to the above.

□

We can now show how to build a combinatorial game category out of a polarized game category. Moreover, we shall see that the outcome cgc will be an instance of this construction.

**Proposition 5.3.** Suppose that $\mathbf{X_o} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+} \mathbf{X_p}$ is a polarized game category Then one can define a cgc $\mathcal{P}(\mathbf{X_o} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+} \mathbf{X_p})$, with the category given by $\mathbf{X_p} \times \mathbf{X_o}$, $M$ as above, and

$$\{(x_i, y_i)_{i \in I} | (x'_j, y'_j)_{j \in J}\} := \left( \coprod_I y_i, \prod_J x'_j \right)$$

*Proof.* Since product and coproducts are functors, the above is also a functor. To check the correspondence, take $G = (x_i, y_i), H = (x'_j, y'_j), G' = (w_k, z_k), H' = (w'_l, z'_l)$. Then we have

$$\begin{array}{c}
\{G|H\} \longrightarrow \{G'|H'\} \\ \hline\hline
\{(x_i, y_i)|(x'_j, y'_j)\} \longrightarrow \{(w_k, z_k)|(w'_l, z'_l)\} \\ \hline\hline
(\coprod_I y_i, \prod_J x'_j) \longrightarrow (\coprod_K z_k, \prod_L w'_l) \\ \hline\hline
\coprod_I y_i \longrightarrow \coprod_K z_k, \prod_J x'_j \longrightarrow \prod_L w'_l \\ \hline\hline
(y_i \twoheadrightarrow \coprod_K z_k)_{i \in I}, \prod_J x'_j \twoheadrightarrow w'_l)_{l \in L} \\ \hline\hline
[(x_i, y_i) \twoheadrightarrow (\coprod_K z_k, \prod_L w'_l)]_{i \in I}, [(\coprod_I y_i, \prod_J x'_j) \twoheadrightarrow (w'_l, z'_l)]_{l \in L} \\ \hline\hline
(g_i \twoheadrightarrow \{G'|H'\})_{i \in I}, (\{G|H\} \twoheadrightarrow h'_l)_{l \in L}
\end{array}$$

as required. We also need to check the coherence for the composite of two ditupled maps. Suppose we have two ditupled maps

$$\{(a_i, b_i)_I | (c_j, d_j)_J\} \xrightarrow{[(s_i)_I | (r_l)_L]} \{(a'_k, b'_k)_K | (c'_l, d'_l)_L\} \xrightarrow{[(s'_k)_K | (r'_n)_N]} \{(a''_m, b''_m)_M | (c''_n, d''_n)\}$$

We need this composite to be equal to

$$(\dagger) \quad [(s_i; [(s'_k)_K | (r'_n)_N])_I | ([(s_i)_I | (r_l)_L]; r'_n)_N]$$

The first expression reduces to

$$\left(\coprod_I b_i, \prod_J c_j\right) \xrightarrow{((s_i)_I,(r_l)_L))} \left(\coprod_K b'_k, prod_L c'_l\right) \xrightarrow{((s'_k)_K,(r'_n)_N)} \left(\coprod_M b''_m, \prod c''_n\right)$$

which reduces further to

$$\left(\coprod_I b_i \xrightarrow{(s_i)_I} \coprod_K b_k \xrightarrow{(s'_k)_K} \coprod_M b''_m,\ \prod c_j \xrightarrow{(r_l)_L} \prod_L c'_l \xrightarrow{(r'_n)_N} \prod_N c''_n\right)$$

However, by the coherence axioms for a polarized game category, this becomes

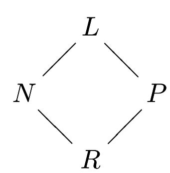
$$\left(\coprod_I \xrightarrow{(s_i;(s'_k)_K)_I} \coprod_M b''_m, \prod_J c_j \xrightarrow{((r_l)_L;r'_n)} \prod_N c''_n\right)$$

which is the expanded version of (†), as required. □

**Example 5.4.**

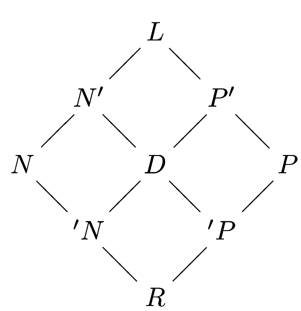
Any category **X** with products and coproducts is a polarized game category (where the module is the identity) and hence, by above, we get a cgc structure on $\mathbf{X} \times \mathbf{X}$.

**Example 5.5.** In particular, taking the simplest non-trivial category with products and coproducts, $\mathbf{X} = \{0 \leq 1\}$, and applying the construction above, we get the outcome lattice




where $(0,0) = R, (1,0) = N, (0,1) = P, (1,1) = L$ (think of 0 as Right wins, 1 as Left wins - the first slot determines who wins if Left goes first, and the second slot determines who wins if Right goes first). It is easily checked that the $\{|\}$ structure on this lattice defined by $\mathcal{P}$ is the same as the $\{|\}$ structure given in example 4.7.

**Example 5.6.** Taking $\mathbf{X} = \{R \leq D \leq L\}$ (where $D$ represents draw) gives the loopy outcome category (Siegel 2009, p. 97)



**Example 5.7.** Recall that the initial object in **polgam** is the finite Abramsky-Jagadeesan polarized game category **AJ**. Applying the games construction to this category gives consecutive-move-ban games. A consecutive-move-ban game is one in which both Left and Right could have starting plays, play alternates so that $G^{LL}$ and $G^{RR}$ are empty, and every option of $G$ has a consecutive-move-ban. If we have an object of $\mathcal{P}(\coprod_I h_i, \prod g_j)$ (where $\coprod h_i$ is a player game, $\prod_J g_j$ an opponent game), it is sent to the consecutive-move-ban game $\{(h_i)_I|(g_j)_J\}$. By the construction of ditupling in $\mathcal{P}$, ditupling in $\mathcal{P}(\mathbf{AJ})$ removes any $G^{LL}$ or $G^{RR}$ options.

Consecutive-move-ban games have appeared in the combinatorial game literature as a useful tool to study misère games: for more detail, see (Ottaway 2009).

There is also a functor from **cgc** to **polgam** (which, unfortunately, is not adjoint to $P$). It defines the opponent category of a cgc to be those objects which have no left option, and the player category of a cgc to be those objects which have no Right options.

**Proposition 5.8.** There is a 2-functor $\mathbf{cgc} \xrightarrow{F} \mathbf{polgam}$ which sends a combinatorial game category $\mathbf{C}$ to the polarized game category with

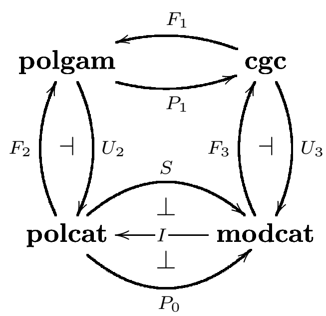
— $\mathbf{X_o}$ is the full subcategory of $\mathbf{C}$ consisting of objects of the form $\{\emptyset|(h_j)_J\}$,
— $\mathbf{X_p}$ is the full subcategory of $\mathbf{C}$ consisting of objects of the form $\{(g_i)_I|\emptyset\}$,
— the module arrows are the module arrows of $\mathbf{C}$,
— coproduct is given by $\coprod_I x_i := \{(x_i)_I|\emptyset\}$,
— product is given by $\prod_I y_i := \{\emptyset|(y_i)_I\}$,
— projection and injection are given by the projection and injection of the cgc,
— tupling and cotupling are both given by ditupling of the cgc.

*Proof.* The coherence rules for a polarized game category all follow easily from the coherence rules for the CGC. □

**Example 5.9.** Applying $F$ to the initial object of **cgc** gives a polarized game category where the opponent objects are all games with no Left option, and the player objects all games with no Right option. Note, however, that this polarized game category is not the initial object in **polgam**, as not every opponent game is a product of player games, nor is every player game a coproduct of opponent games.

**Example 5.10.** Suppose $\mathbf{C}$ is a category with products and coproducts. It is easy to check that if we apply $FP$ to the polarized game category $\mathbf{C} \xrightarrow{I_C} \mathbf{C}$, we get a polarized game category which is isomorphic to $\mathbf{C} \xrightarrow{I_C} \mathbf{C}$. In particular, if we apply $F$ to the four- or nine-element outcome categories, we get back the two- and three-element lattices from which they were built.

If we include the results from (Cockett and Seely 2007) on the free polarized game category, we have the following functors between polarized categories, polarized games, module categories, and combinatorial game categories:




Each $F_i$ is a free functor, each $U_i$ a forgetful functor, and the $P_i$ functors construct module categories out of polarized categories. The only commutativity is $P_0U_2 = U_3P_1$.

## 6. Idempotents in Combinatorial Game Categories

As mentioned in the discussion of combinatorial games, one of the most fundamental tools of combinatorial game theory is the canonical form of a game $G$. It is a game $G'$, which is equivalent to $G$, but is "simplest", in the sense that there are no dominated or reversible moves, and all of the options of the game are in canonical form. In this section, we will show that the notion of the canonical form of a game is deeply linked to categorical notions. In particular, for any object $G$ in the category **games**, there is a unique "maximal" idempotent on $G$, which contains all of the dominated and reversible moves of $G$. Moreover, we shall show that idempotents split in the category of games, and the splitting of this idempotent gives the canonical form of the game $G$.

We begin by describing the general theory of these "maximal" idempotents, then show how this theory applies to the category **games**. The general theory holds in any finite-set-enriched category in which idempotents split.

Categories enriched in finite sets have a number of rather special properties and a notable one is that they always have a fully retracted skeleton. An object is **fully retracted** in case its only idempotent endomorphism is the identity. In a finitely enriched category in which idempotents split, every object has, up to isomorphism, a unique retract which is fully retracted.

Recall that a category is enriched in finite sets in case it is an ordinary (**sets**-enriched) category in which all the homsets are finite. This does not mean the number of objects is finite as the category of finite sets, $\mathbf{sets}_f$, is certainly finite-set-enriched, yet by no means has a finite number of objects. Indeed, any $\mathbf{sets}_f$-concrete category (i.e. a category with a faithful functor to finite sets) will be finite set enriched so that the category of finite groups, rings or fields are all finite set enriched.

A peculiar property that finite set enriched categories have is that every endomorphism, if raised to a high enough power, will start to repeat itself (as there are only finitely many such maps). This allows us to associate with each map an idempotent; a category in which one can associate with each endo-map an idempotent of this form is said to be retractive. If, in addition, each object has an associated fully retracted object the category is said to be fully retractive.

A **retractive inverse**, $g : B \longrightarrow A$, for a map $f : A \longrightarrow B$ is a map such that $gfg = g$. To explain why this is called a "retractive" inverse, consider the following situation where $s$ and $s'$ are sections and so have right inverses $r$ and $r'$ respectively, then $rs'$ has $r's$ as a retractive inverse as $r'srs'r's = r's$. Thus, whenever two objects have a retract in common they will be connected by a map which has a retractive inverse. Notice in this case, although this does not happen in general, also $r's$ has a retractive inverse $rs'$. In this case we shall say that the pair of maps are **mutual retractive inverses**.

When idempotents split, being connected by a map with a retractive inverse implies that the domain and codomain of the map have a common retract. To see this, first note that if $f$ has a retractive inverse $g$ then both $fg$ and $gf$ are idempotents and in the idempotent splitting

$$g : gf \longrightarrow fg \quad \text{and} \quad fgf : fg \longrightarrow gf$$

but also $g(fgf) = gf$ and $(fgf)g = fg$ so that these make these idempotents isomorphic. This means we have a pair of sections exhibiting a common retract of the objects.

Notice that if $g$ is a retractive inverse for $f$ then $f$ need not be a retractive inverse of $g$. However $fgf$ will be a retractive inverse of $g$ as $(fgf)g(fgf) = fgf$. Thus when $g$ is a retractive inverse of $f$ then $g$ and $fgf$ are always mutual retractive inverses.

Retractive inverses need not be unique but they do enjoy the following weak uniqueness property: if $g$ and $g'$ are retractive inverses of $f$ such that $fg = fg'$ and $gf = g'f$ then $g = g'$, as $g = gfg = gfg' = g'fg' = g'$. This is as might be expected as $gf$ and $fg$ are, from our analysis, supposed to split via the same object (up to isomorphism).

An endomorphism $h : A \longrightarrow A$ has a **central retractive inverse** $g$ in case $g$ is a retractive inverse of $h$ such that $hg = gh$: the "central" prefix refers to the fact that the two idempotents generated are the same. Notice that this means that $g^n h^n = (gh)^n = gh$ and $h^n g^n = (hg)^n = hg$ and has the consequence that if $g$ and $g'$ are central retractive inverses of $f$ with $g^n = g'^n$ for some $n \geq 1$ then $g = g'$. This is because:

$$\begin{aligned} g &= gfg = (gf)^n g = g^n f^n g = g'^n f^n g = (g'f)^n g = g'fg \\ &= g'(fg)^n = g'f^n g^n = g'f^n g'^n = g'(fg')^n = g'fg' = g' \end{aligned}$$

The cenral retractive inverses of $h$ can be ordered: suppose $g_1$ and $g_2$ are as above, then define $g_1 \leq g_2$ if $g_1 h g_2 = g_2 = g_2 h g_1$. This is clearly a reflexive relation. It is transitive since if $g_1 \leq g_2$ and $g_2 \leq g_3$ then

$$g_1 h g_3 = g_1 h g_2 h g_3 = g_2 h g_3 = g_3 \quad \text{and} \quad g_3 h g_1 = g_3 h g_2 h g_1 = g_3 h g_2 = g_3$$

and antisymmetric since if $g_1 \leq g_2$ and $g_2 \leq g_1$ then $g_1 = g_1 h g_2 = g_2$.

A central retractive inverse $r$ of $h : A \longrightarrow A$ is a **least central retractive inverse**, if for any other central retractive inverse $g$ of $f$ we have $r \leq g$ (that is $gfr = g = rfg$). Clearly if $h$ has a least central retractive inverse it must be unique.

Clearly any automorphism has as its least retractive inverse its ordinary inverse. Any idempotent $e$ has its least reflexive retractive inverse $e$. To see this first we note that a reflexive retractive inverse of an idempotent is always an idempotent as $g = gegeg = ggeeg = ggeg = gg$. $e$ is clearly a reflexive retractive inverse of itself but for any other such inverse $g$ we have

$$eeg = eegeg = geeeg = geg = g = geg = geeeg = gegee = gee$$

so that $e$ is the least central retractive inverse.

We shall say that a category is **retractive** in case each endomorphism $f : A \longrightarrow A$ has a least central retractive inverse $r_f$ such that $r_{fg} f = f r_{gf}$.

Clearly every groupoid is a retractive category, but there is also an important source of examples in finitely enriched categories:

**Proposition 6.1.** Any finite-set-enriched category is retractive.

*Proof.* Suppose $f : A \longrightarrow A$ then there is are smallest numbers $k, h, h', m > 0$ such that $f^k = f^{k+h}$ and $m \cdot h = k + h'$. Set $r_f = f^{2 \cdot m \cdot h - 1}$ then certainly $f r_f = r_f f$ but also

$$\begin{aligned} r_f f r_f &= f^{2 \cdot m \cdot h - 1} f f^{2 \cdot m \cdot h - 1} \\ &= f^{2 \cdot m \cdot h} f^{k + h' - 1} \\ &= f^{k + 2 \cdot m \cdot h} f^{h' - 1} \\ &= f^{k + h' - 1} \\ &= r_f \end{aligned}$$

To show $r_f$ is least, we suppose we have a central retractive inverse $g$. We must show that $r_f f g = g = g f r_f$. As $r_f$ and $g$ commute it suffices to show $r_f f g = g$. For this we have:

$$\begin{aligned} g &= gfg = (gf)^{2 \cdot m \cdot h} g \\ &= g^{2 \cdot m \cdot h} r_f f g \\ &= g^{2 \cdot m \cdot h} r_f f r_f f g \\ &= (gf)^{2 \cdot m \cdot h} r_f f g \\ &= g f r_f f g \\ &= r_f f g. \end{aligned}$$

It remains to show that $r_{xy}x = xr_{yx}$. For this we observe:

$$(xy)^{2\cdot m\cdot h-1}x = x(yx)^{2\cdot m\cdot h-1}$$

so that if $r_{yx} = (yx)^{2\cdot m\cdot h-1}$ we are done.

**Lemma 6.2.** In any category, if $fg$ repeats with cycle length $h$ after step $k$ (i.e. we have $(fg)^k = (fg)^{k+h}$) then $gf$ repeats with cycle length $h$, and it starts repeating at or before $k+1$ steps.

*Proof.* If $fg$ starts repeating at $k$ then for any $k' > 1$ we have

$$(gf)^{k+k'+h} = f(fg)^{k+h+(k'-1)}g = f(fg)^{k+(k'-1)}g = (gf)^{k+k'}.$$

As this works for all $k' > 1$ it follows that $gf$ repeats no later than $k+1$. □

Now $(yx)^{2\cdot m\cdot h-1} = (yx)^{2\cdot(k+h')-1}$ if the cycle length of $yx$ is less or equal to $k$ (that is $k$ or $k-1$), and we are done. However, if the cycle length is $k+1$ and $h' = 1$, we must use the fact that

$$(yx)^{2\cdot(k+h')-1} = (yx)^{2\cdot((k+1)+h)-1} = r_{yx}.$$

This completes the proof of proposition 6.1. □

An object is fully retracted in case its only idempotent endomorphism is the identity map.

**Lemma 6.3.** In a retractive category:

1 If two fully retracted objects are connected (that is, there are maps both ways between them) then all maps between them are isomorphisms.
2 The endomorphisms of a fully retracted object form a group.
3 Any two fully retracted objects which are retracts of the same object are isomorphic.

*Proof.* Suppose $A$ and $B$ are fully retracted and $f : A \longrightarrow B$ and $g : B \longrightarrow A$ then $r_{fg}fg = fgr_{fg} = 1_A$ so $fg$ is an isomorphism. This means that $f$ is a section. But similarly $gf$ is an isomorphism so $f$ is a retraction, and so is an isomorphism.

Two fully retracted objects which are retractions of the same object are connected, and so are isomorphic. □

We shall call a category **fully retractive** in case the category is retractive and every object can be fully retracted.

**Lemma 6.4.** A retractive category is fully retracted in case every object has an idempotent $e$ which splits such that any other idempotent $e'$ with $ee' = e'e$ has $ee' = e$.

*Proof.* The splitting of $e$ gives a fully retracted object as any idempotent on that object would induce an idempotent $e'$ which commutes with $e$ on the original object and would have $ee' = e'$. □

**Corollary 6.5.** Every finite-set-enriched category in which idempotents split is fully retractive.

*Proof.* The number of idempotents on an object is finite. Define a preorder on idempotents by $e \leq e'$ if $ee' = e$. This is clearly reflexive. It is transitive as $e \leq e' \leq e''$ means $ee' = e$ and $e'e'' = e'$ so that $ee'' = (ee')e'' = e(e'e'') = ee' = e$. This preorder must have least elements: pick such a least element $e_0$. Now suppose $ee_0 = e_0e$ then $e_0e = e_0$ as $e_0$ is miNimal. Thus $e_0$ exhibits the property required by lemma 6.4. □

6.1. *Idempotents in* **games**

We now return to particular considerations of the category **games**. Note that **games** is finite-set enriched, so the above theory will apply, so long as we can show that idempotents split in **games**. This result is well-known to game-theorists, though not in this form.

**Proposition 6.6.** Idempotents split in the category **games**.

*Proof.* We proceed by induction on the birthday of the game. So suppose that $G \xrightarrow{e} G$ is an idempotent, and assume that any idempotent on a game with birthday less than that of $G = \{(g_i)_I | (h_j)_J\}$ splits. If Right chooses a move $h_j$, Left's response falls into one of three categories:

1 some $k \neq j$ and an arrow $h_k \longrightarrow h_j$, ($h_k$ "dominates" $h_j$),
2 an idempotent $h_j \xrightarrow{e_j} h_j$,
3 or some $h_j^L$, and an arrow $G \longrightarrow h_J^L$ ($h_j$ is "reversible"),

(Similarly for any choice of move $g_i$ by the Right player). Define $G'$ by taking $G$ and, for each $j$ and each of the above cases,

1 eliminate $h_j$,
2 split the idempotent $e_j$, and replace $h_j$ with the split object $h'_j$,
3 replace $h_j$ with the list of Right options of $h_j^L$.

It is easy to see that such a definition of $G'$ gives canonical maps $G \xrightarrow{e_1} G' \xrightarrow{e_2} G$ such that $e_2e_1 = e$: for both $e_1$ and $e_2$, we follow the strategy $e$. If our response has been removed we choose its dominated option, if it has been replaced by its idempotent splitting, we use the split map, and if it has been reversed, we use the reversed strategy. □

Thus, by above, every object $G$ in **games** has a fully retracted retract $G'$. We now show that this must be the canonical form of $G$.

**Proposition 6.7.** In **games**, an object $G = \{(g_i)_I | (h_j)_J\}$ is fully retracted if and only if it is in canonical form.

*Proof.* We prove this by induction. Assume that for every game with birthday less than that of $G$, the proposition holds.

For the right-to-left implication, assume $G$ is not in canonical form. Since each of its options are in canonical form, it either has a dominated option or a reversible option. Suppose it has a dominated Right option, $h_k \leq h_j$. Define a strategy $e$ on $G$ which is the identity strategy for any choice by Right except $h_j$. In the case of $h_j$, respond with $h_k$.

This gives a non-trivial idempotent on $G$. If it has a reversible option $G \leq h_j^L$, we define a strategy $e$ on $G$ which is the identity strategy for any choice except $h_j$. If $h_j$ is chosen, respond with $h_j^L$. This also gives a non-trivial idempotent on $G$. Dominated or reversible Left options are treated similarly.

For the left-to-right implication, assume $G$ has a non-trivial idempotent. Assume that the non-trivial option is the choice of either some $h_k \xrightarrow{f} h_j$ (where f is not the identity), or some $h_j^L$ (Left options are similar). In the first case, either $f$ is itself an idempotent, or $h_k \leq h_j$ for $j \neq k$. If $f$ is an idempotent, this contradicts our inductive assumption. Otherwise, we have a dominated or reversible option, so $G$ is not in canonical form. □

Thus, the notion of the canonical form of a game is a particular example of a phenomenon which happens in any finite-set-enriched category in which idempotents split. To conclude this section, we give two counter-examples to further understand the notion of canonical form.

**Example 6.8.** The following counter-example shows that the sum of two games in canonical form need not be in canonical form. Take $G = \{0|0\} = *$ and $H = \{1|0\}$. Then $G + H = \{H, 1 + *|H, *\}$. However, $* < H$, so $H$ is a dominated option.

**Example 6.9.** The following example shows that two games in canonical form may have more than one arrow between them. Take $H = \{5||\{1, 1 + *| - 5\}\}$, and $G = \{|\} = 0$. $G$ is obviously in canonical form, and it is easy to check that $H$ is also (the presence of the 5 and -5 ensure that there are no reversible options). There are two arrows $G \longrightarrow H$: once Right moves, Left can choose either 1 or $1 + *$.

## 7. Loopy Games

A "loopy game" is one in which a player can return to a previous game position. This raises two questions. The first is determining the outcome of such a game: who wins a line of play which endlessly cycles back on itself? The second is structural: is there a category of loopy games? The difficulty with such a categorical structure is composition: when we try to define the "swivel chair strategy" for the composite $G \xrightarrow{f_1} H \xrightarrow{f_2} K$, we could end up with an infinite loop in the $H$ terms, never resolving our response in either $G$ or $K$. As we shall see, solving the first problem also solves the second: if we can define who wins which loops, we can get a categorical structure. However, there are different ways of defining who wins such loops.

In this section, we will look at the different approaches to dealing with this problem, and what categorical structure they contain. Interestingly, the combinatorial game theory community has developed a different approach from the proof theory/computer science community; here, we will be able to compare and contrast the two approaches.

One initial approach is to consider all infinite plays as draws. This gives nine outcome classes for each game, determined by whether Left wins, loses or draws playing first or second (we have previously shown that this expanded outcome lattice is also a combinatorial game lattice). One can then put a partial order on all loopy games just as for normal games:

$$G \leq H \text{ if } o(G+X) \leq o(H+X) \ \forall \text{ loopy games } X.$$

Our question is then to ask whether there is a notion of arrow between loopy games which generalizes this partial order. This is essentially the question Aaron Siegel asks in his survey of loopy games: "Can one specify an *effective* equivalent definition of $[G \geq H]$?" (Siegel 2009, p. 97).

One would like to define an arrow $G \longrightarrow H$ to be a strategy for Left, playing second in $-G+H$, that at least achieves a draw for Left. As mentioned above, the difficulty with this is the composition: if we have arrows $G \longrightarrow H \longrightarrow K$ and attempt to use the usual definition of composition, we find that we may end up with an infinite loop between the $H$ and $-H$, never giving a response in the game $-G+K$. The first solution given by game theorists to this problem is to ban all infinite cycles that could occur in alternating play.

**Definition 7.1.** A loopy game $G$ is a *stopper* if there is no infinite alternating sequence of moves in $G$.

When restricted to the stoppers, the definition of arrows given above does define a category, and the existence of an arrow provides an alternate definition of $\leq$. Moreover, actual games have this condition. One example is the game of Fox and Geese. The foxes are allowed to move freely around the board, while the geese must always move forward. Thus, if the fox is allowed to play continually, one could end up with an infinite set of moves. However, if one is playing alternately, there can never be an infinite cycle, as the geese always move forward.

In general, however, not all games will be stoppers. The second way game theorists deal with the problem is to specify who wins infinite loops: either all loops are won by Right, or all are won by Left.

**Definition 7.2.** If $G$ is a loopy game, define $G^+$ to be the game where all infinite plays are wins for Left, and $G^-$ to be the game where all infinite plays are wins for Right. Say that a loopy game is "fixed" if it is either $G^+$ or $G^-$ for some $G$. A sum $G_1+G_2+\cdots+G_n$ is a win for a player if they win in every component, and a draw otherwise.

The definition of $\leq$ is then modified so that $X$ varies over all fixed or free loopy games. An arrow $G \longrightarrow H$ is then a survival strategy in both $-G^+ + H^+$ *and* $-G^- + H^-$ (where taking the $-$ of a fixed loopy game reverses who wins infinite plays). This definition of arrow gives a categorical structure on the set of all loopy games. Moreover, the "swivel chair" theorem (Siegel 2009, p. 104) then says that $G \leq H$ if and only if there is an arrow from $G \longrightarrow H$.

However, there is a third solution to the problem of loopy games: each possible loop in a game comes pre-assigned as either being a win for Left or a win for Right. That is, the data for a game contains not only what moves one can make from that game, but also an assignment of Left or Right to every position in the game which could be returned to by players. Arrows are survival strategies; that is, strategies on $-G + H$ so that Left wins in at least one component where there is an infinite play. If we modify $\leq$ to range over all loopy games of this type, the existence of $G \longrightarrow H$ is equivalent to $G \leq H$ (see later).

The advantage of this third approach is greater flexibility. By assigning each loop as either a win for Left or a win for Right, one can distinguish between different types of loops that may occur in a game. An example is a situation in Checkers where one player can trap another in a corner. In this case, infinite play will occur. However, the situation looks to be more of an advantage to the player who has trapped the other. Thus, we could assign such a loop as a win for the player who trapped the other player's pieces.

In general, however, not all loopy games easily allow such an assignment. In the game of Philosopher's football, players play on a $n \times m$ board with a ball initially placed in the middle. On their turn, a player may either place a stone anywhere on the board, or jump the ball over a sequence of stones, as in checkers. The goal is to get the ball off the end of your side of the board. Situations can arise in which the ball returns over and over to the same position. One could say that such a loop is a win for a player in whose territory the ball loops more often. However, if the ball loops equally through both player's territories, one must assign this game to be a win for one player or the other, in a somewhat arbitrary fashion.

From the point of view of category theory, however, the loopy games which have an assignment of either Left or Right to each player are far preferable, as they have a universal property: they are inductive/coinductive data types, also known as initial and terminal algebras, or least and greatest fixed points.

### 7.1. *Definition of Loopy Games*

To describe the category of these "fixed" loopy games, we use the description of games which views them as trees. If we view games are trees, then loopy games are represented by trees with backedges.

**Definition 7.3.** A *loopy game* $G$ is a tree $E \xrightarrow{h,t} V$ with backedges, as well as

- a function $E \xrightarrow{p} \{R, L\}$ (which indicates which edges belong to which player),
- a function $w$ from the set of vertices which are the codomain of some backedge to $\{R, L\}$ (which indicates who wins if that vertex is infinitely looped through).

The negative of such a game is easy to describe:

**Definition 7.4.** For a loopy game $G$, $-G$ is the loopy game with the same tree structure, but in which $p$ and $w$ are the opposite of those for $G$.

To describe morphisms between these games, we need to describe a legal play on a pair of games $(G, H)$.

**Definition 7.5.** A *play* $\sigma$ on a pair of loopy games $(G, H)$ is a list of edges of the disjoint union of the trees for $G$ and $H$ such that

— the sublist of edges from $G$ forms a rooted path through $G$,
— the sublist of edges from $H$ forms a rooted path through $H$.

The sublist of $G$ edges describes the moves that players make in the game $G$, and the sublist of $H$ edges the moves the players make in $H$. We can then describe what it means for Left to survive a play which has infinite many edges:

**Definition 7.6.** If a play $\sigma$ on $(G, H)$ is of infinite length, say that *Left survives* $\sigma$ so long as at least one of the $G$ or $H$ sublists loops infinitely often through a vertex $v$ with $w(v) = L$.

We can then describe strategies as a set of plays "closed" under moves by Right and responses by Left.

**Definition 7.7.** A *survival strategy for Left playing second* on $(G, H)$ is a set of plays $s$ such that

— for all $\sigma \in s$ of even length, if $e$ is a Right edge, and $\sigma * e$ is a play, then $\sigma * e \in s$,
— for all $\sigma \in s$ of odd length, there exists a Left edge $e$ such that $\sigma * e \in s$,
— Left survives each $\sigma \in s$ of infinite length.

**Definition 7.8.** The *category of loopy games*, **loopy**, has:

— objects loopy games,
— morphisms $G \longrightarrow H$ survival strategies for Left playing second on $(-G, H)$,
— identity given by the copycat strategy,
— composition given by the usual "swivel chair strategy".

We would like to show that **loopy** is a cgc. To this end, we also need to define the first-player survival strategies.

**Definition 7.9.** A *survival strategy for Left playing first* on $(G, H)$ is a set of plays $s$ such that

— for all $\sigma \in s$ of odd length, if $e$ is a Right edge, and $\sigma * e$ is a play, then $\sigma * e \in s$,
— for all $\sigma \in s$ of even length, there exists a Left edge $e$ such that $\sigma * e \in s$,
— Left survives each $\sigma \in s$ of infinite length.

We can then show:

**Proposition 7.10.** The category **loopy** has structure that makes it into a cgc.

*Proof.* We define the module arrows to be the survival strategies for Left playing first on $(-G, H)$.

Suppose $(g_i)_I$ and $(h_j)_J$ are loopy games. We define the diproduct $\{(g_i)_I|(h_j)_J\}$ to be the tree which has the $g_i$'s and $h_j$'s as subtrees, along with, for each $i$, a Left edge $e_i$ from the root to $g_i$, and for each $j$, a Right edge $f_j$ from the root to $h_j$.

Suppose we have module arrows $(g_i \xrightarrow{s_i} \{(g'_k)_K|(h'_l)_L\})_I$ and $(\{(g_i)_I|(h_j)_J\} \xrightarrow{r_l} h'_l)_L$. We define the ditupled arrow $\{(g_i)_I|(h_j)_J\} \xrightarrow{(s_i|r_l)} \{(g'_k)_K|(h'_l)_L\}$ to be the strategy

$$\bigcup_{i \in I}(e_i * \sigma : \sigma \in s_i) \cup \bigcup_{l \in L}(f_l * \sigma : \sigma \in r_l)$$

Suppose we have an arrow $G \xrightarrow{s} g_i$. We define the injection $G \xrightarrow{\sigma_i \cdot f} \{(g_i)_I|(h_j)_J\}$ to be the strategy

$$\bigcup(e_i * \sigma : \sigma \in f),$$

and the projection is defined similarly. It is straightforward to check that these operations satisfy the required coherences.

□

It is important to note that while **loopy** does have the structure of a combinatorial games category, it does not naturally support the same monoidal structure as the category **games**. For example, take the loopy games $G$ and $H$ where $G$ has a single vertex and backedges to it for both Left and Right, with the vertex designated as a Left win; $H$ is defined similarly, except the vertex is a Right win. The natural game sum of these two games gives a single vertex with backedges for Left and Right: however, there is no canonical choice for whether this vertex is won by Left or Right. Thus, the natural monoidal structure on **games** does not extend to a monoidal structure on **loopy**. This gives another important example of why the important structure for combinatorial games is not the compact monoidal structure, but the combinatorial games structure described in this paper, as **loopy** is a cgc, but not naturally monoidal.

The loopy games have a particularly nice property: the ones designated as Right wins are inductive data types, and the ones designated as Left wins are coinductive data types.

**Definition 7.11.** Let $\mathbf{C}$ be a category, and $\mathbf{C} \xrightarrow{F} \mathbf{C}$ an endofunctor. An *inductive data type* for $F$ is an object $\mu F$, together with a map $F(\mu F) \xrightarrow{\psi} \mu F$ such that for any other object $X \in \mathbf{C}$ and map $FX \xrightarrow{f} X$, there exists a unique map $\mu F \xrightarrow{\bar{f}} X$ such that

$$\begin{array}{ccc} F(\mu F) & \xrightarrow{\psi} & \mu F \\ {\scriptstyle F(\bar{f})}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ FX & \xrightarrow[f]{} & X \end{array}$$

commutes.

A *coinductive data type* for $F$ is an object $\nu F$, together with a map $\nu F \xrightarrow{\phi} F(\nu F)$ such that for any other object $X \in \mathbf{C}$ and map $X \xrightarrow{f} FX$, there exists a unique map $X \xrightarrow{\bar{f}} \nu F$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & FX \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle F(\bar{f})} \\ \nu F & \xrightarrow[\phi]{} & F(\nu F) \end{array}$$

commutes.

**Example 7.12.** For the identity functor, an inductive data type is an initial object, while a coinductive data type is a terminal object.

**Example 7.13.** In **set**, an inductive data type for the functor $X \mapsto X+1$ is the natural numbers, where $\psi$ sends $*$ to 0, and a natural number to its successor.

A coinductive data type for $X \mapsto X+1$ is the set $N \cup \{\omega\}$, and $\phi$ is the predecessor function: $0 \mapsto *$, $n \mapsto n-1$, $\omega \mapsto \omega$.

**Example 7.14.** In **set**, if $A$ is any set, an inductive data type for the functor $X \mapsto 1 + (A \times X)$ is the set of finite lists of elements of $A$. A coinductive data type is the set of finite or countably infinite lists of elements of $A$.

**Example 7.15.** In **set**, an inductive data type for the functor $X \mapsto 1 + (X \times X)$ is the set of all binary trees.

We now describe the functors for which loopy games are the inductive or coinductive data types.

**Definition 7.16.** A loopy functor $\mathbf{loopy} \xrightarrow{F} \mathbf{loopy}$ is a functor of the form

$$\mathbf{loopy} \xrightarrow{\Delta} \mathbf{loopy}^I \times \mathbf{loopy}^J \xrightarrow{(F_i)_I \times (G_j)_J} \mathbf{loopy}^I \times \mathbf{loopy}^J \xrightarrow{\{|\}} \mathbf{loopy}$$

where each $F_i$ and $G_j$ is either a loopy functor or an identity functor.

Note the recursive definition. The first loopy functors built up in this way are

$$X \mapsto \{X|\emptyset\}, X \mapsto \{\emptyset|X\}, X \mapsto \{X|X\}$$

and then other loopy functors are built up from those.

**Definition 7.17.** Suppose that $F$ is a loopy functor. To define $\nu F$, let $X$ be an arbitrary game, and consider the game $F(X)$. We build up the tree for $\nu F$ as follows: each Left option of $F(X)$ is either a diproduct or an $X$. If the option is a diproduct, add a Left edge to $\nu F$; if the option is an $X$, add a backedge to the root of $\nu F$. Do the same with Right edges, and continue on until the game $F(X)$ is exhausted. Finally, label the root of $\nu F$ as a win for Left. $\mu F$ is defined similarly, but with the root a win for Right.

**Proposition 7.18.** If $F$ is a loopy functor, then $\nu F$ is a coinductive data type for $F$, and $\mu F$ is an inductive data type for $F$.

*Proof.* The arrow $\nu F \xrightarrow{\phi} F(\nu F)$ is given by the copycat strategy, as by the definition above, $\nu F$ has the same moves as $F(\nu F)$. Now, suppose we are given a map $X \xrightarrow{f} FX$. From this, we need to build a map $X \xrightarrow{\bar{f}} \nu F$. Note that until an $X$ is encountered in $FX$, the structure of $FX$ is the same as that of $\nu F$. Thus, we follow the strategy $f$ until either ourselves or our opponent chooses an $X$ in $FX$. Thus, there is some follower of $X$, $X_a$, with either $X_a \longrightarrow X$ or $X_a \nrightarrow X$. Thus, by composing with $X \xrightarrow{f} FX$, we get either $X_a \longrightarrow FX$ or $X_a \nrightarrow FX$. We then follow this strategy to continue giving moves to define $\bar{f}$.

Repeating this process, either we run out of moves in $X$, or we encounter a loop in $X$. In either case, we are guaranteed an Left in $\nu F$, and thus guaranteed a survival strategy.

It is easy to see that $\bar{f}$ is the unique map that makes the diagram commute in

$$\begin{array}{ccc} X & \xrightarrow{f} & FX \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle F(\bar{f})} \\ \nu F & \xrightarrow[\phi]{} & F(\nu F) \end{array}$$

as $\phi$ is essentially the identity, and $F(\bar{f})$ is also the copycat strategy until we get to $\bar{f}$, at which point it simply follows that strategy. ☐

7.2. *Conclusions*

The theory presented here is merely a starting point for future structural investigations into game theory. We now know the basic definition of a combinatorial game category. This has allowed us to relate many constructions in game theory. For example, we have shown that the outcome lattices, games born by day n, games with a consecutive move ban, and loopy games all have the same overall structure as the category of games itself. One future consideration will be Misére games (games where the last player to move *loses*). These are considerably more difficult to analyze than normal play games. Moreover, there is no obvious categorical structure one can put on the set of Misére games (Allen 2009). However, some success has been achieved by restricting attention to certain subsets of the set of all Misére games (see, for example, (Plambeck and Siegel 2008)). It would be interesting to see determine how such subsets relate to combinatorial game categories.

7.3. *Acknowledgements*

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**References**

Abramsky, S. (1997) Semantics of Interaction. In *Lecture Notes in Computer Science*, **1059**, 1–31. Springer-Verlag.

Albert, M. H, Nowakowski, R. J, and Wolfe, D. (2009) *Lessons in Play.* A. K. Peters.

Allen, M. R. (2009) *An Investigation of Partizan Misére Games.* Dalhousie University PhD thesis.

Berlekamp, E. R, Conway, J. H, and Guy, R. K. (1984) *Winning Ways for your Mathematical Plays (Vol. 1).* A. K. Peters.

Calistrate, D. Paulhaus, M. and Wolfe, D. (2002) On the lattice structure of finite games. In *More Games of No Chance*, 25–30.

Cockett, J. and Seely, R. (2001) Finite sum-product logic. *Theory and Applications of Categories* **8** (5), 63–99.

Cockett, J. and Seely, R. (2007) Polarized Category Theory, Modules, and Game Semantics. *Theory and Applications of Categories* **18** (2), 4–101.

Demaine, E. and Hearn, B. (2009) *Games, Puzzles, and Computation.* A. K. Peters.

Fraser, W, Hirshberg, S, and Wolfe, D. (2005) The Structure of the Distributive Lattice of Games born by Day n. *Integers: Electronic Journal of Combinatorial Number Theory* **5** (2).

Joyal, A. (1977) Remarques sure la théorie des jeux deux personnes. *Gazette des Sciences Mathématiques du Québec* **4**, 46–52.

Ottaway, P. (2009) *Combinatorial games with restricted options under normal and Misére play.* Dalhousie University PhD thesis.

Plambeck, E. and Siegel, A. (2008) Misere quotients for impartial games. *Journal of Combinatorial Theory, Series A*, 593–622.

Siegel, A. (2009) Coping with Cycles. In *Games of No Chance 3*, 91–123. Cambridge University Press.