Cayley-Dickson algebras and loops

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Abstract

The Cayley-Dickson process is used in connection with square array representations of the Cayley-Dickson algebras. This involves an array operation on square arrays distinct from matrix multiplication. The arrays give a convenient representation of the octonion division algebra and a description of octonion multiplication. The connection between this description of pure octonion multiplication and seven dimensional real space using products related to the commutator and associator of the octonions is extended to the other Cayley-Dickson algebras and the appropriate real vector space.

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1 Introduction

The complex numbers and the quaternion numbers have representations as certain two dimensional and four dimensional matrices over the real numbers. Additionally the quaternions can be represented as certain two dimensional matrices over the complex numbers. The octonion numbers are non-associative and no basic matrix representation of this type can exist. By examining the Cayley-Dickson process a square array representation is found for the octonions with a multiplication distinct from matrix multiplication. This representation will include the above representations for the complex numbers and the quaternions. It will additionally represent all the algebras over the real numbers constructed from the Cayley-Dickson process. For the octonions it appears that this representation was first presented in [26] for general alternative division rings.

Octonion multiplication can be presented using modular arithmetic on its indices. The presentation of the octonions in the present paper will connect the indices with a three dimensional vector space over the finite field of two elements. The representation is extended to all Cayley-Dickson algebras and the proper vector space over the finite field. The presentation of the octonions also connects the study of seven dimensional real space with the standard presentation of three dimensional real space. The purely imaginary octonions and the connection to seven dimensional space is examined using a cross product related to the commutator and a product of three vectors related to the associator of the octonions. These products are then extended to all Cayley-Dickson algebras as they are examples of flexible algebras.

The positive and negative basis elements of a Cayley-Dickson algebras form an algebraic loop. The sixteen dimensional Cayley-Dickson algebra known as the sedenions provides an example of the use of the geometry of its subloops. The geometry of subloops for each Cayley-Dickson loop is a finite projective incidence geometry over the field of two elements. These loops when connected with the cross product on the algebras provide the associated real vector space with a orthonormal basis that is closed under cross product

2 The Cayley-Dickson process

The *Cayley-Dickson process* for the real numbers $\mathbb{R}$ is an iterative process that forms algebras over $\mathbb{R}$ with a *conjugation* involution. For a more general description of the process see [25, 3].

Let $\mathbb{A}_0 = \mathbb{R}$ and define conjugation for every $a \in \mathbb{R}$ by $\overline{a} = a$.

Let $\mathbb{A}_{k+1} = \mathbb{A}_k \oplus \mathbb{A}_k$ for $k+1 \in \mathbb{N}$.

Operations are now defined on $\mathbb{A}_{k+1}$ using operations for $\mathbb{A}_k$ which are derived from $\mathbb{R}$. Addition is component-wise with multiplication and conjugation defined as follows:

$$(a, b) \cdot (c, d) = (ac - \overline{d}b, da + b\overline{c})$$
$$\overline{(a, b)} = (\overline{a}, -b)$$

The basic properties of conjugation are the following:

$$\overline{\overline{a}} = a$$
$$\overline{a + b} = \overline{a} + \overline{b}$$
$$\overline{ab} = \overline{b}\,\overline{a}$$

It is well known that $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$ and $\mathbb{A}_3 = \mathbb{O}$ are the complex, quaternion and octonion *division algebras* respectively. Together with $\mathbb{R}$ these are the four division algebras formed from this process as all other *Cayley-Dickson algebras* have zero divisors. Each algebra derived through the Cayley-Dickson process has a *norm* or *modulus* for each element but only the division algebras have the property that the norm of the product of two numbers is the product of the norm of each number. The division algebras then are *normed division algebras* and are the only finite dimensional normed division algebras over $\mathbb{R}$. Additionally all Cayley-Dickson algebras have the property that each non-zero element has an unique inverse.

As the Cayley-Dickson process is an inductive process it can be used to prove certain results by induction using these pairs. Each algebra individually might need a more specialized representation.

Define an *array operation* $\circ$ on 2×2 arrays as follows:

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \circ \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \cdot \alpha_2 + \gamma_2 \cdot \beta_1 & \beta_2 \cdot \alpha_1 + \beta_1 \cdot \delta_2 \\ \alpha_2 \cdot \gamma_1 + \delta_1 \cdot \gamma_2 & \gamma_1 \cdot \beta_2 + \delta_2 \cdot \delta_1 \end{bmatrix}$$

This operation needs a well-defined multiplication and addition on its entries and is matrix multiplication if the entry multiplication is commutative. In general the operation is not associative. In order to use this operation for the Cayley-Dickson process the correct arrays need to be defined.

An ordered pair from the Cayley-Dickson process (α, β) can be thought of as the array

$$\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$

The first row is the ordered pair and the second row is the conjugate pair $(\overline{\alpha}, -\beta)$ in reverse order with the conjugate of the second component.

Define the set of these arrays as follows:

$$\mathbf{B}_{k+1} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{A}_k \right\}$$

The arrays need to retain the algebraic properties of $\mathbb{A}_{k+1}$. In the Cayley-Dickson process addition is component-wise, so addition of these arrays must be standard matrix addition. The fact that $(\mathbf{B}_{k+1}, +)$ is an abelian group follows from the fact $(\mathbb{A}_k, +)$ is an abelian group with closure following from the conjugation properties. Scalar multiplication by $\mathbb{R}$ is allowed as each element of $\mathbb{R}$ commutes with each element of $\mathbb{A}_k$. Then $\mathbf{B}_{k+1}$ is a vector space over $\mathbb{R}$. In order to make it into an algebra requires the array operation.

The array multiplication on two arrays of $\mathbf{B}_{k+1}$ is the following:

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ -\overline{\beta_1} & \overline{\alpha_1} \end{bmatrix} \circ \begin{bmatrix} \alpha_2 & \beta_2 \\ -\overline{\beta_2} & \overline{\alpha_2} \end{bmatrix} = \begin{bmatrix} \alpha_1 \cdot \alpha_2 - \overline{\beta_2} \cdot \beta_1 & \beta_2 \cdot \alpha_1 + \beta_1 \cdot \overline{\alpha_2} \\ -\alpha_2 \cdot \overline{\beta_1} - \overline{\alpha_1} \cdot \overline{\beta_2} & -\overline{\beta_1} \cdot \beta_2 + \overline{\alpha_2} \cdot \overline{\alpha_1} \end{bmatrix}$$

The operation gives the product from the Cayley-Dickson process in the first row and a form of the conjugate product in the second row. Moreover this operation is closed by the conjugation properties. The distributive properties are also satisfied as they are based on the distributive properties of $\mathbb{A}_k$.

The array of the *conjugate* of $\left[\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{smallmatrix}\right]$. This allows the establishment of the following *conjugation map* on these arrays:

$$\overline{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} = \begin{bmatrix} \overline{A} & -B \\ -C & \overline{D} \end{bmatrix}$$

So if the individual elements of the array are from $\mathbb{A}_k$, the array itself represents elements of $\mathbb{A}_{k+1}$ with addition and multiplication defined above. This stepped approach to the iterations that gives these algebras makes each representation more convenient. Otherwise the algebras are the arrays $\mathbf{B}_{k+1}$ of the arrays $\mathbf{B}_k$ with a recursive multiplication. The importance of this representation is that it is an algebra representation different from the basis representation as a vector space.

For $a \in \mathbb{R}$ define $N(a) = |a|^2$. Then the *algebraic norm* of $\alpha \in \mathbb{C}$ is

$$N(\alpha) = \alpha\overline{\alpha} = a^2 + b^2 = N(a) + N(b), \qquad \alpha = (a, b)$$

In general this gives the norm of $\mathbb{A}_{k+1}$ in terms of the previous norm $\mathbb{A}_k$ and the norm is always a real number. A number and its conjugate commute as this is true in $\mathbb{C}$ and this means the norm of an array can also be thought of as the determinant if the array was considered a matrix. This does not mean that the determinant of the product of two arrays is the product of the determinants of two arrays because of possible zero divisors. When this determinant property is true it does give the two, four and eight square identities that define the complex, quaternion and octonion numbers respectively. The last two identities come from Gauss' identity for complex numbers and a similar identity for quaternions, see [12].

3 The octonions

Let $\mathbf{O}$ be a set of 2×2 arrays with entries from the quaternions $\mathbb{H}$ of the following form:

$$\mathbf{O} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{H} \right\}$$

By the Cayley-Dickson process above $(\mathbf{O}, +, \circ)$ is certainly the octonion algebra and properties of the octonions can be examined by relying on the quaternion algebra. Some properties are now listed that are true for the octonions or at least for the non-zero octonions.

Alternative Properties:

$$a(ab) = a^2 b \quad \text{and} \quad ba^2 = (ba)a$$

Flexible Property:

$$a(ba) = (ab)a$$

Inversive Properties:

$$a^{-1}(ab) = b \quad \text{and} \quad b = (ba)a^{-1}$$

From the *alternative properties*, the octonions are an *alternative division algebra*. From the above division algebras over $\mathbb{R}$ the only proper alternative division algebra. The *flexible* or *reflexive property* is also considered a third alternative property. $(\mathbf{O}^* = \mathbf{O} - \{0\}, \circ)$ is a *Moufang loop* and in this specific case we refer to $\circ$ as a *loop operation* and the non-zero arrays as forming a *matrix loop*. In a loop any of the Moufang properties imply the alternative, flexible and *inversive properties*, see [4]. The inversive properties are also known as the *inverse properties* and these properties can be used to prove that the octonions are a normed division algebra as is done in [9].

To find a convenient representation for the octonions each element of $\mathbf{O}$ uses the standard representation for the quaternions. Each matrix is decomposed into a vector space representation with eight basis elements. An isomorphism then is established to give the octonions a usual but not standard form:

$$\begin{bmatrix} a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} & e + f\mathbf{i} + g\mathbf{j} + h\mathbf{k} \\ -e + f\mathbf{i} + g\mathbf{j} + h\mathbf{k} & a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + c\begin{bmatrix} \mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{bmatrix} + d\begin{bmatrix} \mathbf{k} & 0 \\ 0 & -\mathbf{k} \end{bmatrix}$$
$$+ e\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} + g\begin{bmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix} + h\begin{bmatrix} 0 & \mathbf{k} \\ \mathbf{k} & 0 \end{bmatrix}$$
$$\sim a\mathbf{i}_0 + b\mathbf{i}_1 + c\mathbf{i}_2 + d\mathbf{i}_3 + e\mathbf{i}_4 + f\mathbf{i}_5 + g\mathbf{i}_6 + h\mathbf{i}_7 \in \mathbb{O}$$

The loop operation must be used even on the above loop matrices as the entries do not commute. This leads to the following multiplication table that gives the sixteen element *octonion loop*, $\mathbb{O}_\mathrm{L}$ and the multiplication of any two elements of the octonion algebra $\mathbb{O}$:

Octonion Multiplication

$\cdot$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_6$	$\mathbf{i}_7$
$\mathbf{i}_0$	1	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_6$	$\mathbf{i}_7$
$\mathbf{i}_1$	$\mathbf{i}_1$	-1	$\mathbf{i}_3$	$-\mathbf{i}_2$	$\mathbf{i}_5$	$-\mathbf{i}_4$	$-\mathbf{i}_7$	$\mathbf{i}_6$
$\mathbf{i}_2$	$\mathbf{i}_2$	$-\mathbf{i}_3$	-1	$\mathbf{i}_1$	$\mathbf{i}_6$	$\mathbf{i}_7$	$-\mathbf{i}_4$	$-\mathbf{i}_5$
$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_1$	-1	$\mathbf{i}_7$	$-\mathbf{i}_6$	$\mathbf{i}_5$	$-\mathbf{i}_4$
$\mathbf{i}_4$	$\mathbf{i}_4$	$-\mathbf{i}_5$	$-\mathbf{i}_6$	$-\mathbf{i}_7$	-1	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{i}_5$	$\mathbf{i}_5$	$\mathbf{i}_4$	$-\mathbf{i}_7$	$\mathbf{i}_6$	$-\mathbf{i}_1$	-1	$-\mathbf{i}_3$	$\mathbf{i}_2$
$\mathbf{i}_6$	$\mathbf{i}_6$	$\mathbf{i}_7$	$\mathbf{i}_4$	$-\mathbf{i}_5$	$-\mathbf{i}_2$	$\mathbf{i}_3$	-1	$-\mathbf{i}_1$
$\mathbf{i}_7$	$\mathbf{i}_7$	$-\mathbf{i}_6$	$\mathbf{i}_5$	$\mathbf{i}_4$	$-\mathbf{i}_3$	$-\mathbf{i}_2$	$\mathbf{i}_1$	-1

It is essentially the table found in [21] for general alternative division algebras. As $\mathbb{O}_\mathrm{L}$ is a subloop of the previous loop it is also a Moufang loop. It has three stronger properties known as the *extra properties* and an additional property known as the *C property*, see [13].

The form of the matrices implies that the table can actually be built from the quaternion table in the following way. The upper left 4×4 table is just the quaternion table. Ignoring the row and column of $\mathbf{i}_4$ and the identity, there are three additional 3×3 tables that need to be formed. The lower right table is the negative of the quaternion 3×3 table except for the main diagonal. The lower left table is the negative of the quaternion table with each index raised by adding four. The upper right table is the negative transpose of the lower left table which means that it is the transpose of the quaternion table with each index raised by adding four. Multiplication by $\mathbf{i}_4$ is easily established from the matrices.

A description of the multiplication of the imaginary loop elements of $\mathbb{O}_\mathrm{L}$ is found by using the Fano plane, the finite projective plane of order two. Multiplication will occur cyclically around the lines with the arrows indicating a positive product. One importance of this description is the properties of a projective plane. As two distinct points determine a line, those two elements are a system of generators for a subalgebra isomorphic to $\mathbb{H}$. As two distinct lines intersect in an unique point, those two quaternion subalgebras intersect in an unique subalgebra isomorphic to $\mathbb{C}$. Any triangle then is a system of generators for the octonions $\mathbb{O}$. The Fano plane is also interesting in that it have seven points, seven lines and seven quadrangles, each the complement of a line in the Fano plane. The quadrangles are the affine planes of order two.

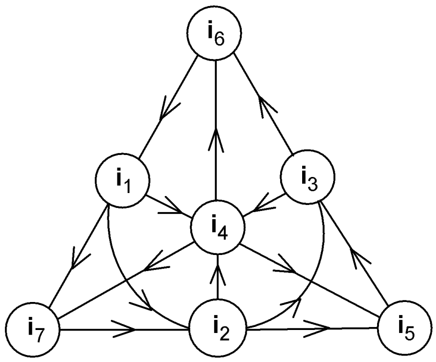

Figure 1. The Octonion Fano Plane

Of historical note is the resemblance of the presentation of the above objects to the presentation found in [8], including the derived form of the eight square identity.

4 Analytic geometry and the octonions

In order to exploit the above representation of the octonions, their connection to seven dimensional real space is explored. This follows some of the considerations found in [2].

The *pure octonions* are $Im(\mathbb{O}) = \{u : u \in \mathbb{O} \text{ and } \overline{u} = -u\}$, then $\mathbb{O} = \mathbb{R} \oplus Im(\mathbb{O})$.

The *dot product* $\bullet$ and *cross product* $\times$ are defined as they would for the pure quaternions. The formal definitions are 7.1 and 7.2 respectively. For the cross product this involves the *commutator*, $[a, b]$, of $\mathbb{O}$.

The dot product is a non-degenerate symmetric bilinear form and two elements of $\mathbb{O}$ are *orthogonal* if $u \bullet v = 0$. The cross product is an non-degenerate alternating bilinear form. Specifically the cross product has the anti-commutative property that $u \times v = -v \times u$. From [2] two pure octonions are orthogonal to their cross product and then the seven dimensional cross product retains that property of the standard cross product. This property will be shown to hold in general in the last section.

Proposition 4.1. *For all $u, v \in Im(\mathbb{O})$, $u \times v$ is zero (vector) if and only if $v = ru$ with $r \in \mathbb{R}$.*

Proposition 4.2. *For all $u, v \in Im(\mathbb{O})$, $u \times v$ is orthogonal to both u and v.*

There is a vector definition for the cross product. Direct computation shows that the following is an equivalent definition subject to representation of $\mathbb{O}$.

Let $u, v \in Im(\mathbb{O})$ with

$$u = b_1\mathbf{i}_1 + c_1\mathbf{i}_2 + d_1\mathbf{i}_3 + e_1\mathbf{i}_4 + f_1\mathbf{i}_5 + g_1\mathbf{i}_6 + h_1\mathbf{i}_7$$
$$v = b_2\mathbf{i}_1 + c_2\mathbf{i}_2 + d_2\mathbf{i}_3 + e_2\mathbf{i}_4 + f_2\mathbf{i}_5 + g_2\mathbf{i}_6 + h_2\mathbf{i}_7$$

Then

$$
\begin{aligned}
u\times v &= (b_1,c_1,d_1,e_1,f_1,g_1,h_1)\times(b_2,c_2,d_2,e_2,f_2,g_2,h_2)\\
&= \begin{vmatrix}\mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3\\ b_1 & c_1 & d_1\\ b_2 & c_2 & d_2\end{vmatrix} + \begin{vmatrix}\mathbf{i}_1 & \mathbf{i}_4 & \mathbf{i}_5\\ b_1 & e_1 & f_1\\ b_2 & e_2 & f_2\end{vmatrix} + \begin{vmatrix}\mathbf{i}_2 & \mathbf{i}_4 & \mathbf{i}_6\\ c_1 & e_1 & g_1\\ c_2 & e_2 & g_2\end{vmatrix} + \begin{vmatrix}\mathbf{i}_3 & \mathbf{i}_4 & \mathbf{i}_7\\ d_1 & e_1 & h_1\\ d_2 & e_2 & h_2\end{vmatrix}\\
&+ \begin{vmatrix}\mathbf{i}_6 & \mathbf{i}_1 & \mathbf{i}_7\\ g_1 & b_1 & h_1\\ h_2 & b_2 & h_2\end{vmatrix} + \begin{vmatrix}\mathbf{i}_7 & \mathbf{i}_2 & \mathbf{i}_5\\ h_1 & c_1 & f_1\\ h_2 & c_2 & f_2\end{vmatrix} + \begin{vmatrix}\mathbf{i}_5 & \mathbf{i}_3 & \mathbf{i}_6\\ f_1 & d_1 & g_1\\ f_2 & d_2 & g_2\end{vmatrix} \qquad (4.1)
\end{aligned}
$$

These seven ordered triples come from the seven lines of the Fano plane, each ordered by the positive multiplication without allowing for the cycling on the lines. The restriction to three dimensional space then retains the standard definition and the cross product is the sum of the seven cross products for each three dimensional space related to the quaternion subalgebras whose imaginary basis elements are those lines of the Fano plane.

The *associator* of $\mathbb{O}$, $[a,b,c]$, is trilinear and alternating from the alternative properties. The pure octonions are also closed under associator so this allows for the following definition of an associator *tri-product* which is an alternating trilinear form:

$$
\begin{aligned}
&\rangle\cdot,\cdot,\cdot\langle : Im(\mathbb{O})\times Im(\mathbb{O})\times Im(\mathbb{O}) \longrightarrow Im(\mathbb{O})\\
&\rangle u,v,w\langle = \frac{1}{2}[u,v,w] = \frac{1}{2}((uv)w - u(vw))
\end{aligned}
$$

The tri-product is non-degenerate as the nucleus of $\mathbb{O}$ is $\mathbb{R}$ and is defined in general in 7.3.

A vector definition subject to representation does exist as direct computation shows that the following is equivalent.

Let $u,v,w \in Im(\mathbb{O})$ with

$$
\begin{aligned}
u &= b_1\mathbf{i}_1 + c_1\mathbf{i}_2 + d_1\mathbf{i}_3 + e_1\mathbf{i}_4 + f_1\mathbf{i}_5 + g_1\mathbf{i}_6 + h_1\mathbf{i}_7\\
v &= b_2\mathbf{i}_1 + c_2\mathbf{i}_2 + d_2\mathbf{i}_3 + e_2\mathbf{i}_4 + f_2\mathbf{i}_5 + g_2\mathbf{i}_6 + h_2\mathbf{i}_7\\
w &= b_3\mathbf{i}_1 + c_3\mathbf{i}_2 + d_3\mathbf{i}_3 + e_3\mathbf{i}_4 + f_3\mathbf{i}_5 + g_3\mathbf{i}_6 + h_3\mathbf{i}_7
\end{aligned}
$$

Then

$$
\begin{aligned}
\rangle u,v,w\langle &= \rangle(b_1,c_1,d_1,e_1,f_1,g_1,h_1),(b_2,c_2,d_2,e_2,f_2,g_2,h_2),(b_3,c_3,d_3,e_3,f_3,g_3,h_3)\langle\\
&= \begin{vmatrix}\mathbf{i}_1 & \mathbf{i}_4 & \mathbf{i}_2 & \mathbf{i}_7\\ b_1 & e_1 & c_1 & h_1\\ b_2 & e_2 & c_2 & h_2\\ b_3 & e_3 & c_3 & h_3\end{vmatrix} + \begin{vmatrix}\mathbf{i}_2 & \mathbf{i}_4 & \mathbf{i}_3 & \mathbf{i}_5\\ c_1 & e_1 & d_1 & f_1\\ c_2 & e_2 & d_2 & f_2\\ c_3 & e_3 & d_3 & f_3\end{vmatrix} + \begin{vmatrix}\mathbf{i}_3 & \mathbf{i}_4 & \mathbf{i}_1 & \mathbf{i}_6\\ d_1 & e_1 & b_1 & g_1\\ d_2 & e_2 & b_2 & g_2\\ d_3 & e_3 & b_3 & g_3\end{vmatrix} + \begin{vmatrix}\mathbf{i}_4 & \mathbf{i}_5 & \mathbf{i}_6 & \mathbf{i}_7\\ e_1 & f_1 & g_1 & h_1\\ e_2 & f_2 & g_2 & h_2\\ e_3 & f_3 & g_3 & h_3\end{vmatrix}\\
&+ \begin{vmatrix}\mathbf{i}_5 & \mathbf{i}_7 & \mathbf{i}_1 & \mathbf{i}_3\\ f_1 & h_1 & b_1 & d_1\\ f_2 & h_2 & b_2 & d_2\\ f_3 & h_3 & b_3 & d_3\end{vmatrix} + \begin{vmatrix}\mathbf{i}_6 & \mathbf{i}_5 & \mathbf{i}_2 & \mathbf{i}_1\\ g_1 & f_1 & c_1 & b_1\\ g_2 & f_2 & c_2 & b_2\\ g_3 & f_3 & c_3 & b_3\end{vmatrix} + \begin{vmatrix}\mathbf{i}_7 & \mathbf{i}_6 & \mathbf{i}_3 & \mathbf{i}_2\\ h_1 & g_1 & d_1 & c_1\\ h_2 & g_2 & d_2 & c_2\\ h_3 & g_3 & d_3 & c_3\end{vmatrix} \qquad (4.2)
\end{aligned}
$$

The seven ordered quadruples are the seven quadrangles of the Fano plane ordered in the following way. Examining $\mathbf{i}_4, \mathbf{i}_5, \mathbf{i}_6, \mathbf{i}_7$, these elements are ordered in the column headed by $\mathbf{i}_4$ in the octonion multiplication table. The last three elements $\mathbf{i}_5, \mathbf{i}_6, \mathbf{i}_7$ come from rows $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ respectively and these elements are ordered in this way in the Fano plane. Starting with $\mathbf{i}_1$ the other three positive elements in that column are $\mathbf{i}_2, \mathbf{i}_4, \mathbf{i}_7$ which are in rows $\mathbf{i}_3, \mathbf{i}_5, \mathbf{i}_6$ respectively. As the ordering of these elements is $\mathbf{i}_5, \mathbf{i}_3, \mathbf{i}_6$, the ordering of their product relative to $\mathbf{i}_1$ is $\mathbf{i}_4, \mathbf{i}_2, \mathbf{i}_7$.

The subsets of $Im(\mathbb{O})$ corresponding to these quadruples are also closed under associator. The set corresponding to $\{\mathbf{i}_4, \mathbf{i}_5, \mathbf{i}_6, \mathbf{i}_7\}$ is the most intriguing as it gives the orthogonal complement of the standard three dimensional space in seven dimensional space. The subset of pure octonions with just these basis vectors are closed under associator and the tri-product of any three distinct basis vectors is the fourth vector or its negative.

Properties of tri-product can be established relying on properties of $\mathbb{O}$ such as alternative, flexible and Moufang properties. Additional properties of the associator of $\mathbb{O}$ can be found in [5] or [23] as properties of alternative rings and two useful properties are listed,

$$[a^2, b, c] = a[a, b, c] + [a, b, c]a$$
$$[a, b][a, b, c] = -[a, b, c][a, b]$$

Proposition 4.3. *If $u \times v$ is zero for $u, v \in Im(\mathbb{O})$, then the tri-product of u, v, w is zero for all $w \in Im(\mathbb{O})$.*

Corollary 4.4. *If the tri-product of u, v, w is non-zero, then each cross product $u \times v$, $v \times w$ and $w \times u$ is non-zero.*

Proposition 4.5. *For all $u, v \in Im(\mathbb{O})$, the tri-product of $u, v, u \times v$ is zero.*

Proposition 4.6. *For all $u, v, w \in Im(\mathbb{O})$, the tri-product of u, v, w is orthogonal to each of u, v and w.*

Proposition 4.7. *For all $u, v, w \in Im(\mathbb{O})$, the tri-product of u, v, w is orthogonal to each cross product $u \times v$, $v \times w$ and $w \times u$.*

Further properties for the products can be derived from properties of the commutator and associator of a non-associative ring. The following lemma is from [27] and gives various generalizations of the Jacobi identity.

Lemma 4.8. *Let R be a non-associative ring. For all $a, b, c \in R$,*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = [a, b, c] + [b, c, a] + [c, a, b] - [c, b, a] - [a, c, b] - [b, a, c]$$

If R is flexible then

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 2[a, b, c] + 2[b, c, a] + 2[c, a, b]$$

If R is alternative then

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 6[a, b, c]$$

Proposition 4.9. *For all $u, v, w \in Im(\mathbb{O})$,*

$$(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 3 \cdot \rangle u, v, w \langle$$

The next lemma contains a fundamental identity of the associator that is true for all non-associative rings. It can be found in [27] and is used in [5]. With additional structure it gives to a relationship between the commutator and associator.

Lemma 4.10. *Let R be a non-associative ring. For all $a, b, c, d \in R$,*

$$[a, b, cd] + [ab, c, d] = a[b, c, d] + [a, bc, d] + [a, b, c]d$$

If R is flexible, then

$$[a, b, [c, d]] + [[a, b], c, d] = [a, [b, c, d]] + [a, [b, c], d] + [[a, b, c], d]$$

If R is alternative, then

$$[a, [b, c, d]] + [a, [b, c], d] + [[a, b, c], d] = [c, [d, a, b]] + [c, [d, a], b] + [[c, d, a], b]$$

Proof. For $a, b, c, d \in R$,

$$[a, b, cd] + [ab, c, d] = a[b, c, d] + [a, bc, d] + [a, b, c]d$$
$$[d, c, ba] + [dc, b, a] = d[c, b, a] + [d, cb, a] + [d, c, b]a$$

If R is flexible then by adding the two equations and applying the property of the associator gives the result. Then

$$[a, b, [c, d]] + [[a, b], c, d] = [a, [b, c, d]] + [a, [b, c], d] + [[a, b, c], d]$$
$$[b, a, [d, c]] + [[b, a], d, c] = [b, [a, d, c]] + [b, [a, d], c] + [[b, a, d], c]$$

If R is alternative then by subtracting the equations and applying the cyclic property of the associator gives the result. ☐

The lemma leads to a proposition for the pure octonions.

Proposition 4.11. *For all $u, v, w, z \in Im(\mathbb{O})$,*

$$u \times \rangle v, w, z\langle + \rangle u, v \times w, z\langle + \rangle u, v, w\langle \times z = v \times \rangle u, z, w\langle + \rangle v, u \times z, w\langle + \rangle v, u, z\langle \times w$$

Linear forms of two of the Moufang properties give another property of the octonions:

$$[[a, b], c, d] + [[d, b], c, a] = [[a, b, c], d] + [[d, b, c], a]$$

This property will be used in a later section and leads to a further proposition for the pure octonions.

Proposition 4.12. *For all $u, v, w, z \in Im(\mathbb{O})$,*

$$\rangle u \times v, w, z\langle + \rangle z \times v, w, u\langle = \rangle u, v, w\langle \times z + \rangle z, v, w\langle \times u$$

Relationships between the dot and cross product can be found in [28],[29], see also [20] or [12].

In [29], Zorn gives the basis for seven dimensional space using three defining vectors and the appropriated cross products with the seven vector given by $(u \times v) \times w$. Using the tri-product, with the set of generators of the algebra $u = \mathbf{i}_1, v = \mathbf{i}_2, w = \mathbf{i}_4$, the seven basis vectors are

$$u, \quad v, \quad u \times v, \quad w, \quad u \times w, \quad v \times w, \quad \rangle u, v, w\langle$$

In the context of the related finite geometry the tri-product is completing the frame of the triangle u, v, w. The basis is not only orthonormal but is closed under cross and tri-product when extended to negatives and zero.

More generally three vectors from seven dimensional space with tri-product non-zero determine pair-wise three non-zero cross products and together with the tri-product give a total of seven non-zero vectors. The cross products and tri-product are orthogonal to their defining vectors and the cross products are orthogonal to the tri-product. Two defining vectors and their cross product have together tri-product zero.

The computer algebra system Maple was used for determinant computations.

5 General algebras and loops

The presence of zero divisors in the other Cayley-Dickson algebras does not allow for all the structure of the division algebras. From [24] all Cayley-Dickson algebras do satisfy the flexible/reflexive property and are examples of flexible algebras. The associator for each algebra has the property that $[a,b,c]=-[c,b,a]$. Additionally each of algebras are *nicely-normed* as defined in [2]. Then for each element $a+\overline{a}\in\mathbb{R}$ and for each non-zero element $a^{-1}=\frac{1}{N(a)}\overline{a}$. For the commutator and associator this implies that

$$[a,b]=-[\overline{a},b], \qquad [a,b,c]=-[\overline{a},b,c]$$

for any component. These properties are used to demonstrate that each Cayley-Dickson algebra $\mathbb{A}_k$ satisfies the *weak inversive properties* for non-zero elements

$$a^{-1}(ab)=a(a^{-1}b) \quad \text{and} \quad (ba^{-1})a=(ba)a^{-1} \quad \text{and} \quad a^{-1}(ab)=(ba)a^{-1}$$

In order to distinguish between the alternative and non-alternative algebras, a tetra-linear mapping is defined on the Cayley-Dickson algebras. The *commu-associator* $[a,b,c,d]$ is defined using linear forms of two of the Moufang identities and is therefore trivial for the octonions as follows:

$$\begin{aligned}[a,b,c,d] &= [ab,c,d]-[a,b,c]d+[db,c,a]-[d,b,c]a\\ &\quad +[d,c,ba]-d[c,b,a]+[a,c,bd]-a[c,b,d]\\ &= [[a,b],c,d]-[[a,b,c],d]+[[d,b],c,a]-[[d,b,c],a]\end{aligned}$$

Further results suggest that each Cayley-Dickson algebra $\mathbb{A}_k$ has a $k+1$-linear mapping trivial for $\mathbb{A}_k$ but non-trivial for $\mathbb{A}_{k+1}$ that would distinguish the two algebras. The present commu-associator does have some additional properties.

Proposition 5.1. *For the commu-associator of every Cayley-Dickson algebra,*

(i) $\forall a,b,c,d\in\mathbb{A}_k,\ [a,b,c,d]=[d,b,c,a]$

(ii) $\forall a,b,c,d\in\mathbb{A}_k,\ [a,b,c,a]=-[a,c,b,a]$

Proof. It follows from the fundamental identity for non-associative rings and the flexible property that

$$\begin{aligned}&[ab,c,a]+[a,b,ca]=a[b,c,a]+[a,b,c]a\\ &[ac,b,a]+[a,c,ba]=a[c,b,a]+[a,c,b]a\end{aligned}$$

which imply the second property. □

In any Cayley-Dickson algebra the canonical basis elements and their negatives form a loop. These *Cayley-Dickson loops* are iteratively generated as follows.

Let $\mathrm{L}_0=\{1,-1\}$ and

$$\mathrm{L}_{k+1}=\{(a,0):a\in\mathrm{L}_k\}\cup\{(0,b):b\in\mathrm{L}_k\}$$

In this way $\mathrm{L}_{k+1}\subset\mathbb{A}_{k+1}$. Using the Cayley-Dickson multiplication and multiplication by zero each $(\mathrm{L}_{k+1},\cdot)$ is shown to be a loop by induction relying on that fact that $(\mathrm{L}_k,\cdot)$ is a loop and has the zero product property as it has no zero divisors. Additionally the negative and conjugate of an element of these loops is also an element. The fourth roots of unity, the quaternion group and the octonion loop $\mathbb{O}_{\mathrm{L}}$ are the beginning of the process. The norm of each element is one and if an element is not in L_0 its conjugate is its negative and its square is negative one. These elements are the *imaginary loop elements*.

The Cayley-Dickson loops satisfy the inversive, alternative and flexible properties but in general each loop is not Moufang. These properties could also be established if L_0 was replaced by $\mathbb{R}^*$.

Proposition 5.2. *Each Cayley-Dickson loop satisfies the inversive properties.*

Proof. Assuming that L_k satisfies the inversive properties, in order to show the left inversive property $a^{-1}(ab) = b$, let $A, B \in \mathrm{L}_{k+1}$. The inductive step is done in four cases depending on the form of A and B. Suppose $A = (a, 0)$ and $B = (0, b)$ with $a, b \in \mathrm{L}_k$. Then

$$A^{-1} \cdot (A \cdot B) = \frac{1}{N[a]}(\overline{a}, 0) \cdot [(a, 0) \cdot (0, b)] = \frac{1}{N[a]}(0, (ba)\overline{a}) = \frac{1}{N[a]}(0, b(a\overline{a})) = (0, b) = B$$

Suppose $A = (0, a)$ and $B = (0, b)$ with $a, b \in \mathrm{L}_k$. Then

$$A^{-1} \cdot (A \cdot B) = \frac{1}{N[a]}(0, -a) \cdot [(0, a) \cdot (0, b)] = \frac{1}{N[a]}(0, a(\overline{a}b)) = \frac{1}{N[a]}(0, (a\overline{a})b) = (0, b) = B$$

The other two cases are similar. To establish the four cases requires both inversive properties from L_k.

The right inversive property is proved similarly. □

The next result follows directly but also be shown inductively relying on the inversive properties and was first shown in [24].

Proposition 5.3. *Each Cayley-Dickson loop satisfies the alternative properties.*

The alternative properties give the Cayley-Dickson loops some additional properties from the exhaustive list in [14], see also [7]. One of these properties is the C property and this appears to be the best way to describe the Cayley-Dickson loops, though related one sided properties are equivalent. The loops are C property loops that do not in general satisfy the extra properties and are therefore not in general Moufang.

The Cayley-Dickson loops are not only power-associative but di-associative with the groups specified by the following theorem.

Theorem 5.4. *Any pair of elements of a Cayley-Dickson loop generates a subgroup of the quaternion group.*

Proof. If $a, b \in \mathrm{L}_{k+1}$ and if the numbers are such that a, b and ab are each not ± 1 then $ab = -ba$. Then $k \geq 1$ and the alternative properties imply that $a(ab) = a^2 b = -b$ and $(ab)a = b$. Then a and b can be used to give a quaternion table and they generate a group isomorphic to the quaternion group. If any of the three numbers are real then they generate a group isomorphic to either the fourth roots of unity, the square roots of unity or 1. □

This theorem can used to give structure to the imaginary loop elements similar to the Fano plane. A geometry can be established for the imaginary loop elements by identifying a number with its negative. Lines are of the form $\{\pm a, \pm b, \pm ab\}$ and each line corresponds with a quaternion subgroup of the Cayley-Dickson loop. The incidence structure formed is known as a Steiner triple system for the larger loops and such relationships are previously known. One reference on the study of flexible algebras through the use of a Steiner basis is [22].

One classical example of a Steiner triple system is the incidence of points and lines in a finite dimensional projective geometry over the finite field $\mathbb{F}_2$. The Fano plane is the smallest example of a Steiner triple system but the above geometry begins with the most basic example. The *geometry of subloops* of the imaginary elements of the fourth roots of unity, the quaternion group and the octonion loop $\mathbb{O}_{\mathrm{L}}$ are a point, the line $PG(1, \mathbb{F}_2)$ and the plane $PG(2, \mathbb{F}_2)$ respectively. The next Cayley-Dickson loop is the sedenion loop $\mathbb{S}_{\mathrm{L}}$ and its geometry of subloops is $PG(3, \mathbb{F}_2)$. This is established in [7] though not using geometric terminology. In order to establish the general result, each Cayley-Dickson algebra needs a representation and the representation is defined recursively relying on the previous algebra.

With $a_n \in \mathbb{R}$, $\mathbf{i}_n \in \mathbb{A}_k$ for $0 \le n < 2^k$ and $k \ge 0$ one has

$$\sum_{n=0}^{2^k-1} a_n \begin{bmatrix} \mathbf{i}_n & 0 \\ 0 & \overline{\mathbf{i}_n} \end{bmatrix} + \sum_{n=0}^{2^k-1} a_{n+2^k} \begin{bmatrix} 0 & \mathbf{i}_n \\ -\overline{\mathbf{i}_n} & 0 \end{bmatrix} = \sum_{n=0}^{2^{k+1}-1} a_n \mathbf{i}_n \quad \in \mathbb{A}_{k+1}$$

For convenience the arrays are abbreviated as the pairs $(\mathbf{i}_n, 0)$ and $(0, \mathbf{i}_n)$. For $\mathbb{A}_{k+1}$ and for a non-negative integer $n = \sum_{i=0}^{k} n_i 2^i$ define the *index vector* $\overline{n} = [n_0, n_1 \ldots, n_k]$. Let $\mathbf{i}_n \in \mathrm{L}_{k+1}$. As $|\mathrm{L}_{k+1}| = 2 \cdot 2^{k+1}$, one has $0 \le n < 2^{k+1}$. If $0 \le n < 2^k$ then $\mathbf{i}_{\overline{n}} = (\mathbf{i}_{\overline{n}}, 0)$, where the indices are equal except for an extra zero entry. If $2^k \le n < 2^{k+1}$ then $\mathbf{i}_{\overline{n}} = (0, \mathbf{i}_{\overline{n}_*})$, where the indices are equal except for an extra entry one.

Lemma 5.5. *For all $\mathbf{i}_m, \mathbf{i}_n \in \mathrm{L}_k \subset \mathbb{A}_k, k \ge 1$, one has $\{\pm \mathbf{i}_{\overline{m}} \mathbf{i}_{\overline{n}}\} = \{\pm \mathbf{i}_{\overline{n}} \mathbf{i}_{\overline{m}}\} = \{\pm \mathbf{i}_{\overline{m}+\overline{n}}\}$.*

Proof. For all $a, b \in \mathrm{L}_k$, ba equals either $\pm ab$. Show for all $\mathbf{i}_m, \mathbf{i}_n \in \mathrm{L}_k$, that $\mathbf{i}_{\overline{m}} \mathbf{i}_{\overline{n}}$ equals either $\pm \mathbf{i}_{\overline{m}+\overline{n}}$.

This is true for L_1.

Assume true for $k = l$ and show for $k = l + 1$.

Let $\mathbf{i}_m, \mathbf{i}_n \in \mathrm{L}_{l+1}$.

Case 1. If $0 \le m, n < 2^l$, then

$$\mathbf{i}_{\overline{m}} \mathbf{i}_{\overline{n}} = (\mathbf{i}_{\overline{m}}, 0) \cdot (\mathbf{i}_{\overline{n}}, 0) = (\mathbf{i}_{\overline{m}} \mathbf{i}_{\overline{n}}, 0) = (\pm \mathbf{i}_{\overline{m}+\overline{n}}, 0) = \pm \mathbf{i}_{\overline{m}+\overline{n}}$$

Case 2. If $0 \le m < 2^l$ and $2^l \le n < 2^{l+1}$, then

$$\mathbf{i}_{\overline{m}} \mathbf{i}_{\overline{n}} = (\mathbf{i}_{\overline{m}}, 0) \cdot (0, \mathbf{i}_{\overline{n}_*}) = (0, \mathbf{i}_{\overline{n}_*} \mathbf{i}_{\overline{m}}) = (0, \pm \mathbf{i}_{\overline{n}_*+\overline{m}}) = \pm \mathbf{i}_{\overline{m}+\overline{n}}$$

Case 3. If $0 \le n < 2^l$ and $2^l \le m < 2^{l+1}$, then

$$\mathbf{i}_{\overline{m}} \mathbf{i}_{\overline{n}} = (0, \mathbf{i}_{\overline{m}_*}) \cdot (\mathbf{i}_{\overline{n}}, 0) = (0, \mathbf{i}_{\overline{m}_*} \overline{\mathbf{i}_{\overline{n}}}) = (0, \pm \mathbf{i}_{\overline{m}_*+\overline{n}}) = \pm \mathbf{i}_{\overline{m}+\overline{n}}$$

Case 4. If $2^l \le m, n < 2^{l+1}$, then

$$\mathbf{i}_{\overline{m}} \mathbf{i}_{\overline{n}} = (0, \mathbf{i}_{\overline{m}_*}) \cdot (0, \mathbf{i}_{\overline{n}_*}) = (-\overline{\mathbf{i}_{\overline{m}_*}} \mathbf{i}_{\overline{n}_*}, 0) = (\pm \mathbf{i}_{\overline{m}_*+\overline{n}_*}, 0) = \pm \mathbf{i}_{\overline{m}+\overline{n}}$$

In each case the last equality comes from the specific vector index extended appropriately. So the result is true for L_{l+1} and all L_k. □

The lemma then leads to the following theorem.

Theorem 5.6. *The geometry of the subloops of the Cayley-Dickson loop L_{k+1} is $PG(k, \mathbb{F}_2)$, $k \ge 0$.*

Proof. The projective geometries over $\mathbb{F}_2$ are the geometry of vector subspaces over $\mathbb{F}_2$. $\pm \mathbf{i}_0$ corresponds to the zero space and the geometry has been established for $k = 0$. For $k > 0$, any line from the loop L_{k+1} is of the form

$$\{\pm \mathbf{i}_m, \pm \mathbf{i}_n, \pm \mathbf{i}_m \mathbf{i}_n\} = \{\pm \mathbf{i}_{\overline{m}}, \pm \mathbf{i}_{\overline{n}}, \pm \mathbf{i}_{\overline{m}+\overline{n}}\}, \qquad m \ne n, m \ne 0, n \ne 0$$

In the vector space over $\mathbb{F}_2$ the span of two distinct non-zero vectors is

$$< \overline{m}, \overline{n} >= \{\overline{0}, \overline{m}, \overline{n}, \overline{m} + \overline{n}\}$$

As in projective geometry the zero space is not a point and the geometry is determined by the incidence of points and lines, the result is established. □

The theorem does not determine the type of subloops or equivalence of subloops. It does establish the number and relationships among subloops of a particular cardinality in L_{k+1}. The next section provides an example that non-equivalent subloops of the same cardinality exist.

Additionally the theorem suggests the existence of a $k+1$-linear, non-trivial mapping for each $\mathbb{A}_{k+1}$. The mapping come form the fact that in $PG(k, \mathbb{F}_2)$ a fundamental frame of consists of $k+2$ points. The linear mapping could then lead to a product to complete a partial frame of $k+1$ points as the cross and tri-products complete the frames for the pure quaternions and octonions respectively.

6 The sedenions

The sixteen dimensional Cayley-Dickson algebra over $\mathbb{R}$, the *sedenions* $\mathbb{S}$, is now examined. The sedenions have zero divisors and $\mathbb{S}$ is therefore not a division algebra. One example of a product which is zero follows with entries in $\mathbb{O}$. It was made using intersecting lines in opposite directions in the octonion Fano plane,

$$\begin{bmatrix} \mathbf{i}_1 & \mathbf{i}_5 \\ \mathbf{i}_5 & -\mathbf{i}_1 \end{bmatrix} \circ \begin{bmatrix} \mathbf{i}_7 & \mathbf{i}_3 \\ \mathbf{i}_3 & -\mathbf{i}_7 \end{bmatrix} \tag{6.1}$$

Beginning with the above representation for $\mathbb{O}$ the appropriate arrays are defined and decomposed. An isomorphism is then established for sixteen basis elements. This leads to a table of sedenion multiplication that can be built from the octonion table and is also found in [7]. The table gives the thirty two element *sedenion loop* $\mathbb{S}_{\mathrm{L}}$. Given the size of the table it is presented in four pieces with the first being the octonion table.

Sedenion (ii)

$\cdot$	$\mathbf{i}_8$	$\mathbf{i}_9$	$\mathbf{i}_{10}$	$\mathbf{i}_{11}$	$\mathbf{i}_{12}$	$\mathbf{i}_{13}$	$\mathbf{i}_{14}$	$\mathbf{i}_{15}$
$\mathbf{i}_0$	$\mathbf{i}_8$	$\mathbf{i}_9$	$\mathbf{i}_{10}$	$\mathbf{i}_{11}$	$\mathbf{i}_{12}$	$\mathbf{i}_{13}$	$\mathbf{i}_{14}$	$\mathbf{i}_{15}$
$\mathbf{i}_1$	$\mathbf{i}_9$	$-\mathbf{i}_8$	$-\mathbf{i}_{11}$	$\mathbf{i}_{10}$	$-\mathbf{i}_{13}$	$\mathbf{i}_{12}$	$\mathbf{i}_{15}$	$-\mathbf{i}_{14}$
$\mathbf{i}_2$	$\mathbf{i}_{10}$	$\mathbf{i}_{11}$	$-\mathbf{i}_8$	$-\mathbf{i}_9$	$-\mathbf{i}_{14}$	$-\mathbf{i}_{15}$	$\mathbf{i}_{12}$	$\mathbf{i}_{13}$
$\mathbf{i}_3$	$\mathbf{i}_{11}$	$-\mathbf{i}_{10}$	$\mathbf{i}_9$	$-\mathbf{i}_8$	$-\mathbf{i}_{15}$	$\mathbf{i}_{14}$	$-\mathbf{i}_{13}$	$\mathbf{i}_{12}$
$\mathbf{i}_4$	$\mathbf{i}_{12}$	$\mathbf{i}_{13}$	$\mathbf{i}_{14}$	$\mathbf{i}_{15}$	$-\mathbf{i}_8$	$-\mathbf{i}_9$	$-\mathbf{i}_{10}$	$-\mathbf{i}_{11}$
$\mathbf{i}_5$	$\mathbf{i}_{13}$	$-\mathbf{i}_{12}$	$\mathbf{i}_{15}$	$-\mathbf{i}_{14}$	$\mathbf{i}_9$	$-\mathbf{i}_8$	$\mathbf{i}_{11}$	$-\mathbf{i}_{10}$
$\mathbf{i}_6$	$\mathbf{i}_{14}$	$-\mathbf{i}_{15}$	$-\mathbf{i}_{12}$	$\mathbf{i}_{13}$	$\mathbf{i}_{10}$	$-\mathbf{i}_{11}$	$-\mathbf{i}_8$	$\mathbf{i}_9$
$\mathbf{i}_7$	$\mathbf{i}_{15}$	$\mathbf{i}_{14}$	$-\mathbf{i}_{13}$	$-\mathbf{i}_{12}$	$\mathbf{i}_{11}$	$\mathbf{i}_{10}$	$-\mathbf{i}_9$	$-\mathbf{i}_8$

Sedenion (iii)

$\cdot$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_6$	$\mathbf{i}_7$
$\mathbf{i}_8$	$\mathbf{i}_8$	$-\mathbf{i}_9$	$-\mathbf{i}_{10}$	$-\mathbf{i}_{11}$	$-\mathbf{i}_{12}$	$-\mathbf{i}_{13}$	$-\mathbf{i}_{14}$	$-\mathbf{i}_{15}$
$\mathbf{i}_9$	$\mathbf{i}_9$	$\mathbf{i}_8$	$-\mathbf{i}_{11}$	$\mathbf{i}_{10}$	$-\mathbf{i}_{13}$	$\mathbf{i}_{12}$	$\mathbf{i}_{15}$	$-\mathbf{i}_{14}$
$\mathbf{i}_{10}$	$\mathbf{i}_{10}$	$\mathbf{i}_{11}$	$\mathbf{i}_8$	$-\mathbf{i}_9$	$-\mathbf{i}_{14}$	$-\mathbf{i}_{15}$	$\mathbf{i}_{12}$	$\mathbf{i}_{13}$
$\mathbf{i}_{11}$	$\mathbf{i}_{11}$	$-\mathbf{i}_{10}$	$\mathbf{i}_9$	$\mathbf{i}_8$	$-\mathbf{i}_{15}$	$\mathbf{i}_{14}$	$-\mathbf{i}_{13}$	$\mathbf{i}_{12}$
$\mathbf{i}_{12}$	$\mathbf{i}_{12}$	$\mathbf{i}_{13}$	$\mathbf{i}_{14}$	$\mathbf{i}_{15}$	$\mathbf{i}_8$	$-\mathbf{i}_9$	$-\mathbf{i}_{10}$	$-\mathbf{i}_{11}$
$\mathbf{i}_{13}$	$\mathbf{i}_{13}$	$-\mathbf{i}_{12}$	$\mathbf{i}_{15}$	$-\mathbf{i}_{14}$	$\mathbf{i}_9$	$\mathbf{i}_8$	$\mathbf{i}_{11}$	$-\mathbf{i}_{10}$
$\mathbf{i}_{14}$	$\mathbf{i}_{14}$	$-\mathbf{i}_{15}$	$-\mathbf{i}_{12}$	$\mathbf{i}_{13}$	$\mathbf{i}_{10}$	$-\mathbf{i}_{11}$	$\mathbf{i}_8$	$\mathbf{i}_9$
$\mathbf{i}_{15}$	$\mathbf{i}_{15}$	$\mathbf{i}_{14}$	$-\mathbf{i}_{13}$	$-\mathbf{i}_{12}$	$\mathbf{i}_{11}$	$\mathbf{i}_{10}$	$-\mathbf{i}_9$	$\mathbf{i}_8$

Sedenion (iv)

$\cdot$	$\mathbf{i}_8$	$\mathbf{i}_9$	$\mathbf{i}_{10}$	$\mathbf{i}_{11}$	$\mathbf{i}_{12}$	$\mathbf{i}_{13}$	$\mathbf{i}_{14}$	$\mathbf{i}_{15}$
$\mathbf{i}_8$	-1	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_6$	$\mathbf{i}_7$
$\mathbf{i}_9$	$-\mathbf{i}_1$	-1	$-\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_5$	$\mathbf{i}_4$	$\mathbf{i}_7$	$-\mathbf{i}_6$
$\mathbf{i}_{10}$	$-\mathbf{i}_2$	$\mathbf{i}_3$	-1	$-\mathbf{i}_1$	$-\mathbf{i}_6$	$-\mathbf{i}_7$	$\mathbf{i}_4$	$\mathbf{i}_5$
$\mathbf{i}_{11}$	$-\mathbf{i}_3$	$-\mathbf{i}_2$	$\mathbf{i}_1$	-1	$-\mathbf{i}_7$	$\mathbf{i}_6$	$-\mathbf{i}_5$	$\mathbf{i}_4$
$\mathbf{i}_{12}$	$-\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_6$	$\mathbf{i}_7$	-1	$-\mathbf{i}_1$	$-\mathbf{i}_2$	$-\mathbf{i}_3$
$\mathbf{i}_{13}$	$-\mathbf{i}_5$	$-\mathbf{i}_4$	$\mathbf{i}_7$	$-\mathbf{i}_6$	$\mathbf{i}_1$	-1	$\mathbf{i}_3$	$-\mathbf{i}_2$
$\mathbf{i}_{14}$	$-\mathbf{i}_6$	$-\mathbf{i}_7$	$-\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_2$	$-\mathbf{i}_3$	-1	$\mathbf{i}_1$
$\mathbf{i}_{15}$	$-\mathbf{i}_7$	$\mathbf{i}_6$	$-\mathbf{i}_5$	$-\mathbf{i}_4$	$\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_1$	-1

The connection between the imaginary loop elements of $\mathbb{S}_{\mathrm{L}}$ and the three dimensional projective geometry over $\mathbb{F}_2$ can be used to understand subalgebras of the sedenions. A partition of projective three space into lines is known as a *spread*. Two such partitions follow that use sequences as the lines have been given an orientation:

$$\{[\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3], [\mathbf{i}_6, \mathbf{i}_{11}, \mathbf{i}_{13}], [\mathbf{i}_7, \mathbf{i}_9, \mathbf{i}_{14}], [\mathbf{i}_5, \mathbf{i}_{10}, \mathbf{i}_{15}], [\mathbf{i}_4, \mathbf{i}_8, \mathbf{i}_{12}]\}$$
$$\{[\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3], [\mathbf{i}_{13}, \mathbf{i}_{10}, \mathbf{i}_7], [\mathbf{i}_{14}, \mathbf{i}_{11}, \mathbf{i}_5], [\mathbf{i}_{15}, \mathbf{i}_9, \mathbf{i}_6], [\mathbf{i}_4, \mathbf{i}_8, \mathbf{i}_{12}]\}$$

The spreads each correspond to five copies of the quaternion group which only intersect in the square roots of unity or five copies of the quaternion algebra that intersect only in the real subalgebra. Additionally the two spreads each contain a ruling family of lines for the same hyperbolic quadric in $PG(3, \mathbb{F}_2)$. For $1 \leq n \leq 15$,

$$n = x_0 \cdot 2^0 + x_1 \cdot 2^1 + x_2 \cdot 2^2 + x_3 \cdot 2^3$$

which then is the index vector $[x_0, x_1, x_2, x_3]$ for the index n. The hyperbolic quadric is the following and it contains the nine points on the non repeating lines of the two spreads:

$$x_0^2 + x_0 x_1 + x_1^2 + x_2^2 + x_2 x_3 + x_3^2 = 0$$

Another way to understand the sedenions is by examining an *incidence tetrahedron*, such as $\mathbf{i}_5, \mathbf{i}_6, \mathbf{i}_7, \mathbf{i}_{12}$. Every other point is on a face of the tetrahedron with one exception $\mathbf{i}_8$. Together these five points are the points of an elliptic quadric in $PG(3, \mathbb{F}_2)$. The elliptic quadric is

$$x_0^2 + x_0 x_1 + x_1^2 + x_2^2 + x_2 x_3 = 0$$

As there are fifteen points in $PG(3, \mathbb{F}_2)$ there are fifteen planes corresponding to fifteen subloops of the sedenion loop $\mathbb{S}_{\mathrm{L}}$ each with sixteen elements. It is established in [7] that only eight of these subloops are octonion loops with the remaining seven known as *quasi-octonion loops* which are not Moufang loops. The basic octonion Fano plane gives an octonion subloop and the seven planes through the point $\pm\mathbf{i}_8$ give the other seven octonion subloops. The quasi-octonion loops each have a Fano plane description distinct from the octonion loops and each includes a line of the basic octonion Fano plane. The **Figures 2, 3** represent the two remaining planes through the line $[\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3]$ which are examples of these descriptions. The planes correspond to the dual coordinate vectors $[0, 0, 1, 0]^{\mathrm{t}}$ and $[0, 0, 1, 1]^{\mathrm{t}}$ respectively. The remaining descriptions are easily made with a line from the basic Fano plane as the circular arc.

Given the structure of $PG(3, \mathbb{F}_2)$ other geometric ideas can be applied to the subloops of $\mathbb{S}_{\mathrm{L}}$.

A cross and tri-product can be defined for the imaginary elements of $\mathbb{S}$ as in 7.2 and 7.3 respectively. This cross product has lost some properties of the standard cross product. For the pure sedenions this is demonstrated by the zero divisors 6.1 written in vector form as follows:

$$(\mathbf{i}_1 + \mathbf{i}_{13}) \times (\mathbf{i}_7 + \mathbf{i}_{11}) = 0$$

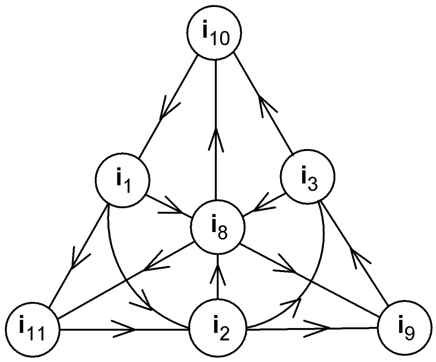

Figure 2. An Octonion Subloop

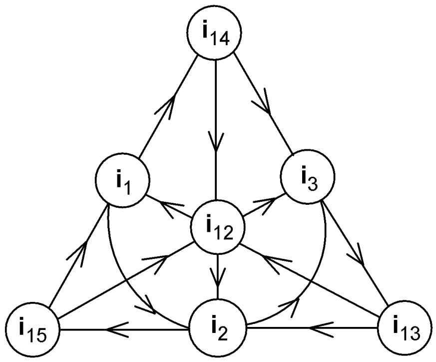

Figure 3. A Quasi-Octonion Subloop

The tri-product also has lost some properties even for the basis elements as is indicated by

$$\rangle\mathbf{i}_2,\mathbf{i}_1,\mathbf{i}_{12}\langle = \rangle\mathbf{i}_{12},\mathbf{i}_2,\mathbf{i}_1\langle = 0 \quad \text{but} \quad \rangle\mathbf{i}_2,\mathbf{i}_{12},\mathbf{i}_1\langle = \mathbf{i}_{15}$$

Despite these difficulties it is possible to give the fifteen imaginary basis vectors from a system of generators corresponding to an incidence tetrahedron. With $u = \mathbf{i}_1, v = \mathbf{i}_2, w = \mathbf{i}_4, z = \mathbf{i}_8$, beginning with basis of the pure octonions the remaining basis vectors are

$$z, \quad u\times z, \quad v\times z, \quad \rangle u,v,z\langle, \quad w\times z, \quad \rangle u,w,z\langle, \quad \rangle v,w,z\langle, \quad \mathbf{i}_{15}$$

Using the commu-associator

$$4\mathbf{i}_{15} = [\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_4,\mathbf{i}_8], \quad -4\mathbf{i}_{15} = [\mathbf{i}_1,\mathbf{i}_4,\mathbf{i}_2,\mathbf{i}_8], \quad \text{but} \quad [\mathbf{i}_2,\mathbf{i}_4,\mathbf{i}_8,\mathbf{i}_1] = 0$$

What prevents a well defined quadruple vector product is understanding how the eight components of the commu-associator lead to four non-zero elements when the commu-associator is non-zero.

7 General products

From the comments in [29], generalizations of the cross product for other Cayley-Dickson algebras seem appropriate. The general tri-product can be defined as all Cayley-Dickson algebras satisfy the flexible property. The Cayley-Dickson algebras involved could have zero divisors and these products will not retain all the properties of the products related to the division algebras.

For the Cayley-Dickson algebra $\mathbb{A}_k$, $k > 0$, define its set of *purely imaginary elements* by

$$Im(\mathbb{A}_k) = \{u : u \in \mathbb{A}_k \text{ and } \overline{u} = -u\} \quad \text{and} \quad \mathbb{R} \oplus Im(\mathbb{A}_k)$$

The *dot product* is first defined for $\mathbb{A}_k$ or equivalently for 2^k-dimensional real space by

$$\bullet : \mathbb{A}_k \times \mathbb{A}_k \longrightarrow \mathbb{R}, \qquad u \bullet v = < u, v > = Re(u\overline{v}) = \frac{1}{2}(u\overline{v} + v\overline{u}) \tag{7.1}$$

For $u, v \in Im(\mathbb{A}_k)$, one has

$$u \bullet v = -\frac{1}{2}(uv + vu)$$

The *cross* and *tri-product* are defined relying on the following proposition. The commu-associator is also closed for imaginary elements as it is defined using the commutator and associator.

Proposition 7.1. *For all $u, v, w, z \in Im(\mathbb{A}_k)$,*

$$[u, v] \in Im(\mathbb{A}_k), \quad [u, v, w] \in Im(\mathbb{A}_k) \quad \textit{and} \quad [u, v, w, z] \in Im(\mathbb{A}_k)$$

Define the following operations:

$$\times : Im(\mathbb{A}_k) \times Im(\mathbb{A}_k) \longrightarrow Im(\mathbb{A}_k)$$

$$u \times v = \frac{1}{2}[u, v] = \frac{1}{2}(uv - vu) \tag{7.2}$$

$$\rangle \cdot, \cdot, \cdot \langle : Im(\mathbb{A}_k) \times Im(\mathbb{A}_k) \times Im(\mathbb{A}_k) \longrightarrow Im(\mathbb{A}_k)$$

$$\rangle u, v, w \langle = \frac{1}{4}([u, v, w] - [w, v, u]) = \frac{1}{2}[u, v, w] = \frac{1}{2}((uv)w - u(vw)) \tag{7.3}$$

So the product of purely imaginary elements can be written in terms of the inner and cross product as for the pure quaternions,

$$uv = - < u, v > + (u \times v)$$

While it is true that the dot product, cross product and tri-product are non-degenerate multilinear forms, the tri-product does not have all the properties that would usually describe it as an alternating trilinear form. These basic properties are restatement of the properties of the associator:

(i) $\forall u, v, w \in Im(\mathbb{A}_k), \quad \rangle u, v, w \langle = - \rangle w, v, u \langle$
(ii) $\forall u, v, w \in Im(\mathbb{A}_k), \forall r \in \mathbb{R}, \quad r \cdot \rangle u, v, w \langle = \rangle ru, v, w \langle = \rangle u, rv, w \langle = \rangle u, v, rw \langle$
(iii) $\forall u, v, w, z \in Im(\mathbb{A}_k), \quad \rangle u, v, w + z \langle = \rangle u, v, w \langle + \rangle u, v, z \langle$
(iv) $\forall u, v \in Im(\mathbb{A}_k), \quad \rangle u, v, u \langle = 0$
(v) $\forall u, v \in Im(\mathbb{A}_k), \quad \rangle u, 0, v \langle = \rangle u, v, 0 \langle = 0$

The general cross product can be represented through the sum of the appropriate number of cubic determinants similar to 4.1 but a representation for the general tri-product similar to 4.2 seems more difficult. The number of cubic determinants for the cross product equals the number of lines in the corresponding finite geometry. As indicated previously, the cross product of two independent vectors can be zero. The orthogonality relationship for the cross product holds in general from the flexible and weak inversive properties.

Proposition 7.2. *For all $u, v \in Im(\mathbb{A}_k)$, $u \times v$ is orthogonal to both u and v.*

Proof. Let $u, v \in Im(\mathbb{A}_k)$. Then

$$\begin{aligned} -4 \cdot u \bullet (u \times v) &= u[u,v] + [u,v]u = u(uv - vu) + (uv - vu)u \\ &= u(uv) - u(vu) + (uv)u - (vu)u = (vu)\overline{u} - \overline{u}(uv) = 0 \end{aligned}$$

The result is similar for v. □

Properties of the tri-product similar to the properties of the pure octonions seem more difficult as the author's proofs rely on properties of the octonions. One property of the dot and cross product comes from the fact that the associator of imaginary numbers is closed and the product of three vectors:

$$(uv)w = -(u \times v) \bullet w - < u, v > w + (u \times v) \times w$$

Proposition 7.3. *For all $u, v, w \in Im(\mathbb{A}_k)$, $u \bullet (v \times w) = (u \times v) \bullet w$.*

One convenient relationship between the cross and tri-products comes from a generalized Jacobi identity.

Proposition 7.4. *For all $u, v, w \in Im(\mathbb{A}_k)$,*

$$(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = \rangle u, v, w\langle + \rangle v, w, u\langle + \rangle w, u, v\langle$$

Also it follows from the flexible property that the following relation holds.

Proposition 7.5. *For all $u, v, w, z \in Im(\mathbb{A}_k)$,*

$$\rangle u, v, w \times z\langle + \rangle u \times v, w, z\langle = u \times \rangle v, w, z\langle + \rangle u, v \times w, z\langle + \rangle u, v, w\langle \times z$$

From the fact that the standard basis elements and their negatives are the loop L_k with the property that $ab = \pm ba$ gives the following

Theorem 7.6. *For $Im(\mathbb{A}_k)$, the imaginary loop elements of L_k are closed under cross product when extending to include zero.*

Closure under the tri-product is related to the type of subloops of sixteen elements. Using the relationship between the basis for $Im(\mathbb{A}_k)$, ordered numerically, and the appropriate vector space over $\mathbb{R}$ with the canonical basis of vectors gives the following

Corollary 7.7. *For the $2^k - 1$ dimensional vector space over $\mathbb{R}$, the canonical basis vectors are closed under cross product and when extended by negatives and the zero vector.*

Restating this to include the earlier result on finite geometries gives another

Corollary 7.8. *For the $2^k - 1$ dimensional vector space over $\mathbb{R}$ with canonical basis vectors, the geometry of its axes is $PG(k-1, \mathbb{F}_2)$ under cross product.*

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