Nuclear Physics B360 (1991) 362–396
North-Holland

NONABELIONS IN THE FRACTIONAL QUANTUM HALL EFFECT

Gregory MOORE

Department of Physics, Yale University, New Haven, CT 06511, USA

Nicholas READ

Departments of Applied Physics and Physics, Yale University, New Haven, CT 06520, USA

Received 31 May 1990
(Revised 5 December 1990)

Applications of conformal field theory to the theory of fractional quantum Hall systems are discussed. In particular, Laughlin's wave function and its cousins are interpreted as conformal blocks in certain rational conformal field theories. Using this point of view a hamiltonian is constructed for electrons for which the ground state is known exactly and whose quasihole excitations have nonabelian statistics; we term these objects "nonabelions". It is argued that universality classes of fractional quantum Hall systems can be characterized by the quantum numbers and statistics of their excitations. The relation between the order parameter in the fractional quantum Hall effect and the chiral algebra in rational conformal field theory is stressed, and new order parameters for several states are given.

1. Introduction

The past few years have seen a great deal of interest in two-dimensional many particle and $(2+1)$-dimensional field-theoretic systems from several motivations. These include the fractional quantum Hall effect, high-temperature superconductivity and the anyon gas, conformal field theory in $1+1$ dimensions and its relation to $2+1$ Chern–Simons–Witten (CSW) theories, knot invariants, exactly soluble statistical mechanical models in $1+1$ dimensions, and general investigations of particle statistics in two space dimensions [1–6]. A common theme in most of these investigations is the richness of representations of the braid group, $\mathscr{B}_n$, which replaces the permutation group as the group describing particle statistics in two dimensions. In particular, in the fractional quantum Hall effect (FQHE) it was suggested early on that the fractionally charged quasiparticle excitations obey fractional statistics [7,8], that is adiabatic interchange of two identical quasiparticles produces a phase not equal to ± 1. In other words, in a suitable gauge, the wave functions transform under interchange of quasiparticles as a one-dimensional, i.e. abelian representation of the braid group, in a way not possible in

0550-3213/91/$03.50

higher dimensions because there the only one-dimensional representations of the permutation group correspond to Bose or Fermi statistics. Mathematically, it is known that higher-dimensional representations of $\mathscr{B}_n$ exist, and it has recently been shown that some of these are quite acceptable as a description of particle statistics [2]. In these representations, the wave function of a set of excitations of specified position and quantum numbers becomes a vector, and each exchange of these "particles" gives a matrix, i.e. nonabelian action on this vector. It is interesting to ask whether there exist in nature exotic two-dimensional systems whose elementary excitations include some transforming as nonabelian representations of $\mathscr{B}_n$. Particles defining nontrivial abelian representations of $\mathscr{B}_n$ are known as "anyons" and it seems apt to call these new objects "nonabelions". Fractional quantum Hall systems are the best candidates for such behavior since fractional statistics is already believed to occur there. This idea is further supported by the development of Ginzburg–Landau (GL) theories of the FQHE where the action contains a Chern–Simons (CS) term [4, 9, 10]. Actions similar to the low-energy limit of these theories, but with more complicated gauge groups have recently been shown to be related to the "holomorphic half" of rational conformal field theories (RCFTs) which in turn provide a wealth of nonabelian braid group representations [1, 5].

In this paper we will discuss how the GL–CSW–RCFT connection suggests a new viewpoint on the strongly correlated ground states of the FQHE, and in particular we will use this viewpoint to construct wave functions for the exact ground state of a certain hamiltonian and for quasiparticle excitations which are nonabelions. The main idea is that certain conformal blocks (i.e. the holomorphic square roots of correlation functions) in $(1+1)$-dimensional conformal field theories can be interpreted as wave functions for the electrons in the FQHE ground and excited states, the latter including quasiparticles with non-trivial statistics.

Beyond these specific examples, we will give some arguments for a wider point of view, in which elementary excitations of ground states of incompressible FQHE systems are to be characterized by their quantum numbers and braiding properties (statistics). These properties are rather rigidly constrained, they cannot be perturbed at all by small changes in the hamiltonian of the system, and at the same time seem to give enough information to distinguish physically different systems. This suggests that incompressible FQHE systems fall into classes which we may call "universality classes" by analogy with critical phenomena where the scaling fields of the critical system, together with their corrections, play a similar role. Indeed, this analogy becomes a correspondence when we use the mathematical relationship outlined above. An explicit representative of each class of systems can be constructed using this relationship.

(To avoid confusion, note that these FQHE systems are not themselves critical; it is only the braiding properties of $1+1$ RCFT which are being used. The critical properties of the transitions *between* different states, whether at the same or different filling factor, are an important but separate issue which will not be discussed in this paper.)

It will be seen that this point of view is really an analysis of the (dynamical, spectrum-generating) symmetry. It amounts to an analysis of the possible kinematical properties of the incompressible ground states and their excitations. Dynamical questions, such as how to find the ground state of a given hamiltonian, or how to calculate excitation energies, will not be discussed. This is in accordance with the usual procedure in physics, that kinematics and symmetries of a problem are studied before dynamics. We hope that the general point of view given here will aid in constructing field theoretic representations of FQHE states with the help of which questions such as the stability of different states can be studied. This might be useful in understanding such longstanding problems as the nonappearance of FQHE plateaus at even-denominator σ_{xy} in high magnetic fields which is "explained" in numerical calculations but for which there is no convincing physical picture.

The rest of this paper is organized as follows. In sect. 2 we review some relevant background on both rational conformal field theory and the theory of the fractional quantum Hall effect, and outline our view of their relation. In sect. 3 we discuss Laughlin's states and their hierarchical descendants from the RCFT point of view. The corresponding field theory for filling factor $\nu = 1/q$ is identified as the level $q/2$ rational torus, and the implications for FQHE states on higher-genus Riemann surfaces are discussed. In sect. 4 we address spin-singlet ground states; the principal example of such a state is Halperin's state, which we show corresponds to the level $k = 1$ SU(2) Wess–Zumino–Witten theory [11] combined with the rational torus. In sect. 5 we construct perhaps the simplest of all states with nonabelion quasiparticles, at even-denominator filling factor, using a combination of the rational torus and the Ising model. Conformal field theory leads to a hamiltonian for which our state is the exact ground state. Sect. 6 contains final discussion and speculations.

A few aspects of our constructions have been discussed previously. Banks et al. [12] explored the idea of relating anyon wave functions to conformal blocks in unpublished work. Wen, in a series of papers [13], has used Witten's work on Chern–Simons gauge theory [5] to discuss ground-state degeneracy of FQHE and chiral spin liquid systems on two-dimensional surfaces of genus larger than zero, and has also applied conformal field theory to the study of *edge* excitations of these systems.

2. Adiabatic transport, statistics, order parameters and extended algebras

This section summarizes relevant background material on the fractional quantum Hall effect and conformal field theory and gives our general picture of how they are related. Some readers may prefer to read it in conjunction with sect. 3.

2.1. ADIABATIC TRANSPORT, NONABELIAN STATISTICS AND THE FUSION ALGEBRA IN INCOMPRESSIBLE SYSTEMS

The fractional quantum Hall effect (FQHE) [3], i.e. a plateau in the Hall resistance, is observed in two-dimensional electron gases in high magnetic fields only when the mobile charged excitations have a gap in their excitation spectrum, so the system is incompressible (in the absence of disorder). Therefore the theory of the FQHE begins with the search for ground states of the interacting electron system which exhibit such a gap. In this paper, our goal is not to solve any particular hamiltonian, but to characterize the general properties such states must have if they exist. Accordingly, we will begin by assuming that we have an "incompressible FQHE system" defined as follows. We take a system of electrons confined in a two-dimensional layer, with a strong perpendicular magnetic field. We assume that the field is sufficiently large in comparison with other energies, in particular the interactions between electrons, that the electrons of each spin may be assumed to fill an integer number of Landau levels and partially fill a "last" Landau level (often the lowest); excitations of electrons to higher Landau levels or holes in lower Landau levels than the last can be neglected. (This may not always be strictly true, but we expect that the physics of the states is not affected by some admixture of excited Landau levels.) In this case the hamiltonian reduces to potential energy terms, up to constants; we will take only translationally invariant hamiltonians. The spins of the electrons in the last Landau level may not necessarily be fully polarized by the magnetic field, i.e. the total spin in the ground state may not be given by the value it would take for non-interacting electrons. At high fields, the spins *are* observed to be polarized, and a Landau level polarized parallel to the field is filled before the same Landau level with opposite polarization. In this regime, FQHE plateaus with Hall conductance σ_{xy} a rational number (in units of e^2/h) are observed, the rational having odd denominators only. At lower fields, the partial occupation of the first excited Landau level, the FQHE with even-denominator σ_{xy} is also observed [14] in which not all spins are polarized [15]. We will limit ourselves to the two extreme cases, very strong polarization where spin reversal can be neglected and the problem reduces to that of spinless electrons, and small Zeeman splitting where we may look for spin singlet ground states. Finally, by "incompressible" we mean that all excited states have a finite energy difference from the ground state, including spin excitations in the spin singlet case. We will assume incompressibility in some places in the discussion, although it is not clear that a gap for all types of excitations is necessarily required to observe a Hall plateau (compare "gapless superconductivity"). Also, our states will be fluids, that is they have no long-range positional order (as opposed to solids, which would have gapless phonon modes).

In an incompressible system, one expects that excitations can be localized into wave packets whose quantum numbers differ from those of the ground state only in a finite region, up to exponentially small corrections. In particular, the charged excitations will feel the background magnetic field and so their spectrum will have a Landau level-like structure; in this case a localized wave packet is a "coherent state" since the x and y coordinates of the excitation are non-commuting operators. For a single, charged "quasiparticle" excitation, these coherent states are eigenstates and for several well-separated quasiparticles (in a simultaneous coherent state for all quasiparticles) will be approximate eigenstates. For neutral excitations, a wave packet is, of course, not expected to be an eigenstate. One may try to distinguish "particle-like" from "collective" excitations, the latter having Bose statistics and being typically related to fluctuations of conserved quantities such as charge and spin, thus being neutral and having spin zero or one. The other excitations have either non-trivial charge, spin or other quantum numbers, and/or non-Bose statistics. The latter are very important; we regard statistics as like another quantum number (though strictly speaking it is not one and should not be confused with them) related to the braid group, which can be used to classify and distinguish excitations. This point will be extensively discussed in this paper. We note some ambiguity in whether a neutral Bose excitation should be regarded as "collective" or not. Eventually, our point of view will be that collective excitations are always generated by "charge" densities or currents related to some continuous symmetry which may or may not be present in the underlying electrons but is present in the many-body state. In the two-dimensional conformal field theory point of view to be discussed below, they correspond to currents in a current algebra. The other excitations correspond to nontrivial "primary" fields or their descendants under the chiral algebra (see below). This will sharpen the definitions considerably.

We are interested in the statistics of excitations, that is, how the wave function changes when the locations of excitation wave packets are exchanged. In two space dimensions, such exchanges involve braiding of the world-lines of the excitations and so smooth motions of their positions. In higher dimensions, exchanges are just permutations (all exchange paths are homotopic to one another). Braiding by its very nature involves exchanges along continuous paths, which suggests that exchanges must be done slowly. It is then natural to try to invoke the quantum adiabatic theorem. In our incompressible FQHE systems, we are in good shape to apply this theorem. Consider a state with several well-separated localized excitations. Since excitations are gapped, internally exciting some mode within a localized excitation should cost a finite amount of energy. If we choose a lowest energy internal state of each excitation, for given quantum numbers (drawn from whatever are available in our particular system) of each, the remaining states close in energy should correspond to changes in position of the localized excitations. Thus a small change in position of an excitation gives a state partially orthogonal to the original state. A key point is that, even when we specify positions and quantum numbers (drawn from some set) of each excitation, we may still have a vector space (a subspace of Hilbert space) of approximately degenerate states whose dimension is larger than one. Therefore we must consider adiabatic transport (on the positions of the excitations) within this isolated degenerate manifold. Let us transport excitations with all separations remaining large. Since all virtual excitations which might mediate interactions between our excitations have energy gaps, we expect that the energy of our states remains constant on transport. The phase resulting from the energy of the states is in any case removable [16]. We then consider transport around a closed loop in the configuration space, which means that excitations return to their original positions or are exchanged with identical excitations. In the case where the space of states is one-dimensional, the result of transport is a phase factor, known as Berry's phase. In the general, higher-dimensional case the result is a unitary matrix operation on the state vector we started with [17]. In either case, the effect on the wave function can be expressed as the exponential of the line integral of a vector potential (possibly matrix-valued) [16, 17]. Matrices for all possible different closed loops will not commute, because if they did we could diagonalize them and decompose the vector space into one-dimensional subspaces. The degeneracy of these spaces would have to be accidental and so nongeneric. The generic case will involve noncommuting matrices. We stress that the result of adiabatic transport is gauge invariant; gauge transformations merely shift unitary factors between the states and the hamiltonian. For charged excitations, the result of transport contains a phase given by the exponential of i times the charge of each excitation times the area of the loop it swept out, due to the magnetic flux enclosed by the loop. The remaining matrix or phase should depend only on the homotopy class of the loop in configuration space.

"Statistics" means the effect of exchange of identical excitations (other than the magnetic flux piece). The one-dimensional case, where the effect is a phase, is the usual case of Bose, Fermi or fractional statistics. The higher-dimensional (nonabelian) case is not nearly as familiar. (An early discussion of the general ideas involved is given in ref. [18].) Particles obeying fractional statistics are called "anyons", and those obeying nonabelian matrix statistics "nonabelions". For an elementary interchange of two excitations (one where the loop swept out does not enclose other excitations) we have [1], omitting any flux factor,

$$\psi_{p;i_1\dots i_s\dots i_r\dots i_n}(z_1\dots z_{i_s}\dots z_{i_r}\dots z_n)=\sum_q B_{pq}[i_1\dots i_n]\psi_{q;i_1\dots i_n}(z_1\dots z_n) \quad (2.1)$$

for the interchange of particles r, s $(r<s)$, where p, q label a basis in the vector space, and B is a matrix on the p, q labels, not on the quantum numbers (observables), $\{i_a;\ a=1,\dots,n\}$, which label the individual particles. Here we have defined the matrix B even for exchange of nonidentical excitations $(i_r \neq i_s)$. The braid group for a set of identical braids is generated by all possible elementary interchanges. The group for distinguishable braids may be defined as finally

bringing each object back to its original position and is called the pure braid group. We are interested in an intermediate case where the braids carry labels some of which may be identical. The symmetry group we want brings the braids back to their original positions up to permutations of braids with identical labels. The structure of the pure or intermediate braid group is best described in terms of the generators of the full braid group with some additional constraints. The matrices B furnish a representation of this full braid group. Their dependence on $\{i_a\}$ and p, q is specialized by imposing locality properties in both conformal field theory [1] and in particle statistics [2]; this will be discussed later. The operation of taking one excitation completely around another gives a matrix operation called monodromy which is given loosely speaking by B^2.

Another kind of operation on a wave function for a set of excitations is called "fusion" or "operator product". It is natural to make new excitations by bringing together two others. Two excitations close together should look like one excitation with quantum numbers given by some kind of "sum" of the two original excitations. Examples are charges, which add, and spins, where we must use the Clebsch–Gordan formulas. More generally, we write symbolically

$$\phi_j \times \phi_k = \sum_i N_{jk}^i \phi_i \,. \tag{2.2}$$

Here i runs over a set X including the "identity" which is the trivial excitation, i.e. its creation operator is 1. ϕ_j, ϕ_k represent two excitations of generic quantum numbers j, k and N_{jk}^i are integer coefficients. The wave function obtained by "fusing" ϕ_j, ϕ_k is a linear combination of wave functions with a single excitation at z. $N_{jk}^i \neq 0$ means that an excitation of type ϕ_i appears in the linear combination, $N_{jk}^i > 1$ that there is more than one way to fuse ϕ_j and ϕ_k to get ϕ_i (thus the space of resulting wave functions with ϕ_i at z potentially has dimension > 1, depending on the other excitations present). In the example where the quantum numbers are ordinary charge only, $N_{j,k}^i = \delta_{i,j+k}$ while for spin $N_{j,k}^i$ is 1 if spin i appears in the decomposition of the tensor product of spins j and k and zero otherwise (in general (2.2) is *not* tensor product). Clearly fusing of excitations should be commutative, so

$$N_{jk}^i = N_{kj}^i$$

and also the order of the successive fusions should not matter (associativity)

$$\phi_i \times (\phi_j \times \phi_k) = (\phi_i \times \phi_j) \times \phi_k \,,$$

$$\sum_l N_{il}^m N_{jk}^l = \sum_n N_{ij}^n N_{nk}^m \,.$$

The coefficients N^i_{jk} can therefore be regarded as matrix elements in an algebra of matrices which is commutative and associative; it is called Verlinde's algebra [19]. We have already assumed that two excitations close together "look like" their sum, and that braiding matrices depend on the homotopy class of the path. The latter implies that successive braidings obey the relations of the braid group (one of which has the form of the Yang–Baxter equation) while the former means that taking an excitation around a pair before or after fusing makes no difference. These consistency conditions are physically plausible because they rest on locality; the structure of the states and their B and N properties is "topological" and local because the underlying physics is local – we have a short-range hamiltonian and the basic correlations are short range (actually, the realistic hamiltonian has a $1/r$ Coulomb interaction; while not strictly short range, it is short range enough for most purposes, and replacing it by a short-range interaction should not affect the existence of incompressible fluid states). The consistency conditions place important constraints on the possible braid group representations. They are built in to conformal field theory [1] and also play a key role in the analysis of Fröhlich et al. [2]. In spatial dimensions greater than two the analogous conditions ultimately lead to Fermi or Bose statistics being the only acceptable permutation group representations, once other labels are identified as group theoretic in origin [20]. We will not try to prove that these conditions hold in general in the FQHE situation, though we conjecture that they do; instead they will appear automatically in our constructions based on conformal field theory.

We now explain very briefly where nontrivial braiding appears in conformal field theory (CFT). In a two-dimensional conformal field theory, we have correlation functions of fields $\phi_i(z, \bar{z})$ (where $z = x + iy$) which are real functions but may be decomposed as

$$\left\langle \prod_{a=1}^{n} \phi_{i_a}(z_a, \bar{z}_a) \right\rangle = \sum_p \left| \mathscr{F}_{p;i_1\dots i_n}(z_1, \dots, z_n) \right|^2,$$

where the conformal block functions $\mathscr{F}_p$ are holomorphic in their arguments and we have taken the "diagonal" case for simplicity. Here p labels members of a basis of functions $\mathscr{F}_{p;i_1\dots i_n}(z_1, \dots, z_n)$ which form a vector space for each n-tuple $(z_1, \dots, z_n)$ (more precisely, a vector bundle over the complex n-dimensional manifold with coordinates $(z_1, \dots, z_n)$). $\mathscr{F}_p$ transforms just like ψ_p in (2.1) [1]. Furthermore, fusion of operators $\phi_i(z)$ (obtained, heuristically, by factorizing $\phi_i(z, \bar{z}) = \phi_i(z)\bar{\phi}_i(\bar{z})$ in this diagonal theory) is defined exactly as above. Thus the natural correspondence that this suggests is that *holomorphic* wave functions of particle systems in two spatial dimensions might be conformal blocks of some conformal field theory in two-dimensional space-time. In the simpler of the examples to be discussed later (namely the so-called "parent" states), it is clear that the wave functions of excitations in FQHE systems vary holomorphically in the quasiparticle positions, but it is not so clear in general. We will, however, argue that wave functions can usually be brought to such a holomorphic form, or else can be built as holomorphic and antiholomorphic factors, each of which are conformal blocks.

In the case when the wave functions vary holomorphically the adiabatically-obtained vector potential (whether scalar or matrix) can be uniquely characterized as the unique connection compatible with both the induced inner product on the restricted space of states and the complex structure [21]. This suggests that the use of adiabatic transport could be eliminated and a more general formulation given which would not require the existence of energy gaps. These ideas should have very wide applicability in two-dimensional many-body systems.

2.2. ORDER PARAMETERS AND CHIRAL ALGEBRAS

The Laughlin state for N particles in the lowest Landau level, denoted $|0_L; N\rangle$ has coordinate representation in the symmetric gauge (we set the magnetic length to 1 throughout) [22]

$$\prod_{i<j}(z_i - z_j)^q \exp\left[-\tfrac{1}{4}\sum_i |z_i|^2\right] \tag{2.3}$$

(where now $i = 1, \ldots, N$) and has filling factor $\nu = 1/q$ with q an odd integer for fermions. This state has long-range order in the operator [4]

$$\Psi^\dagger(z) \equiv \psi^\dagger(z) U^q(z) \mathrm{e}^{-|z|^2/4},$$

where $\psi^\dagger(z)$ creates a particle in the lowest Landau level and $U(z)$ is a Laughlin's quasihole operator [22], in first quantization

$$U(z) = \prod_{i=1}^{N}(z_i - z_j). \tag{2.4}$$

Furthermore, the unnormalized Laughlin state can be written as a Bose condensate in $\Psi^\dagger$ [4]:

$$|0_L; N\rangle = \left(\int \mathrm{d}^2 z\, \Psi^\dagger(z)\right)^N |0\rangle, \tag{2.5}$$

where $|0\rangle$ is the vacuum state (no particles). By taking linear combinations of states $|0_L; N\rangle$ with different N one can obtain a state with a non-zero expectation value for $\Psi^\dagger(z)$. In ref. [4] a classical Ginzburg–Landau (GL) theory for $\langle\Psi\rangle$ was developed, which involves also a vector potential $\mathscr{A}$ identical to that obtained by adiabatic transport as discussed above, and a scalar potential $\mathscr{A}_0$. The action contains a Chern–Simons term which gives the vortex excitations fractional charge and statistics in agreement with direct calculations on the Laughlin states. While the explicit demonstration of long-range order is for the Laughlin state, and the derivation of the GL theory assumed such order but not a particular ground state [4,9,10], one expects the order to persist in the exact ground state at $\nu = 1/q$ of a hamiltonian more general than that for which Laughlin's state is exact, as long as the ground state is incompressible. We expect similar order parameters and GL theories to exist for all incompressible FQHE ground states. Such order parameter operators will add both charge and flux, in the ratio ν, as for the Laughlin state. The statistics of the operator, found from adiabatic transport and use of canonical anticommutators for the creation operators, must be bosonic so that these "pseudoparticles" can Bose condense in the FQHE ground state. The form of these operators for a few other states will be described later in this paper.

Three-dimensional gauge theories (abelian or nonabelian), but with the Chern–Simons (CS) term as the only term in the action, have also appeared recently as "topological field theories" [5] and have been shown to reproduce the braiding and fusing properties of corresponding two-dimensional CFTs. Given a quantum field theory version of the GL theory of the FQHE, it is reasonable to argue that, in the presence of quasiparticle (vortex) excitations, low-energy adiabatic transport can be represented by keeping only the CS term in the action, since all other terms vanish far from a quasiparticle. The quasiparticles then appear as charged sources whose worldlines are Wilson lines in the pure CS gauge theory [5], giving another view of the connection with CFT.

A quantum field theory with a CS term for the Laughlin state can be produced by a similar gauge transformation of the electron system [10]. (Note that the resulting scalar field is not quite the same as Ψ and does *not* have long-range order in the Laughlin state, but only algebraic order [9].) A heuristic long-wavelength field theory of the FQHE is then obtained by coarse-graining arguments. The power-lay decay of the order can be reproduced in this field theory [23]. It is possible to reproduce the Laughlin wave function itself by solving this field theory for a specific form of hamiltonian [24], but this derivation is not yet in our opinion fully convincing.

The role of the CS theory is to produce the appropriate Friedan–Shenker (FS) vector bundles [25] describing braiding of quasiparticles. Actual wave functions (or conformal blocks) are sections of these bundles. The existence of an order parameter in the Laughlin state means that destruction of an electron, i.e. creation of a hole, is equivalent to creation of q quasiholes of charge $-e/q$ [4]. Hence not only the statistical properties of quasihole states, but also wave functions in the electron coordinates themselves should be given by conformal blocks. This is one of the main points in this paper.

There are several ramifications of this point. First, if we wish to take a CFT and produce a FQHE system, we must demand that among the spectrum of primary fields is one that behaves as a fermion with the quantum numbers of the electron.

Second, we can produce a "representative" wave function, essentially by condensing the order parameter operator (built out of the primary field just mentioned) in a way analogous to (2.5). It is a "representative" of an infinite family of possible wave functions, all with identical braiding and fusing properties, because in a conformal field theory there are infinitely many "secondary" fields for each primary field [26]. The secondary fields are often called descendants of the primary fields. In particular, the Virasoro algebra is always present, allowing us to generate descendants which are essentially linear combinations of derivatives with respect to z of the correlations of the primary fields. We expect that this freedom can be used, for example, to move some of the zeroes of the wave function slightly away from the particles, where they are sitting in the Laughlin state (observe the multiple zeroes $z_i = z_j$ in eq. (2.3)). This is the expected form of a general incompressible state at $\nu = 1/q$ with the zeroes bound close to but not all exactly at $z_i = z_j$; it is responsible for the long-range order in Ψ. This freedom in the construction will be advantageous if we are ever to solve arbitrary hamiltonians in this framework. It is the basis for the universality of the braiding and fusing properties of the quasiparticle excitations. Each family is a "universality class"; universality classes are to be distinguished solely by the quantum numbers of the ground state and excitations, and by the braiding and fusing algebras; in other words by the corresponding CFT. In the literature there are many constructions of different "trial" states; it would be interesting to see which of these are actually in the same universality classes.

We do not know if all universality classes of FQHE behavior are given by some CFT or CSGT, but we conjecture that this is so. This question is related to other conjectures involving these objects [1], and an affirmative answer would open the way to a classification of FQHE systems.

A third ramification concerns the fully extended or chiral algebra of the CFT and by implication the FQHE system. In the fusion rule algebra (2.2) of a CFT, it may happen that some subset of fields ϕ_i form a closed subalgebra. If in addition, all other fields are local with respect to these fields (that is transport of one around the other produces a phase factor of 1) then all the fields can be regarded as falling into representations (multiplets) of an extended algebra, consisting of the above-mentioned subset as well as whatever algebra we started with (which always includes the Virasoro algebra). Thus fusion of a field ϕ_i with one of the extending fields produces a field $\phi_{i'}$ whose monodromy and fusing matrices with other fields are identical to those of ϕ_i by the consistency conditions mentioned above. All fields that differ only by fusion with extending fields can be put into the same multiplet of the extended algebra. The fusion algebra (2.2) can then be rewritten for the irreducible representations $[\phi_i]$ generated from ϕ_i by the extended algebra. Note that the extending fields themselves now appear in the "identity" representation. We usually prefer to work with algebras that are extended as far as they can be in this way. The best studied case, the case of rational conformal field theory

(RCFT) occurs when the CFT contains only a finite number of distinct representations of this chiral algebra, so the index set X is finite (we often let ϕ_i denote irreducible representations of the fully extended algebra from now on). An important example of CFT (rational or irrational) arises whenever a continuous symmetry is present; the symmetry is generated by integrals of current density fields, which generate a chiral algebra called an affine or Kac–Moody Lie algebra (or current algebra). The algebra contains a parameter k, called the level: rational CFTs with this algebra exist whenever k is itself rational (positive integral if unitarity is required). Another, very simple, example is the so-called rational torus. Suppose $i \in \mathbb{Z}$ denotes charge, so that there is a U(1) current algebra, the fusion algebra is $N_{jk}^{i} = \delta_{i,j+k}$, and that ϕ_{2N} is local with respect to the other ϕ_i. Then $\phi_{\pm 2N}$ are the extending fields, and representations of the full chiral algebra are fields $[\phi_i]$ where i and $i+2N$ are identified, so the fusion algebra is now given by addition modulo $2N$. The notation N for the level is conventional: usually it is assumed that N is integral, which makes ϕ_{2N} bosonic (no confusion should arise from the use of the same symbol N for both the level and the number of particles). More generally, RCFTs are usually defined as possessing only bosonic extending fields, whereas our definition in terms of locality allows also fermionic extending fields. In this case we could speak of a chiral superalgebra. This occurs in the rational torus for N half-integral and turns out to be relevant for the FQHE for electrons, for the following reasons.

There is a close relation between any order parameter in any FQHE system and the chiral algebra of the corresponding CFT. Consider a quasiparticle excitation of the FQHE ground state, at position z. Acting with the order parameter operator on this state at z' close to z, should according to the notion of an order parameter and the GL picture only produce some unimportant "collective", bosonic excitations of the state, which will not change the braiding and fusing of this composite object with other quasiparticles. Now the order parameter operator can be assumed to consist of one or more electron creation operators close to z', and one or more quasihole operators which (speaking globally) add flux but not charge or spin. One or more of the electron operators can be removed from the order parameter operator at z', because they merely create particles in gaussian wave packets which do not affect the structure of the many-particle state. Then the net operator applied to the quasiparticle state has (locally) non-zero charge and possibly spin and may be fermionic but still does not change the braiding and fusing properties in any significant way. Noting that this operator is itself a valid excitation, we see that it can be regarded as an extending field which should be included in the chiral algebra along with the charge density and spin density operators. We will see below that in the example of the Laughlin state at filling factor $\nu = 1/q$, this leads directly to the identification of the corresponding RCFT as the rational torus at level $N = q/2$. Thus the idea of a FQHE at rational filling ν, as characterized by the existence of one or more order parameter operators, corresponds directly with the idea of the chiral algebra in rational conformal field theory (the condition of a finite number of representations in the latter will be discussed further below). Therefore we may speak of the chiral algebra of the FQHE system. As a corollary, we note that if just one electron operator is removed from the order parameter in the above argument, we obtain an excitation with the quantum numbers and Fermi statistics of a hole, which lies in the chiral algebra. The chiral algebra is therefore in fact always a chiral superalgebra for the FQHE for electrons (for a FQHE in a system of charged bosons in a magnetic field, it could be simply a bosonic chiral algebra). From the construction of a "representative" ground state by condensing the order parameter as in eq. (2.5), destruction of any electron in a state containing quasiparticles is equivalent to acting with this same extending field, which is by definition local with respect to any quasiparticle, i.e. a single-valued function of $(z'-z)$. The fact that the hole operator lies in the chiral algebra therefore automatically implies that the electron wave function for a quasiparticle excitation is single valued in the electron coordinates, which is obviously a necessary constraint on such wave functions.

2.3. HIGHER-GENUS SURFACES AND EDGE STATES AT BOUNDARIES

It is of considerable interest to consider the FQHE in different geometries, such as a closed oriented two-dimensional manifold of genus g (a sphere with g handles) or manifolds with edges; the latter is of course the experimental situation. After Laughlin's work describing a disc of FQHE fluid in an infinite plane, Haldane [27] considered the sphere and Haldane and Rezayi [28] the torus. Our remarks up to now completely describe a homogeneous FQHE fluid filling the surface of a sphere. An interesting feature of the torus (genus 1) is that for a hamiltonian that is translationally invariant on the surface (which can be represented as a parallelogram with periodic boundary conditions), all states in the system are q-fold degenerate at filling factor $\nu=p/q$, for any finite system, irrespective of the nature of the ground state [29]. Here ν is the number of electrons divided by the number of flux quanta piercing the surface. This degeneracy is found explicitly for Laughlin's ground state at $\nu=1/q$ on the torus [28]. In general, the degeneracy of the ground state in the thermodynamic limit could be larger (though still a multiple of q), because some energy gaps might tend to zero in this limit.

Understanding the nature of the degeneracy on a torus is of interest in connection with topological approaches to explaining the quantization of Hall conductance in the presence of a random background potential; these approaches require a q-fold degeneracy on the torus, in the thermodynamic limit [30]. Other approaches to the quantization issue involve the existence of gapless "edge states" localized along the boundary of a system [31], which seems to be related to degeneracy on closed manifolds. In the integer quantum Hall effect, that is, for non-interacting electrons, these edge states are well known [32], but they are much less well understood in the FQHE case where interactions are involved. Wen and Niu [13] have discussed these questions in a similar framework to ours below, though in less generality.

Both of these questions have analogues in RCFT and CSGT. While the zero-point function or conformal block of a system on a sphere is nondegenerate, on higher-genus surfaces it is a member of a vector space analogous to that for the n-point functions on the sphere (it is really a vector bundle over the moduli space of Riemann surfaces, but we will not need this). Conformal blocks for n operators ϕ_i feel the effects of these nontrivial backgrounds also. In a beautiful paper, Verlinde [19] has obtained simple formulas for the dimensions of the conformal blocks of the vacuum on a genus-g surface, by arguments that relate them to braiding and fusing as discussed above. A very simple picture of the degeneracy is as follows. If in a FQHE system we have a quasiparticle (or in CFT, an insertion of a primary field) ϕ_i at z, then we can picture it as a source of a fictitious, generalised "electric flux" which is divergenceless except at the quasiparticles. On a surface of genus greater than 0, we can have electric flux flowing round the nontrivial loops on the surface. The flux can be changed by creating a quasiparticle–antiquasiparticle pair, transporting the quasiparticle around a nontrivial cycle on the surface and then reannihilating the pair. This leaves an extra flux along the path of the quasiparticle. This defines Verlinde's operators. Note that addition of flux is given by the fusion rules (2.2), and that the operators for the two loops on each handle do not commute because changing their order is equivalent to taking one quasiparticle around the other. Hence the operators for at most one direction per handle can be diagonalized.

In RCFT, a consequence of Verlinde's arguments is that for a torus, the degeneracy k, say, is equal to the number of distinct primary fields of the chiral algebra, which is finite by definition in a rational theory. For higher genus, the degeneracy grows as k^g only if the monodromy is abelian. When nonabelions are present, the formula becomes more complicated. In the FQHE, it seems reasonable to assume the ground-state generacy is finite in the apparent absence of any spontaneously broken symmetry, and so we expect only a finite number of primary fields (or types of quasiparticles, modulo extending fields) to be present. Thus we are conjecturing that the relationship of the FQHE and RCFT extends also to all the properties of higher genus surfaces. Indeed, it seems to us highly likely that a FQHE system should define a "modular functor" or a "modular tensor category" (see ref. [1]; actually what we really require here is an extension of these concepts to chiral superalgebras as opposed to algebras).

Turning to gapless edge states, it is interesting to note Witten's construction [5] of CS gauge theory on a space in the shape of a disc (see also refs. [1,6]). The edge then forms a (1 + 1)-dimensional conformal field theory, i.e. there are physical gapless excitations. The CFT obtained at the edge is the same one that is related to the braiding and fusing properties in the interior of the disc. This leads us to conjecture that the same holds in the FQHE: the edge states are described physically by the same RCFT that describes the bulk mathematically. However, we will not elaborate further on this point in this paper.

In summary, we have made a number of conjectures which unify many aspects of the FQHE and suggest a classification of the kinematically allowed states through the corresponding RCFTs. In the following sections we will attempt to clarify and justify these ideas by showing that they are true in a number of important known FQHE systems, and that examples of nonabelions can be constructed.

3. Electron wave functions as conformal blocks: Laughlin states and the hierarchy

Let us return to the Laughlin state in the disc geometry:

$$\Psi_{\text{Laughlin}}(z_1,\dots,z_N)=\prod_{i<j}(z_i-z_j)^q \exp\left[-\tfrac{1}{4}\sum|z_i|^2\right], \tag{3.1}$$

where q is an odd integer [3]. In the thermodynamic limit this state $|0_L;N\rangle$ describes a fluid ground state with a uniform number density $\rho_0\equiv\nu/2\pi=1/2\pi q$ inside a radius of order $\sqrt{2N}$. The GL description of this limit for a normalized fluid state $|\alpha\rangle$ of slowly varying density involves a gauge field

$$i\mathscr{A}(z)\sim\int\frac{\langle\alpha|\psi^\dagger\psi(z')|\alpha\rangle}{z-z'}\,\mathrm{d}^2z'. \tag{3.2}$$

In the GL description [4] this gauge field couples to the order parameter (which has charge q; we set the charge of the electron to 1 from now on) and also enters with a Chern–Simons term

$$\frac{q}{4\pi}\int\mathscr{A}\,\mathrm{d}\mathscr{A} \tag{3.3}$$

in the action. If we are interested primarily in statistics of excitations we may expect such topological terms in the action to play a dominant role – since they dominate all other terms at long distances and low energies. On the other hand, it is now well known that CSW theory (i.e. (2 + 1)-dimensional gauge theory with only a CS term in the action) for an abelian gauge field is closely connected to the (1 + 1)-dimensional conformal field theory known as the "rational torus" [1,5]. The rational torus theory is characterized by a "level" N and is denoted by $U(1)_N$*. The level N can be determined in terms of q by comparing the abelian

* See ref. [1] sect. 10 for an explanation of the notation.

representations of $\mathscr{B}_n$ which arise in the FQHE system to those defined by the braiding of Wilson lines in the CSW system, as we have discussed in sect. 2. We will return to this below. However, we believe that in fact a stronger statement can be made, namely that one can expect the actual conformal blocks of the rational torus theory to be related to the electron wave functions of the FQHE system. The reason for this is that both the ground state of the FQHE system and the quantum state obtained in hamiltonian quantization of the CSW theory can be characterized by a condensation of operators associated with singular gauge transformations, as we now explain.

The Landau–Ginzburg action describing the fractional quantum Hall system is derived from the condensation of the order parameter [4]

$$\Psi^{\dagger}(z) = \psi^{\dagger}(z)U^{q}(z)\mathrm{e}^{-|z|^2/4}, \tag{3.4}$$

where ψ is the electron field and U is the quasihole creation operator (see ref. [3] and eq. (2.4)). U can be regarded as the "holomorphic version" of the insertion of a flux quantum, that is, of a gauge transformation which is singular at z. The holomorphic version of the transformation softens the singularity to a simple zero of the wave function of each electron, so that no physical flux is actually added at z and the operator U might be better described as creating a vortex at z (but note that, on a compact surface, U *does* add a net flux quantum [27]).

On the other hand, from the CSW point of view an analogous operation U can be defined by the effect of a singular gauge transformation $g(\theta) = \mathrm{e}^{i\theta}$, where θ is an angular coordinate centered around the point z. Since the Chern–Simons term is invariant only up to boundary contributions, making such a gauge transformation in a path integral with action (3.3) corresponds to the insertion of a Wilson line [6]. Thus, the "ground state" (i.e. the physical gauge invariant state in the CSW Hilbert space) must be unaffected by gauge transformations $g(\theta) = \mathrm{e}^{iq\theta}$. We may say that in the physical state there is a "condensation of Wilson lines," and precisely this condensation is the explanation, from the $(2+1)$-dimensional perspective, of the rationality of the rational torus [6].

Alternatively, starting from the Ginzburg–Landau theory in the formulation of ref. [10] we may attempt to quantize the theory directly (progress in this direction has been made recently by Girvin et al. [24]) by treating the scalar field in first quantization as a collection of N Bose particles, coupled to the fictitious gauge field. The bosons play the rôle of Witten's Wilson lines in pure CS gauge theory [5], and when the gauge field has been integrated out the hamiltonian for the bosons contains covariant derivatives which are the same ones that appear in the U(1) Wess–Zumino model [5,33]. The remaining problem of quantizing the Bose field (or finding the N-particle wave function) is therefore a problem of finding the appropriate section of the FS line bundle, again suggesting the connection with the two-dimensional rational torus CFT.

We may therefore postulate that a "representative" ground state is described by the thermodynamic limit ($N \to \infty$) of the conformal-field-theoretic correlator

$$\Psi_{\text{Laughlin}} = \left\langle \prod_{i=1}^{N} e^{i\sqrt{q}\phi(z_i)} \exp\left[-i \int d^2z' \sqrt{q}\,\rho_0 \phi(z') \right] \right\rangle, \tag{3.5}$$

where $\rho_0 = 1/2\pi q$ is the electron density in the ground state, ϕ is a free massless scalar field in two dimensions

$$\langle \phi(z)\phi(w) \rangle = -\log(z-w), \tag{3.6}$$

and the exponentials are normal ordered as usual. Use of eq. (3.6) in eq. (3.5) indeed reproduces (3.1) in the thermodynamic limit, after a certain gauge transformation has been made. The exponentials in eq. (3.5) represent (two-dimensional, holomorphic) Coulomb charges and so the last factor is needed to ensure charge neutrality; the integral is taken over a disc centered at the origin with radius chosen to satisfy this condition. The self-interaction of this background charge is neglected. The interaction of the $\sqrt{q}$ charges with this factor produces the exponential of a sum of singular integrals of the form

$$-q\rho_0 \int d^2z' \log(z_i - z') \tag{3.7}$$

which must be handled with care. The real part of this integral produces the nonholomorphic gaussian factors in (3.1) (apart from edge effects in the finite system); this of course ensures that the electrons are in lowest Landau level single-particle states. The imaginary part is ill defined because as a function of z it has a branch point at each point in the integration region (see a related discussion in ref. [34]) and arises for the following reason. The expression (3.5) is trying to give us the answer in a gauge where the vector potential is zero, which means it differs by an everywhere-singular gauge transformation from the usual symmetric gauge vector potential for the uniform background magnetic field. The gauge transformation in question has just the form of the exponential of the singular imaginary part of (3.7). We can use this gauge transformation to remove the ill-defined phase and obtain the symmetric gauge Laughlin state as desired. In the following this step will usually be left implicit, and we will often ignore the exponential factor in the wave function and refer to it as holomorphic.

Note that the charge current of the rational torus, i.e. $J(z) = (i/\sqrt{q})\partial\phi(z)$ attributes the correct physical charge to each electron. The large uniformly distributed background charge in (3.5) thus has the physical meaning of the background magnetic field, in view of the proportionality of charge and flux in CS theory. It is amusing to note that the Coulomb gas correlator (3.5) is just the holomorphic version of Laughlin's original mapping of the modulus squared of his wave function to a one-component plasma!

It is important to realize that we are not insisting that (3.1) is the only possible FQHE ground state. In fact, in eq. (3.5) we could have replaced the Coulomb operators, which are charged extending fields of the rational torus chiral algebra, by any linear combination of their descendant fields (the same for each particle i by symmetry) and obtained the same statistical properties of the quasiparticles. The possible descendants include Virasoro secondary fields, which are linear combinations of derivatives of the primary field and so will lead to wave functions in which the zeroes are not all fixed to the particles as they are in Laughlin states. Far from being a special property of Laughlin states, the relationship with CFT and CSGT is a general feature of the physics of the FQHE systems. Laughlin's state, or the expression (3.5), is just a "representative" of a whole "universality class" of states with equivalent braiding (statistics) properties. It is remarkable that these wave functions can be recovered from the GL–CSW formulation.

It was pointed out by Haldane that Ψ_{Laughlin} is in fact the unique incompressible ground state for a special hamiltonian with an interaction involving suitable short-range "pseudopotential" interactions [27,35]. These interactions may be given a conformal field-theoretic interpretation which will prove quite useful in a later section. Haldane's hamiltonian is of the form

$$\mathscr{H} = \sum_i \frac{1}{2m} |-i\nabla_i - eA(z_i)|^2 + \sum_{l=0}^{q-1} V_l \left\{ \sum_{i<j} P_l^{ij} \right\}, \tag{3.8}$$

where V_l are positive constants (the pseudopotentials) and P_l^{ij} is the projection onto the relative angular momentum state of angular momentum l for particles i, j. Mathematically, the operator P_l^{ij} acting on the function $f(z_1, \ldots, z_N)$ is obtained by first expanding f in powers of $z_i - z_j$ in the variables $z_i = \frac{1}{2}(z_i + z_j) + \frac{1}{2}(z_i - z_j)$ and $z_j = \frac{1}{2}(z_i + z_j) - \frac{1}{2}(z_i - z_j)$ about their common center of mass $z[ij] \equiv \frac{1}{2}(z_i + z_j)$. P_l^{ij} then projects onto the lth term of this expansion. If we interpret a function of the z_i as a correlation function, i.e. $f(z_1, \ldots, z_N) \sim \langle \Phi(z_1) \ldots \Phi(z_N) \rangle$, then this expansion is simply the operator product expansion, so the relative angular momentum projection operators are simply the operators extracting certain terms from the operator product expansion. In particular, since

$$e^{i\sqrt{q}\phi(z)} e^{i\sqrt{q}\phi(w)} \sim (z-w)^q e^{2i\sqrt{q}(\frac{1}{2}(z+w))} + \ldots \tag{3.9}$$

begins at a relative angular momentum of q, the Laughlin state is a zero-energy eigenstate of (3.8) and since it is the densest such eigenstate it is incompressible.

We now consider the conformal field theoretic interpretation of excitations about Laughlin's state. Quasihole excitations about Laughlin's state are created by insertion of a flux quantum, i.e. by the operator U (in the GL picture, they are vortices). As explained above, this operation has an analogue in the Chern–Simons theory, namely, the action of a singular gauge transformation $g(\theta) = e^{i\theta}$, where θ is an angular coordinate centered on some point z. In the CSW theory we insert a Wilson line, and the corresponding operation in RCFT is the insertion of the vertex operator $e^{i\phi(z)/\sqrt{q}}$ [6]. This operator is the primary field which generates the basic nontrivial representation of the rational torus chiral algebra, and from an expression similar to (3.9) its operator products also generate the other primary fields of the rational torus at level $N = q/2$, which are $e^{ir\phi(z)/\sqrt{q}}$, where $r = 0, 1, \dots, q-1$ ($r = q$ gives the extending field). Thus, according to the hypothesis that conformal blocks and electron wave functions should be identified, we expect the electron wave function for a quasihole to be

$$\Psi_{\text{quasihole}}(z_1, \dots, z_N; w) = \left\langle e^{i\phi(w)/\sqrt{q}} \prod_{i=1}^{N} e^{i\sqrt{q}\phi(z_i)} \exp\left[-i\int d^2z' \sqrt{q}\,\rho_0 \phi(z')\right] \right\rangle$$

$$= \prod_i (z_i - w) \prod_{i<j} (z_i - z_j)^q \exp\left[-\tfrac{1}{4}\sum |z_i|^2 - \frac{1}{4q}|w|^2\right], \tag{3.10}$$

which is indeed Laughlin's quasihole wave function. As an alternative to the usual charge counting or adiabatic methods of determining the charge of the quasihole, we may note that from the operator product expansion

$$J(z) e^{i\phi(w)/\sqrt{q}} \sim \frac{1/q}{z-w} e^{i\phi(w)/\sqrt{q}} + \dots \tag{3.11}$$

we learn that the charge of the quasihole is $-1/q$ as it should be. (The extra minus sign may be understood in CFT language as follows. If we wish to measure the charge of a quasiparticle we must surround the particle by a line integral of the current. Deforming the line integral away from the quasiparticle we pick up a contribution N from the electron operators $e^{i\sqrt{q}\phi(z)}$ but we also pick up charge $-(N + 1/q)$ from the neutralizing background.) The wave function for several quasiholes is obtained by inserting several of the quasihole vertex operators:

$$\Psi \sim \left\langle \sum_{j=1}^{M} e^{i\phi(w_j)/\sqrt{q}} \prod_{i=1}^{N} e^{i\sqrt{q}\phi(z_i)} \exp\left[-i\int d^2z' \sqrt{q}\,\rho_0 \phi(z')\right] \right\rangle. \tag{3.12}$$

Note these electron wave functions contain the correct prefactor, as a function of the w_i, to describe the fractional statistics of quasihole wave functions – in other words our procedure naturally gives the "fractional statistics gauge" [7, 36]. The statistics of the quasiholes is $\theta/\pi = 1/q$ as is well known. Thus, we can regard

(3.12) as an electron wave function parametrized by the w_i, or we can regard it as the electron coordinate representation of a coherent state in the M quasihole sector of excited states, in the sense of sect. 2. We know from refs. [7,36] that (3.12) gives an acceptable quasihole wave function and so we extend our hypothesis about the equality of wave functions and conformal blocks to the case of quasiparticles. Of course, this is extremely natural from both the GL and CSW points of view.

The identification of the level of the rational torus as $q/2$ is completed by noting that (i) q quasiholes close together have the same charge and statistics as a real hole; (ii) the charge of the quasiholes was quantized in units of $1/q$ because any wave function must be single valued in the electron coordinates, so the q quasihole composite has trivial monodromy with all other excitations; (iii) the latter property coincides with the definition of any field in the fully extended chiral algebras; (iv) these chiral and fusion algebras are exactly those of the rational torus at level $q/2$. Furthermore, the q quasihole composite, together with the creation operator for an electron, is the order parameter $\Psi^\dagger$ for the Laughlin states (see ref. [4] and sect. 2). Here we see clearly the relation between the order parameter and the extending fields, which was already discussed in sect. 2. Some interesting issues are raised by the correspondence since the level N of the rational torus is half odd-integral: $N=q/2$. Usually it is required that the rational torus be well defined on an arbitrary Riemann surface without the introduction of any additional mathematical structure, and this forces N to be an integer [6]. However, it appears that if additional structure, such as a spin structure, *is* provided, then half-integer level can be defined [37]. In the present case, there is no physical reason why the FQHE system should not be defined on surfaces of arbitrary genus, and so the requisite mathematical structure must arise naturally. Thus, a complete understanding of fractional Hall systems in arbitrary topology will involve some notion of "spin theories" along the lines described in ref. [37]. It is illuminating to note that the rational torus with $N=\frac{1}{2}$ describes a non-interacting massless right-moving Dirac fermion (without spin projection) while the case $N=\frac{3}{2}$ has "$N=2$ supersymmetry".

The point of view advocated in this section can be extended to study states on the torus or on even higher-genus Riemann surfaces. The appropriate technology has been well developed in the literature on conformal field theory. For example, interpreting wave functions as conformal blocks allows one to reinterpret some of the results of ref. [28]. The q-fold degeneracy of all states for filling factor $\nu=1/q$ on the torus [29] corresponds to the q distinct representations of the rational torus chiral algebra which can flow around the torus; this result follows just from the fusion rules and the statistics of the quasiholes, following Verlinde [19] (see also ref. [13]).

We should also make some remarks about wave functions for quasi-electrons (i.e. quasiparticles of *positive* charge in our units). These have always presented more difficulty than quasiholes [36]. Since our approach emphasizes holomorphy of wave functions, and since also in the rational torus point of view we regard the charge of quasiholes as defined mod 1 because of the extending field, it seems most natural to make a quasielectron of charge $1/q$ from $q-1$ quasiholes plus an electron creation operator, all located at z'. In other words, it differs from $q-1$ quasiholes by one electron, which is equivalent to the conjugate of the extending field as defined above. This wave function is holomorphic in its dependence on z' through the electrons other than the one localized at its center, which is created in the gaussian (coherent state) packet

$$p(z,\bar{z}') = \frac{1}{2\pi}\exp\left[\tfrac{1}{2}z\bar{z}' - \tfrac{1}{4}|z|^2 - \tfrac{1}{4}|z'|^2\right]$$

centred at z' in which this function is antiholomorphic (if we ignore the gaussian factors); here z represents the coordinate of the added electron itself. This form ensures that the wave function is sufficiently close to being holomorphic in z' for the purpose of adiabatic transport, etc. It is easy to see that this quasi-electron has the correct statistics, bearing in mind that the extending field is fermionic (the rational torus chiral algebra is a superalgebra for half-integral level). From the point of view of CFT, another natural choice would be $e^{-i\phi(z)/\sqrt{q}}$. The use of such alternative forms is expected to make no essential difference.

The above picture of quasihole wave functions is well suited to describing the hierarchy [7, 27]. Starting from an electron wave function with electrons at $z_i^{(0)}$ we may use (3.12) to produce the wave function for a gas of quasiholes at $z_i^{(1)}$. We can then make a new *electron* wave function by projecting the quasiholes into a Laughlin-type state. We do this by taking an inner product, in the quasihole coordinates, between the quasihole gas state and a Laughlin-type state for the quasiholes (neutralizing backgrounds are omitted):

$$\Psi^{(1)}\left(z_1^{(0)},\dots,z_{N_0}^{(0)}\right) = \int \sum_1^{N_1} \mathrm{d}^2 z_i^{(1)} \left\langle \prod_1^{N_1} e^{i\sqrt{2p_2+1/m}\,\phi(\bar{z}_j^{(1)})} \right\rangle_L \times \left\langle \prod_{j=1}^{N_1} e^{i\phi(z_j^{(1)})/\sqrt{m}} \prod_{i=1}^{N_0} e^{i\sqrt{m}\,\phi(z_i^{(0)})} \right\rangle_R . \tag{3.13}$$

By locality – i.e in order that the integrand is single valued – p_2 is determined to be an integer*. The filling fraction is determined by the largest power of $z^{(0)}$ and is seen to be

$$\nu = \frac{1}{m+1/2p_2} \equiv \frac{p}{q}.$$

* The subscripts R, L above refer to right- and left-movers, i.e. to holomorphic and antiholomorphic contributions to a correlation function, and are meant to emphasize the analogy to similar inner products which arise in string theory.

Quasihole excitations of the left-moving part of this state are obtained by insertion of

$$\exp\left[\frac{i}{\sqrt{2p_2+1/m}}\phi(\bar{z})\right]$$

in the left-moving piece of the above wave function, and are quasi-electrons from the electron point of view, i.e. they have charge $1/q$ [7,27] in the original units. Evidently, these excitations have statistics $\theta/\pi = -m/(2p_2m+1)$ (the minus sign comes from the dependence on $\bar{z}$ rather than z). To continue building the hierarchy we now make a gas of these new excitations and project their wave function onto a Laughlin-type state. This requires introducing a new holomorphic scalar field. In this way we can produce a sequence of states $\Psi^{(k)}$. For example the state $\Psi^{(2s-1)}$ is given by

$$\begin{aligned}
&\Psi^{(2s-1)}\left(z_1^{(0)},\dots,z_{N_0}^{(0)}\right)\\
&\quad=\int\prod_{l=1}^{2s-1}\sum_{i=1}^{N_l}\mathrm{d}^2z_i^{(l)}\left\langle\prod_{j=1}^{N_1}\exp\left[\frac{i}{\sqrt{m_1}}\phi\left(z_j^{(1)}\right)\right]\prod_{i=1}^{N_0}\exp\left[i\sqrt{m_1}\,\phi\left(z_i^{(0)}\right)\right]\right\rangle\\
&\quad\times\left\langle\prod_{j=1}^{N_2}\exp\left[\frac{i}{\sqrt{m_2}}\phi\left(\bar{z}_j^{(2)}\right)\right]\prod_{i=1}^{N_1}\exp\left[i\sqrt{m_2}\,\phi\left(\bar{z}_i^{(1)}\right)\right]\right\rangle\\
&\quad\times\left\langle\prod_{j=1}^{N_3}\exp\left[\frac{i}{\sqrt{m_3}}\phi\left(z_j^{(3)}\right)\right]\prod_{i=1}^{N_2}\exp\left[i\sqrt{m_3}\,\phi\left(z_i^{(2)}\right)\right]\right\rangle\\
&\quad\dots\\
&\quad\times\left\langle\prod_{j=1}^{N_{2s-1}}\exp\left[\frac{i}{\sqrt{m_{2s-1}}}\phi\left(z_j^{(2s-1)}\right)\right]\prod_{i=1}^{N_{2s-2}}\exp\left[i\sqrt{m_{2s-1}}\,\phi\left(z_i^{(2s-2)}\right)\right]\right\rangle\\
&\quad\times\left\langle\prod_{i=1}^{N_{2s-1}}\exp\left[i\sqrt{m_{2s}}\,\phi\left(\bar{z}_i^{(2s-1)}\right)\right]\right\rangle, \qquad (3.14)
\end{aligned}$$

where $m_{j+1} = 2p_{j+1} = 1/m_j$, $m_1 = m$ and $N_0 = N$. From the above one may obtain the usual continued fraction expansion for ν [7,27]:

$$\nu = \cfrac{1}{m+\cfrac{1}{2p_2+\dots+\cfrac{1}{2p_{2s-1}+\cfrac{1}{2p_{2s}}}}} \equiv \frac{p}{q}. \qquad (3.15)$$

Quasiholes at the last level are made by inserting

$$\exp\left[\frac{i}{\sqrt{m_{2s+1}}}\phi(\bar{z})\right].$$

Similar expressions hold for an odd number of levels, giving $\Psi^{(2s)}$. Also, quasi-electrons can be used in place of quasiholes at any level and similar results obtained; we will not give explicit results for this case. These constructions precisely reproduce Halperin's version of the hierarchy [7].

The following facts may be established from the above formulas. First, when the continued fraction (3.15) is multiplied up one finds that p and q have no common factors and that q is odd; every such fraction is obtained once only in this way [7,27,35]. If n is the number of levels ($n=2s$ in eq. (3.14)), then p has the same parity as n. Second, a quasihole at the last level has charge $(-1)^n/q$ in electronic units. This object has statistics

$$\frac{\theta}{\pi}=\frac{(-1)^{n-1}}{2p_n+\cfrac{1}{2p_{n-1}+\ldots+\cfrac{1}{2p_2+\cfrac{1}{m}}}}\equiv\frac{p'}{q'}. \tag{3.16}$$

(This expression can also be obtained by generalizing the method of ref. [8] to the hierarchy wave functions.) Then, using elementary results on continued fractions [38], one can prove that $q'=q$ and p' are odd and p' satisfies

$$pp'\equiv 1 \qquad (\mathrm{mod}\ q). \tag{3.17}$$

These conditions clearly fix p' (mod $2q$) and hence θ (mod 2π). Consequences of (3.17) are that a cluster of q quasiholes has total charge -1 and statistics $\theta/\pi=q^2p'/q\equiv 1$ (mod 2), the same as an ordinary hole, while a cluster of p quasiholes has charge $-p/q$ and statistics $\theta/\pi=p^2p'/q\equiv p/q$ (mod 2), the same [8] as a Laughlin quasihole $U(z)$ (a "single flux") acting on this state. (These conclusions were stated by Su [39] though our formula (3.17) has apparently not appeared in the literature previously.) If one *assumes* that these identifications hold and that θ/π has denominator q, then one is led to eq. (3.17) [39].

It should also be possible to extend these ideas to the study of the hierarchy states on the torus.

In summary, we have shown in this section that Laughlin's ground state and quasiparticle excitation wave functions, and their hierarchical extensions, can be interpreted as conformal blocks in certain CFTs, in accordance with the idea that such a connection should exist because of the Ginzburg–Landau–Chern–Simons theorv of the FQHE [4, 9, 10] and the Chern–Simons–CFT relationship [1, 5].

4. Spin-singlet states

4.1. HALPERIN STATE

It is commonly supposed that the spin of the electron is irrelevant in the quantum Hall effect since the large magnetic field polarizes the electrons. As pointed out in ref. [40] this is not necessarily the case at low magnetic fields. Since the effective mass of the electron is only $\sim 7/100$ of the true mass and since the effective g factor is only $\sim 1/4$, the ratio of Zeeman to cyclotron energies is $\sim 7/400$ and the electron spin can be important even in the lowest Landau level. Halperin [40] has proposed a spin-singlet state for electrons with spin up located at $z_i^{\uparrow}$ and spin down located at $z_i^{\downarrow}$:

$$\Psi_{\text{Halperin}}\left(z_1^{\uparrow},\dots,z_{N/2}^{\uparrow},z_1^{\downarrow},\dots,z_{N/2}^{\downarrow}\right)$$
$$= \prod_{i<j}\left(z_i^{\uparrow}-z_j^{\uparrow}\right)^{n+1}\prod_{i<j}\left(z_i^{\downarrow}-z_j^{\downarrow}\right)^{n+1}\prod_{i<j}\left(z_i^{\uparrow}-z_j^{\downarrow}\right)^{n}\exp\left[-\tfrac{1}{4}\sum_i\left(|z_i^{\uparrow}|^2+|z_i^{\downarrow}|^2\right)\right], \tag{4.1}$$

where $n \geqslant 0$ must be an even integer to satisfy the Pauli principle. Here the filling factor is $\nu = 2/(2n+1)$. To see that this state is singlet, we may first consider the case $n=0$, which represents a Landau level filled with electrons of both spins, and clearly must be a singlet. The general case is obtained by multiplying by $\prod(z_i - z_j)^2$ (where the product runs over all pairs) which is totally symmetric and spin independent and so leaves the state a singlet. It is possible that the particle-hole conjugate of this state, with $\nu = 8/5$, has been seen in recent experiments [41].

An analogous discussion to that of sect. 3 can be carried out regarding order parameters and effective gauge fields (see also ref. [42]). If we do not demand manifest SU(2) (spin rotation) symmetry, this can be done straightforwardly. Introduce an operator

$$U_\sigma(z) = \prod_{i=1}^{N/2}\left(z_i^{\sigma}-z\right) \tag{4.2}$$

which acting on Ψ_{Halperin} creates a quasihole of charge $-1/(2n+1)$ and spin $\frac{1}{2}$; this is the basic quasihole of the Halperin state [35] and has statistics $\theta/\pi = (n+1)/(2n+1)$. Then define the operator

$$\Psi_\sigma^{\dagger}(z) = \psi_\sigma^{\dagger}(z)U_\sigma^{n+1}U_{-\sigma}^{n}(z)\mathrm{e}^{-|z|^2/4}\,\mathrm{e}^{i\pi S_z}, \tag{4.3}$$

where $\psi_\sigma^\dagger$ is the creation operator for an electron of spin σ, and the factor involving the total z-component of spin S_z is a "cocycle" needed to make

$$\left[\int \mathrm{d}^2 z\, \Psi_\sigma^\dagger(z), \int \mathrm{d}^2 z'\, \Psi_{\sigma'}^\dagger(z')\right] = 0$$

for all σ, σ'. Ψ_σ condenses in the Halperin state:

$$|0_\mathrm{H}; N\rangle = \left(\int \mathrm{d}^2 z^\uparrow\, \mathrm{d}^2 z^\downarrow\, \Psi_\uparrow^\dagger(z^\uparrow)\Psi_\downarrow^\dagger(z^\downarrow)\right)^{N/2} |0\rangle. \tag{4.4}$$

Note that the integrations over z for $\uparrow$ and $\downarrow$ spin particles are completely independent and the operators can be re-ordered arbitrarily. We have here a two-component order parameter $\langle \Psi_\sigma \rangle$ with independent phases for the two spin directions (the reader is cautioned again that this approach is not manifestly SU(2) invariant, so arbitrary unitary rotations of this spinor order parameter cannot be made). A nonzero expectation is of course only obtained in a state with indefinite charge and S_z in this case. The GL action will contain the order parameter and a two-component gauge field $\mathscr{A}_\sigma$; the latter is best treated by taking + and − components with respect to S_z to obtain a gauge field $\mathscr{A}$ coupling to U(1) charge as in the spin-polarized (charge only) case and a field $\mathscr{A}_\mathrm{s}$ which couples to S_z only. In terms of corresponding CFTs, this is clearly very similar to the treatment of Dirac fermions with (iso-)spin *via* abelian bosonization, while the previous section is like the spinless case. In each case, this becomes an exact correspondence in the case of maximal filling ($\nu = 2$ or 1 respectively) when the quasihole statistics reduce to those of fermions, $\theta = \pi$. This immediately suggests that the Halperin state can be reproduced from CFT correlators like those in eq. (3.5) but using one scalar field ϕ for charge and another ϕ_s for S_z; each electron of course carries both quantum numbers. This calculation is left as an exercise for the reader; we need only remark that the CFT involved is just the direct product of a rational torus with $N = 2n + 1$ for ϕ and $N = 1$ for ϕ_s, and in terms of the basic fields $\exp(ir\phi/\sqrt{2N})$, where $r = 0, 1, \dots, 2N - 1$, the electrons are represented by $r = N = 2n + 1$ for the charge part and $r = 1$ for the spin. Quasiholes must be single valued in the electron coordinates, which leads to the rule that their rational torus representatives must have r in the charge part even for integer spin objects and odd for half integer spin (thus the sum of the r's for spin and charge must be even). Spin and charge have therefore *not* separated in these excitations; the physical charge in our units is $-r/(2n + 1)$. Put another way, the electron representative above extends the algebra (in accordance with the general argument of sect. 2) to a chiral superalgebra, and only the members of this subset are representations of this algebra. (For the charge-only case of sect. 3, the level $N = q/2$ system can be thought of similarly as an extension of a level $N = 2q$ rational torus algebra.) There are q such representations, and so the degeneracy on the torus will again be only q-fold, the minimum it can be according to Haldane [29].

Although this representation is very simple and useful for calculations, it is still desirable to have a manifestly SU(2) invariant formulation. Unfortunately, a direct quantum mechanical formulation (in terms of operators on the electrons) of the order parameter and nonabelian SU(2) gauge field, whose GL theory will contain the nonabelian Chern–Simons term, has so far eluded us. Here we will only write the electron wave functions as the conformal blocks which must result from such a formulation (this uses the work of Witten [5] relating the nonabelian CSGT to nonabelian current algebra), and complete the identification of the RCFT. (Another interesting approach is given in ref. [43].)

We may represent the Halperin state in terms of conformal blocks using the $k=1$ SU(2) WZW model together with the rational torus at level $2n+1$ as follows. Let $V^{\pm}(z)$ be the primary spin-$\frac{1}{2}$ multiplet in the WZW theory, then we have

$$\Psi_{\text{Halperin}} = \left\langle V^+(z_1^{\uparrow})\dots V^+(z_{N/2}^{\uparrow})V^-(z_1^{\downarrow})\dots V^-(z_{N/2}^{\downarrow})\right\rangle_{\mathrm{SU}(2)_{k=1}}$$

$$\times\left\langle \prod_{i=1}^{N}\exp\left[i\sqrt{n+\tfrac{1}{2}}\,\phi(z_i)\right]\exp\left[-i\int \mathrm{d}^2z'\sqrt{n+\tfrac{1}{2}}\,\rho_0\phi(z')\right]\right\rangle, \quad (4.5)$$

where $\rho_0 = 2/2\pi(2n+1)$. The charge current is given by

$$J(z) = \frac{i}{\sqrt{n+\frac{1}{2}}}\,\partial\phi\,.$$

Obviously, the $V^{\pm}$ operators have replaced the $\mathrm{e}^{i\phi_s/\sqrt{2}}$ operators used before for the spin part, while the rational torus for the charge part is unchanged. This isomorphism between the $k=1$ SU(2) WZW and $N=1$ rational torus models is of course well known. The quasihole operators are now built up with the same rules to ensure single-valuedness as in the abelian representation; for example the elementary quasihole is created by

$$V^{\pm}(w)\exp\left(\frac{i\phi}{2\sqrt{n+\frac{1}{2}}}\right). \quad (4.6)$$

From the operator product expansion we may reobtain the fact that they have spin $\frac{1}{2}$ and charge $-1/(2n+1)$. We note that the restrictions on the representations can be simply stated as the fact that we are really dealing with U(2) current algebra, extended by the fermionic field that condenses in (4.5).

Finally, we note that from the SU(2) Ward identities it is manifest that the ground state is a spin singlet.

4.2. HALDANE–REZAYI STATE

Recent experiments have shown clear indications of a FQHE plateau at a filling factor $\nu=\frac{5}{2}$ [14]. Motivated by this result, Haldane and Rezayi [44] proposed a state which is a spin singlet and has $\nu=\frac{1}{2}$ (or $\frac{5}{2}$ on including a completely filled Landau level), and showed that this state is the incompressible ground state of a certain "hollow core" pseudopotential hamiltonian. Some support for the idea of a spin-singlet ground state in this system is provided by later results which showed that not all the electrons in the "last" Landau level are polarized parallel to the magnetic field [15].

The Haldane–Rezayi (HR) state is

$$\Psi_{\mathrm{HR}}\left(z_1^{\uparrow},\ldots,z_{N/2}^{\downarrow}\right)=\prod_{i<j}\left(z_i^{\uparrow}-z_j^{\uparrow}\right)^3\left(z_i^{\downarrow}-z_j^{\downarrow}\right)^3\prod_{i,j}\left(z_i^{\uparrow}-z_j^{\downarrow}\right)\operatorname{per}\left(\frac{1}{z_i^{\uparrow}-z_j^{\downarrow}}\right)$$

$$=\prod_{i<j}\left(z_i-z_j\right)^2\det\left(\frac{1}{\left(z_i^{\uparrow}-z_j^{\downarrow}\right)^2}\right), \tag{4.7}$$

where per is the "permanent" of a matrix, defined in general by

$$\operatorname{per}M_{ij}=\sum_{\sigma\in S_L}\prod_{i=1}^{L}M_{i,\sigma(i)} \tag{4.8}$$

for an $L\times L$ matrix with elements M_{ij}. More generally, similar states can clearly be constructed at filling $\nu=1/q$, with q even, by replacing the exponent 2 in the product in the last line by q. Such states are singlets, from a physical point of view, because the determinant in (4.7) has the form of the well-known real space version of a BCS paired spin-singlet wave function of spin-$\frac{1}{2}$ fermions. Indeed, in the order parameter picture of the FQHE [4], the condensate in the HR state involves spin-singlet pairing of fermions [42], rather than the condensation of singlet bosons we have seen up to now (more on this in sect. 5). This state may be expressed in terms of conformal field theoretic correlators by introducing first-order bosonic ghost systems as used in superstring theory [45]. In particular, let β,γ be a $\lambda=\frac{1}{2}$ bosonic first-order ghost system, i.e. β,γ are free fields with

$$\langle\beta(z)\gamma(w)\rangle=\frac{1}{z-w}=\langle\gamma(w)\beta(z)\rangle \tag{4.9}$$

the only non-zero expectations, and introduce *two* scalar fields ϕ,ω. We may then write

$$\Psi_{\mathrm{HR}}=\left\langle\prod_{i=1}^{N/2}\beta(z_i^{\downarrow})e^{i\omega(z_i^{\downarrow})}\prod_{i=1}^{N/2}\gamma(z_i^{\uparrow})e^{-i\omega(z_i^{\uparrow})}\right\rangle\left\langle\prod_{i=1}^{N}e^{i\sqrt{2}\phi(z_i)}\exp\left[-\int d^2z'\sqrt{q}\rho_0\phi(z')\right]\right\rangle. \tag{4.10}$$

Unfortunately there seems to be no natural conformal-field-theoretic explanation of why this state is a spin singlet. Nevertheless, we may make three remarks. First, the identity

$$\det\left(\frac{1}{z_i - w_j}\right)\operatorname{per}\left(\frac{1}{z_i - w_j}\right) = \det\frac{1}{(z_i - w_j)^2} \tag{4.11}$$

which was used in eq. (4.7) and is needed in the proof of the spin-singlet property [44] may easily be proved using bosonization of the β, γ system [45]. Second, the fields β, γ, ω can be used to form dimension-one currents $-\frac{1}{2}\gamma^2, \frac{1}{2}\beta\gamma, \frac{1}{2}\beta^2$, $e^{\pm i\omega}\gamma, e^{\pm i\omega}\beta, \partial\omega$, which generate an affine superalgebra. It would be of considerable interest to find and test any physical implications of this symmetry. Third, we speculate that the spin part of the HR state may be related to $k = -\frac{5}{4}$ SU(2) current algebra with the electron corresponding to the spin-$\frac{1}{2}$ field of that system (times a rational torus field as in (4.10)), not in the sense that the conformal blocks give the HR wave function (it can be shown that they do not), but in the more general sense that the braiding properties of the excitations are the same. The $k = -\frac{5}{4}$ theory is unique among rational k values in that it has a spin-$\frac{1}{2}$ primary field with fermionic statistics as required in any expression like (4.10).

This speculation is perhaps strengthened by the observation that a state closely analogous to the HR state in structure *is* obtained from conformal blocks of fractional k SU(2) theory. It resembles (4.7) when written in the second form in (4.7) but with a permanent in place of the determinant:

$$\Psi_{\text{per}} = \prod_{i<j}(z_i - z_j)^q \operatorname{per}\left(\frac{1}{z_i^{\uparrow} - z_j^{\downarrow}}\right). \tag{4.12}$$

Here q is odd and again $\nu = 1/q$. This state was proposed in ref. [46]. It is a singlet because a singlet BCS wave function for spin-$\frac{1}{2}$ *bosons* must have an odd parity pairing function, here $1/z$, and q is odd to make it antisymmetric. This state can be obtained from a β–γ system:

$$\Psi_{\text{per}} = \left\langle \prod_{i=1}^{N/2}\beta(z_i^{\downarrow})\prod_{i=1}^{N/2}\gamma(z_i^{\uparrow})\right\rangle\left\langle \prod_{i=1}^{N} e^{i\sqrt{q}\phi(z_i)}\exp\left[-\int d^2z'\sqrt{q}\rho_0\phi(z')\right]\right\rangle. \tag{4.13}$$

In this case there is no mystery about the SU(2) symmetry because the currents $\beta^2, \beta\gamma, \gamma^2$ generate the Kac–Moody algebra at level $k = -\frac{1}{2}$ and the pair β, γ form the spin-$\frac{1}{2}$ representation. This state possesses spin-$\frac{1}{2}$ neutral boson excitations as well as charged quasiparticles, while HR similarly has neutral spin-$\frac{1}{2}$ fermions. These objects are the BCS quasiparticles corresponding to the condensation of pairs in the respective ground states.

5. Conformal blocks as electron wave functions

In this section we reverse the reasoning of the previous discussion, and, starting with certain conformal blocks we ask if they correspond to reasonable electron systems. As we have seen, sensible electron wave functions must be single valued and must satisfy the Pauli principle. If the electrons have spin we may ask that they be in a definite spin state, e.g., in a spin-singlet state. Many conformal field theories may be seen to give rise to conformal blocks satisfying such properties. For example, fractional level sl(2) current algebra naturally produces spin-$\frac{1}{2}$ operators with only abelian monodromy (two examples were given in sect. 4). Another example is provided by spin-$\frac{1}{2}$ descendents of the spin $j = k/2$ field in odd integer ($= k$) level sl(2) current algebra. These functions, when multiplied by an appropriate rational torus correlator to make them single valued, give acceptable electron wave functions. In this section we focus on a simple example of this technique. The system we discuss might not be very realistic, but the purpose of our discussion is simply to give an existence proof that there are "reasonable" electron systems with nonabelian excitations. To this end we study the "pfaffian state" Ψ_{Pf} defined for N spinless electrons by

$$\Psi_{\mathrm{Pf}}(z_1,\dots,z_N) = \mathrm{Pfaff}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^q \exp\left[-\tfrac{1}{4}\sum|z|^2\right]. \tag{5.1}$$

Here q is an even integer and the filling fraction is $\nu = 1/q$. The pfaffian is defined by

$$\mathrm{Pfaff}\, M_{ij} = \frac{1}{2^{L/2}(L/2)!}\sum_{\sigma\in S_L}\mathrm{sgn}\,\sigma\prod_{k=1}^{L/2} M_{\sigma(2k-1),\sigma(2k)} \tag{5.2}$$

for an $L\times L$ antisymmetric matrix whose elements are M_{ij}, or as the square root of the determinant of M. It arises from applying Wick's theorem to real fermion fields, or as the real space BCS wave function for pairing of spinless fermions. The structure of this state was originally inspired by that of the HR state.

We interpret this state as a product of a correlator of energy operators in the Ising model (or Majorana fermions in the holomorphic half of the model) times an appropriate rational torus correlator:

$$\Psi_{\mathrm{Pf}} = \langle \psi(z_1)\dots\psi(z_N)\rangle_{\mathrm{Ising}}\left\langle \prod_i \mathrm{e}^{i\sqrt{q}\phi(z_i)}\exp\left[-i\int \mathrm{d}^2z'\sqrt{q}\,\rho_0\phi(z')\right]\right\rangle. \tag{5.3}$$

(The Majorana fermions ψ here in the Ising model should not be confused with the destruction operator for physical electrons which appears elsewhere.) The reader may well object that it is unreasonable simply to pull wave functions out of a hat and try to extract physics from them. Therefore, our next task is to produce a hamiltonian for which Ψ_{Pf} is an exact incompressible ground state. We will apply the insight that the relative angular momentum operators extract terms from the operator product expansion. Thus from the expansion

$$\begin{aligned}
&\psi(z_1)e^{i\sqrt{q}\phi(z_1)}\psi(z_2)e^{i\sqrt{q}\phi(z_2)} \\
&\quad\sim (z_{12})^{q-1}\Big\{:e^{2i\sqrt{q}\phi(z[12])}: + (z_{12})^2\Big(:e^{2i\sqrt{q}\phi(z[12])}::\psi'\psi(z[12]): \\
&\qquad + \tfrac{1}{4}i\sqrt{q}\,:e^{2i\sqrt{q}\phi(z[12])}\phi''(z[12]):\Big) + O(z_{12}^4)\Big\},
\end{aligned} \tag{5.4}$$

where $z[12] \equiv \frac{1}{2}(z_1 + z_2)$ and $z_{12} \equiv z_1 - z_2$, and an application of the conformal Ward identities we get

$$\begin{aligned}
P_{q+1}^{12}\Psi_{\text{Pf}} = z_{12}^2\Bigg\{ &\sum_{i=3}^{2N} \frac{1+3q/4}{(z[12]-z_i)^2} - q\Bigg(\sum_{i=3}^{2N}\frac{1}{z[12]-z_i}\Bigg)^2 \\
&+ \sum_{i=3}^{2N}\frac{1}{z[12]-z_i}\Bigg(\frac{\partial}{\partial z_i} + \tfrac{1}{4}z_i^*\Bigg)\Bigg\}\big(P_{q-1}^{12}\Psi_{\text{Pf}}\big).
\end{aligned} \tag{5.5}$$

Therefore, defining the operators $\mathscr{O}^{ij}$, e.g. for $i, j = 1, 2$

$$\begin{aligned}
\mathscr{O}^{12} \equiv P_{q+1}^{12} - z_{12}^2\Bigg\{ &\sum_{i=3}^{2N} \frac{4+3q}{(z_1+z_2-2z_i)^2} - 4q\Bigg(\sum_{i=3}^{2N}\frac{1}{z_1+z_2-2z_i}\Bigg)^2 \\
&+ \sum_{i=3}^{2N}\frac{2}{z_1+z_2-2z_i}\Bigg(\frac{\partial}{\partial z_i} + \tfrac{1}{4}z_i^*\Bigg)\Bigg\}P_{q-1}^{12},
\end{aligned} \tag{5.6}$$

we may form the positive definite hamiltonian

$$\mathscr{H} = \sum_i \frac{1}{2m}(-i\nabla - eA)^2 + \sum_{l=0}^{q-2} V_l \sum_{i<j} P_l^{ij} + v\sum_{i<j}(\mathscr{O}^{ij})^\dagger \mathscr{O}^{ij}, \tag{5.7}$$

where v is a small positive constant, possibly representing the effective interactions of other degrees of freedom in the two-dimensional system which have been integrated out from the problem. Since the operator $\mathscr{O}$ involves a first-order differential operator we may expect that Ψ_{Pf} is nondegenerate, for this filling factor.

We now consider excitations around this ground state. The state may be described in the order parameter formalism [4] by

$$\left(\int d^2z\, d^2w\, \frac{1}{z-w}\psi^\dagger(z)U^q(z)\psi^\dagger(w)U^q(w)\,e^{-(|z|^2+|w|^2)/4}\right)^{N/2}|0\rangle. \tag{5.8}$$

(Similar expressions hold in the HR and permanent states of sect. 4.) Since the order parameter is paired, the flux quantum in the GL theory is halved (excitations need only be single valued when dragged around the pair of operators $\psi^\dagger U^q \psi^\dagger U^q$), and thus we expect that there will be excitations with half units of flux. Since the filling factor is $1/q$ these will be quasihole excitations of charge $1/2q$. By flux quantization these excitations themselves can only occur in pairs (not bound). A trial wave function for a pair of excitations with effectively a half quantum of flux each can be written as

$$\Psi_{\text{pair}}(z_1,\dots,z_N;v_1,v_2)$$

$$= \left\{ \sum_{\sigma\in S_n} \frac{\operatorname{sgn}\sigma \prod_{k=1}^{N/2} \left[(z_{\sigma(2k-1)} - v_1)(z_{\sigma(2k)} - v_2) + v_1 \leftrightarrow v_2 \right]}{(v_1 - v_2)^{1/8+1/4q} (z_{\sigma(1)} - z_{\sigma(2)}) \dots (z_{\sigma(N-1)} - z_{\sigma(N)})} \right\}$$

$$\times \prod_{i<j} (z_i - z_j)^q \exp\left[-\tfrac{1}{4} \sum_i |z_i|^2 \right]. \tag{5.9}$$

A calculation (verifying the conformal Ward identities) shows that this wave function is *exactly* reproduced by the insertion of a pair of spin operators as in the following correlator:

$$\Psi_{\text{pair}} = \left\langle \prod_{i=1}^{2N} \psi(z_i) \exp\left[i\sqrt{q}\,\phi(z_i) \right] \sigma(v_1) \exp\left[\frac{i}{2\sqrt{q}} \phi(v_1) \right] \right.$$

$$\left. \times \sigma(v_2) \exp\left[\frac{i}{2\sqrt{q}} \phi(v_2) \right] \exp\left[-i \int d^2z' \sqrt{q}\, \rho_0 \phi(z') \right] \right\rangle \tag{5.10}$$

and thus we interpret the spin operator (times the basic rational torus representation) as the quasihole excitation operator in this system* (the rational torus now has level $2q$ and the square root branch cuts in the RT part of (5.10) are cancelled by the square root as a fermion moves round the spin field).

Assuming the existence of a LG description of this system we expect that once again there will be a Chern–Simons description of the two relevant conformal field theories (the CSW description of the critical Ising model is given in ref. [1]) and that the quasiholes will be described by singular gauge transformations. (When making several such gauge transformations we must choose the cuts carefully.)

* The equality of (5.9) and (5.10) implies an interesting combinatorial identity described in appendix A.

Thus, we expect that the four-quasihole wave functions will be given by

$$\Psi_{\text{quartet}} = \Big\langle \prod_{i=1}^{N} \psi(z_i)\exp\big[i\sqrt{q}\,\phi(z_i)\big] \times \prod_{i=1}^{4} \sigma(v_i)\exp\bigg[\frac{i}{2\sqrt{q}}\phi(v_i)\bigg]\exp\bigg[-i\int d^2z'\,\sqrt{q}\,\rho_0\phi(z')\bigg]\Big\rangle. \quad (5.11)$$

Unfortunately, we have not so far been able to write a formula as explicit as, say, (5.9) for (5.11). As is well known in conformal field theory, conformal blocks of the type (5.11) in fact span a vector space of dimension greater than one. From the fusion rules of the Ising model it is easy to see that there is a two-dimensional space of blocks of the type given in eq. (5.11). In other words, the notation in eq. (5.11) is ambiguous because there are actually two linearly independent such functions Ψ_1, Ψ_2 (the ambiguity is resolved by using the machinery of chiral vertex operators [1]). Intuitively, we may think of these states as defining a strong pairing of quasiholes 1 2 and 3 4 with two distinct ways of joining the groups 1 2 and 3 4 together. If we now transport the quasihole 2 around quasihole 3 the wave functions will change by a nontrivial monodromy matrix. The monodromy of such blocks is well known to be identical to the monodromy of the four spin blocks and so, upon transport of quasihole 2 around quasihole 3 there will be nontrivial mixing of the two degenerate excitations described by

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \to e^{\pm 2\pi i/8}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \quad (5.12)$$

(The form of the matrix is obtained from the explicit four spin blocks in appendix D of the second paper in ref. [1].) This is therefore a "physical" system whose excitations are nonabelions.

Since the combination $\psi^{\dagger}U^{q}$ is always a fermion at $\nu = 1/q$, q even, and so these must pair if they are to have any chance to condense, and since the pfaffian state is the simplest way for them to do so, we feel that it is likely that if an incompressible state is ever observed at these filling factors with full spin polarization, it should be this state. Such a state will inevitably have neutral fermion and charged nonabelion excitations.

6. Conclusions

In this paper we have given a description of certain correlated electron ground states in terms of conformal-field-theoretic conformal blocks, and shown how some methods of conformal field theory can be used to rederive some standard results in the theory of the fractional quantum Hall effect. Furthermore, this point of view can be turned around to produce new and possibly interesting correlated electron ground states, together with model hamiltonians for which these ground states are exact. We have indicated that the ultimate reason these two subjects are related must be found in the relation of two-dimensional conformal field theory to Chern–Simons–Witten theory, on the one hand, and the Landau–Ginzburg description of the fractional quantum Hall effect on the other. We have also argued that incompressible FQHE systems should be classified according to the quantum numbers and statistics of their elementary excitations, as well as their ground-state quantum numbers. Finally, although the system with nonabelion excitations constructed in this paper may seem a little contrived, we have tried to show that it is really rather simple and so we may expect that something like it might eventually be observed. Of course, other natural states may also exist and in general the possibility of nonabelions in fractional quantum Hall effect systems deserves serious consideration.

One of us (G.M.) would like to thank Nathan Seiberg for a previous collaboration on the relation between anyons and RCFT, especially in connection with the relationship of refs. [4,6]. We would like to thank the Aspen Center for Physics where this work was begun. Finally, we would like to thank H. Mathur, A. Morozov, K. Rabe, and R. Shankar for useful discussions. The work of G.M. was supported by DOE under contract DE-AC02-76ER03075. The work of N.R. was supported in part by an Alfred P. Sloan Research Fellowship.

Note added

Recent work by one of us (N. Read [48], see also Blok and Wen [49]) has shed additional light on the hierarchy states. The latter can now be understood as multicomponent Coulomb gas systems which in the framework of this paper would be described as a set of n right-moving scalar fields compactified on a torus $\mathbb{R}^n/\Lambda$, where n is the number of levels in the continued fraction for ν and Λ is an integral lattice. Many new states can be obtained by different choices for Λ, subject to certain rules. Other papers exploring the relationship of conformal field theory and Chern–Simons theory with the quantum Hall effect and "chiral spin liquids" have now appeared [50,51].

Appendix A

A COMBINATORIAL IDENTITY

One can compare the formula (5.9) for the correlation function (5.10) with the formula one would have obtained for the same wave function using the methods of ref. [47], where one interprets the square of the Ising model as the Ashkin–Teller model. From these two formulations we obtain the identity (for N even)

$$\sum_{\sigma\in S_N} \frac{\operatorname{sgn}\sigma}{(z_{\sigma(1)}-z_{\sigma(2)})\dots(z_{\sigma(N-1)}-z_{\sigma(N)})} \prod_{k=1}^{N/2} \left[(z_{\sigma(2k-1)}-v)(z_{\sigma(2k)}-v')+v\leftrightarrow v'\right]$$

$$=2^{N/2}(N/2)!\sqrt{\sum_I 2^{|I|}\left(\operatorname{Pfaff}_{i,j\in I}\frac{1}{z_i-z_j}\right)^2\left[\prod_{j\in I}(z_j-v)(z_j-v')\right](v-v')^{N-|I|}}\,, \tag{A.1}$$

where I runs directly over distinct subsets of $\{1,\dots,N\}$ containing an even number $|I|$ of elements.

References

[1] G. Moore and N. Seiberg, Lectures on RCFT, Proc. 1989 Trieste Spring School on Superstrings; Commun. Math. Phys. 123 (1989) 177

[2] J. Fröhlich, Lectures 1989 Banff Summer School

[3] S. Girvin and R. Prange, eds., The quantum Hall effect (Springer, New York, 1987)

[4] N. Read, Phys. Rev. Lett. 62 (1989) 86

[5] E. Witten, Commun. Math. Phys. 121 (1989) 351

[6] G. Moore and N. Seiberg, Phys. Lett. 220 (1989) 422

[7] B. Halperin, Phys. Rev. Lett. 52 (1984) 1583

[8] D.P. Arovas, J.R. Schrieffer and F. Wilzcek, Phys. Rev. Lett. 53 (1984) 722

[9] S.M. Girvin, *in* The quantum Hall effect (Springer, New York, 1987);
S.M. Girvin and A.H. MacDonald, Phys. Rev. Lett. 58 (1987) 1252

[10] S. C. Zhang, T.H. Hansson and S. Kivelson, Phys. Rev. Lett. 62 (1989) 82

[11] E. Witten, Commun. Math. Phys. 92 (1984) 455

[12] T. Banks, I. Klebanov, J. Lykken and L. Susskind, unpublished

[13] X.G. Wen, Santa Barbara preprints;
X.G. Wen and Q. Niu, Santa Barbara preprint

[14] R. Willett et al., Phys. Rev. Lett. (1987) 1776

[15] J.P. Eisenstein et al., Phys. Rev. Lett. 61 (1988) 997

[16] B. Simon, Phys. Rev. Lett. 51 (1983) 2167

[17] F. Wilczek and A. Zee, Phys. Rev. Lett. 52 (1984) 2111

[18] A.M.L. Messiah and O.W. Greenberg, Phys. Rev. 136 (1964) 248

[19] E. Verlinde, Nucl. Phys. B300 (1988) 360

[20] S. Doplicher, R. Haag and J.E. Roberts, Commun. Math. Phys. 23 (1971) 199; 35 (1974) 49

[21] P. Griffiths and J. Harris, Principles of algebraic geometry (Wiley, New York, 1978) p. 73

[22] R.B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395

[23] D.-H. Lee and M.P.A. Fisher, Phys. Rev. Lett. 63 (1989) 903; private communication

[24] S.M. Girvin, A.H. MacDonald, M.P.A. Fisher, S-J. Rey and J.P. Sethna, preprint

[25] D. Friedan and S. Shenker, Nucl. Phys. B281 (1987) 509

[26] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333

[27] F.D.M. Haldane, Phys. Rev. Lett. 51 (1983) 605

[28] F.D M. Haldane and F.H. Rezayi, Phys. Rev. B31 (1985) 2529

[29] F.D.M. Haldane, Phys. Rev. Lett. 55 (1985) 2095

[30] R. Tao and Y.-S. Wu, Phys. Rev. B30 (1984) 1097;
Q. Niu, D.J. Thouless and Y.-S. Wu, Phys. Rev. B31 (1985) 3372;
R. Tao and F.D.M. Haldane, Phys. Rev. B33 (1986) 3844
[31] C.W.J. Beenakker, Phys. Rev. Lett. 64 (1990) 216;
A.H. MacDonald, Phys. Rev. Lett. 64 (1990) 220
[32] B.I. Halperin, Phys. Rev. B25 (1982) 2185
[33] V. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83
[34] G.W. Semenoff, Phys. Rev. Lett. 63 (1989) 1026
[35] F.D.M. Haldane, The hierarchy of fractional states and numerical studies, *in* The quantum Hall effect (Springer, New York, 1987)
[36] R.B. Laughlin, Elementary theory: The incompressible quantum fluid, *in* The quantum Hall effect (Springer, New York, 1987)
[37] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys., to be published
[38] G.H. Hardy and E.M. Wright, An introduction to the theory of numbers, Fifth edition (Oxford Univ. Press, 1979) p. 129
[39] W.P. Su, Phys. Rev. B34 (1986) 1031
[40] B. Halperin, Helv. Phys. Acta. 56 (1983) 75
[41] J.P. Eisenstein, H.L. Störmer, L. Pfeiffer and K.W. West, Phys. Rev. Lett. 62 (1989) 1540
[42] N. Read, to be published
[43] D.-H. Lee and C.L. Kane, Phys. Rev. Lett. 64 (1990) 1313
[44] F.D.M. Haldane and E.H. Rezayi, Phys. Rev. Lett. 60 (1988) 956; [Erratum: 60 (1988) 1886]
[45] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93
[46] D. Yoshioka, A.H. MacDonald and S.M. Girvin, Phys. Rev. B38 (1989) 3636
[47] P. DiFrancesco, H. Saleur and J.-B. Zuber, Nucl. Phys. B290 (1987) 527
[48] N. Read, Phys. Rev. Lett. 65 (1990) 1502
[49] B. Blok and X.-G. Wen, Phys. Rev. B, to be published
[50] A. Balatsky, Univ. of Illinois preprint
[51] J. Fröhlich and T. Kerler, ETH preprint ETH-TH/90-26