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Emergent (2+1)D topological orders from iterative (1+1)D gauging

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Gauging introduces gauge fields in order to localize an existing global symmetry, resulting in a dual global symmetry on the gauge fields that can be gauged again. By iterating the gauging process on spin chains with Abelian group symmetries and arranging the gauge fields in a 2D lattice, the local symmetries become the stabilizer of the *XZZX*-code for any Abelian group. By twisting the gauging map, we obtain codes that explicitly confine anyons, whose local creating operators violate an odd number of plaquettes. Their fusion results in either mobile dipole excitations twisting only half of the plaquette terms, or complete immobile Sierpiński-like excitations if we twist all the terms. Our construction naturally realizes any gapped boundary by taking different quantum phases of the initial (1+1)D globally symmetric system. In addition, our method also establishes a promising route to obtain higherdimensional topological codes from lower ones and to identify their gapped boundaries and their tensor network representations.

Gauging is fundamental in the Standard Model to comprehend and unify forces. It transforms a system, promoting its global symmetry to a local symmetry by introducing new degrees of freedom known as gauge fields. While the initial motivation for gauging was Lagrangians supported on the continuum with Lie group symmetries, the gauging of quantum lattice Hamiltonians with finite group symmetries has gained significant attention[1–3].

The power of gauging lies in the fact that it connects very distinct phases of matter, which makes it the standard tool to classify quantum phases and to prove the existence of anomalies[4–6]. Since gauging global (1+1)D symmetries results in emergent dual global symmetries (which could be non-invertible for non-Abelian groups[7,8]) this turns gauging into the source of dualities in (1+1)D[9]. In (2+1)D, the emergent symmetries give rise to a very rich phenomena including 1-form and surface symmetries[10]. Gauging has also been generalized to other settings beyond on-site global symmetries, including non-on-site global symmetries[11] and higher form symmetries[6,12–14], leading to the creation of fractal phases[15,16].

All previous gauging and duality setups relate systems in the same physical dimension. In this work, we use gauging to establish a bulk–boundary correspondence: the construction of a (2+1)D topologically ordered system (with local symmetries) from (1+1)D globally symmetric systems.

To achieve this, we iteratively gauge the emergent 1D global symmetries of the new gauge fields for finite Abelian groups. Since the corresponding matter fields are not discarded at every step, we arrange them as the horizontal layers of the newly constructed 2D lattice. Unexpectedly, the local symmetries from each gauging, modified by the composition of the subsequent maps, become the stabilizers of the generalization of the *XZZX*-code[17](a realization of the toric code[18] proposed in[19]) for any Abelian group. By twisting the gauging map by a 2-cocycle[20,21], we explicitly confined anyons that now violate an odd number of plaquette terms and whose fusion results in mobile dipoles or completely immobile Sierpiński-like excitations.

The different gapped boundaries (and hence the condensable anyons at the boundary) of our construction depend on the quantum phase of the initial (1+1)D globally symmetric system. We show this by establishing a connection between boundary Hamiltonian terms and (1+1)D string order parameters evaluated on the initial system. Such connection illuminates the fact that both settings, gapped boundaries of quantum doubles of G and (1+1)D quantum phases with global symmetries, are classified by the same mathematical object.

Since the gauging operator is a tensor network, our 2D construction inherits that structure, giving rise to a subfamily of projected entangled pair states (PEPS)[22] that we refer to as projected entangled

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pair emergent states (PEPES) that satisfy a different version of the virtual symmetry leading to topological ordered PEPS[23,24].

While preparing this manuscript, Ref. 25 appears where D + 1-dimensional qubit Hamiltonians are constructed by coupling D-dimensional Hamiltonians with multiple $\mathbb{Z}_2$ symmetries and their dual models by using generalized Kramers–Wannier dualities. We are not limited to qubit systems nor to order two symmetries.

Results

Gauging

The procedure of gauging maps globally symmetric operators and states $\{O,|\psi\rangle\}$, to local symmetric ones $\{\hat{O},|\hat{\psi}\rangle\}$ such that it preserves their expectation values: $\langle\psi|O|\psi\rangle=\langle\hat{\psi}|\hat{O}|\hat{\psi}\rangle$. It has been proven[3,15] that the gauging of a Hamiltonian with its zero gauge coupling limit can preserve the gap and the ground subspace.

The map is implemented by a gauging operator $\mathcal{G}_0$[2], that maps the initial *matter* Hilbert space $\mathcal{H}_0$ to $\mathcal{H}_0\otimes\mathcal{H}_1$, introducing new degrees of freedom (dof) supported in $\mathcal{H}_1$, called the gauge fields. Given a global symmetry of a finite Abelian group G represented as $\otimes_i u_g^i$ in $\mathcal{H}_0$, where $g\in G$ and i denotes the vertices of a 1D chain, the new introduced Hilbert space is $\mathcal{H}_1=\otimes_{\hat{i}}\mathbb{C}[G]^{\hat{i}}$, where $\hat{i}$ denotes the edge between i and $i+1$ and $\mathbb{C}[G]=\text{span}\{|g\rangle, g\in G\}$. We define in $\mathbb{C}[G]$ the unitary representation of $G\{X_g\}_{g\in G}$ as $X_g|h\rangle=|gh\rangle$ that allows to construct the local symmetry projectors $\mathcal{P}^i=\frac{1}{|G|}\sum_{g\in G}X_{g^{-1}}^{\hat{i}-1}\otimes u_g^i\otimes X_g^{\hat{i}}$. Then the global projector to the local symmetric subspace is $\mathcal{P}=\Pi_i\mathcal{P}^i$ such that the gauging operator is defined by $\mathcal{G}_0=\mathcal{P}(\otimes_{\hat{i}}|e\rangle_{\hat{i}})$, where e denotes the trivial group element and it satisfies

$$\left(X_{g^{-1}}^{\hat{i}-1,1}\cdot u_g^{i,0}\cdot X_g^{\hat{i},1}\right)\mathcal{G}_0=\mathcal{G}_0\quad\forall g\in G,\forall i,\tag{1}$$

where $j=0,1$ denotes the action on $\mathcal{H}_j$.

Finally gauged states are given by $|\hat{\psi}\rangle=\mathcal{G}_0|\psi\rangle$ and gauge operators by $\hat{O}\cdot\mathcal{G}_0=\mathcal{G}_0\cdot O$. As an example let us consider the transverse-field Ising model $H=-J(\sum_i X_iX_{i+1}+gZ_i)$ with global symmetry $\otimes_i Z_i$ that it is mapped under gauging to $\hat{H}=-J(\sum_i X_iZ_{\hat{i}}X_{i+1}+gZ_i)$ with local symmetry $X_{\hat{i}-1}Z_iX_{\hat{i}}$ and an *emergent* global symmetry $\otimes_{\hat{i}}Z_{\hat{i}}$ only supported on the gauge fields. Importantly, as we will show next, this emergent global symmetry is always present after gauging.

The emergent global dual symmetry

Let us define the operator $Z_{\hat{g}}=\sum_h\hat{g}(h)|h\rangle\langle h|$, associated to an irrep $\hat{g}:G\to U(1)$ of G, satisfying $X_g\cdot Z_{\hat{g}}=\hat{g}(g^{-1})\cdot Z_{\hat{g}}\cdot X_g$. Then, the global operator $\otimes_{\hat{i}}Z_{\hat{g}}^{\hat{i},1}$ commutes with the local symmetry of (1), so it does with $\mathcal{P}^i$, and it is a global symmetry of $\otimes_{\hat{i}}|e\rangle_{\hat{i}}$. Therefore, the gauged operators and the gauged states endow the following emergent dual global symmetry:

$$\otimes_{\hat{i}}Z_{\hat{g}}^{\hat{i},1}\cdot\mathcal{G}_0=\mathcal{G}_0\quad\forall\hat{g}\in\hat{G},\tag{2}$$

where the unitary operators $\{Z_{\hat{g}}\}$ are a representation of the dual group $\hat{G}$ of the irreps. In the Supplementary Note 1 we show that for non-Abelian groups the emergent global symmetry is Rep(G) and it comes from the zero gauge flux configuration.

In the literature, gauging also involves decoupling and projecting out the matter, resulting in just gauge fields with a global symmetry, which can be understood as a duality process. In the example, the decoupling process maps $\hat{H}\to\tilde{H}=-J(\sum_i Z_{\hat{i}}+gX_{\hat{i}-1}X_{\hat{i}})$, since $Z_i\to X_{\hat{i}-1}X_{\hat{i}}$ using the local symmetry XZX, which corresponds to the Kramers–Wannier duality[26].

Iterative Abelian gauging

The emergent global $\hat{G}$ symmetry can be gauged as well. To do so we construct the gauging map $\mathcal{G}_1:\mathcal{H}_1\to\mathcal{H}_1\otimes\mathcal{H}_2$ by first introducing $\mathcal{H}_2=\otimes_i\mathbb{C}[\hat{G}]^i$, defining the unitary representation of $\hat{G}\{X_{\hat{g}}\}_{\hat{g}\in\hat{G}}$ as $X_{\hat{g}}|\hat{h}\rangle=|\hat{g}\hat{h}\rangle$, and then projecting $\otimes_i|\hat{e}\rangle_{i,2}$ onto the local symmetric

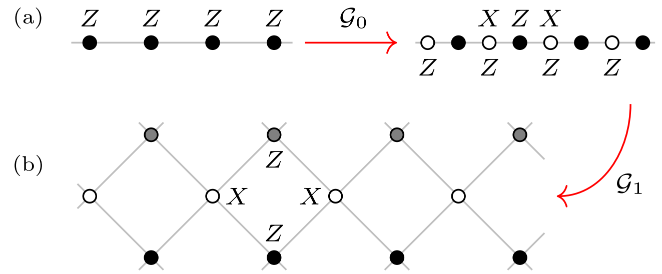

Fig. 1 | Sketch of the iterative gauging for $G=\mathbb{Z}_2$. a The global symmetry generated by $Z^{\otimes N}$ is gauged to a local $X\otimes Z\otimes X$ symmetry and a global $Z^{\otimes N}$ on the gauge fields. **b** Emergent local symmetries after applying the second gauging map.

subspace of $X_{\hat{g}^{-1}}^{i,2}\otimes Z_{\hat{g}}^{\hat{i},1}\otimes X_{\hat{g}}^{i,2}$. After composing both gauging maps $\mathcal{G}_1\circ\mathcal{G}_0:\mathcal{H}_0\to\mathcal{H}_0\otimes\mathcal{H}_1\otimes\mathcal{H}_2$, the initial local symmetry of $\mathcal{G}_0$, see (1), changes to

$$\left(X_{g^{-1}}^{\hat{i}-1,1}\cdot Z_{g^{-1}}^{i,2}\cdot u_g^{i,0}\cdot X_g^{\hat{i},1}\right)\cdot\mathcal{G}_1\circ\mathcal{G}_0=\mathcal{G}_1\circ\mathcal{G}_0,\tag{3}$$

where $Z_g=\sum_{\hat{h}}\hat{h}(g)|\hat{h}\rangle\langle\hat{h}|$ is a representation of G on $\mathbb{C}[\hat{G}]$ and it satisfies $Z_g\cdot X_{\hat{g}}=\hat{g}(g)\cdot X_{\hat{g}}\cdot Z_g$. To get Eq. (3) we just have to check that $\mathcal{G}_1(X_g^\dagger\otimes X_g)=(X_g^\dagger\otimes Z_g^\dagger\otimes X_g)\mathcal{G}_1$ which is how two point symmetric correlation functions (of the global symmetry $\hat{G}$) maps through gauging to string order parameters (of the global symmetry G):

$$X_g^{\hat{i},j}\cdot X_g^{\dagger\hat{i}',j}\xrightarrow{\mathcal{G}_1}X_g^{\hat{i},j}\cdot\left(\prod_{i\leq k<i'}Z_g^{k,j+1}\right)\cdot X_g^{\dagger\hat{i}',j}.\tag{4}$$

See Fig. 1 for an sketch. Again, there is an emergent global symmetry of G after gauging with $\mathcal{G}_1$ realized by Z_g acting on $\mathcal{H}_2$.

Therefore, we can iterate the gauging of the emergent global symmetries defining $\mathcal{G}_j:\mathcal{H}_j\to\mathcal{H}_j\otimes\mathcal{H}_{j+1}$ and compose M gauging maps:

$$\mathcal{G}\equiv\mathcal{G}_{M-1}\circ\cdots\circ\mathcal{G}_1\circ\mathcal{G}_0:\mathcal{H}_0\to\bigotimes_{j=0}^{M}\mathcal{H}_j,\tag{5}$$

where $\mathcal{G}_0$ is related to $\mathcal{G}_j\equiv\mathcal{G}_e$ by $u_g\leftrightarrow Z_g$ with j even and $\mathcal{G}_1=\mathcal{G}_j\equiv\mathcal{G}_o$ when j odd.

2D lattice from iterative 1D gauging

We place every new Hilbert space $\mathcal{H}_{j+1}$ coming from $\mathcal{G}_j:\mathcal{H}_j\to\mathcal{H}_j\otimes\mathcal{H}_{j+1}$, on the next layer of a 2D array. Every local Hilbert space $\mathbb{C}[G]$ will be on the vertices $\{i\}$, and $\mathbb{C}[\hat{G}]$ on the edges $\{\hat{i}\}$ (placed between i and $i+1$). This creates a rotated squared lattice with the following symmetries:

$$\left(X_{g^{-1}}^{\hat{i}-1,j}\cdot Z_{g^{-1}}^{i,j+1}\cdot Z_g^{i,j-1}\cdot X_g^{\hat{i},j}\right)\cdot\mathcal{G}=\mathcal{G}, j\text{ odd}$$
$$\left(X_{\hat{g}^{-1}}^{i-1,j}\cdot Z_{\hat{g}^{-1}}^{\hat{i},j+1}\cdot Z_{\hat{g}}^{\hat{i},j-1}\cdot X_{\hat{g}}^{i,j}\right)\cdot\mathcal{G}=\mathcal{G}, j\text{ even}$$

see Fig. 2. These local symmetries commute since $[Z_g^\dagger\otimes X_g, X_{\hat{g}}^\dagger\otimes Z_{\hat{g}}]=0$. Remarkably, for $G=\mathbb{Z}_2$ they are the stabilizers of the $XZZX$-code[17], which is a different realization of the toric code[18,19].

So our state $\mathcal{G}$ is a common (+1) eigenstate of the aforementioned commuting stabilizer terms which can be seen as the ground state of the topological code Hamiltonian:

$$H_{G,\text{bulk}}^{\text{Emerg.}}=-\sum_{\substack{g\in G\\ e\neq g}}\begin{smallmatrix}& Z_g^\dagger & \\ X_g^\dagger & & X_g\\ & Z_g & \end{smallmatrix}-\sum_{\substack{\hat{g}\in\hat{G}\\ \hat{g}\neq\hat{e}}}\begin{smallmatrix}& Z_{\hat{g}}^\dagger & \\ X_{\hat{g}}^\dagger & & X_{\hat{g}}\\ & Z_{\hat{g}} & \end{smallmatrix}\tag{6}$$

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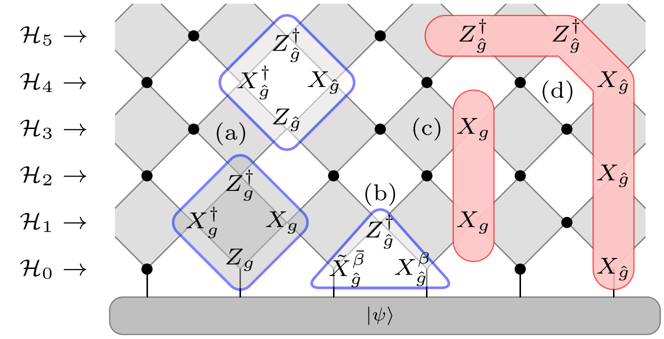


Fig. 2 | Main operators on the emergent 2D system. a Commuting bulk Hamiltonian terms. **b** Generic boundary term. **c** Vertical anyon given by $g \in G$ that could condense on the boundary. **d** Concatenation of horizontal and vertical anyons given by $\hat{g} \in \hat{G}$.

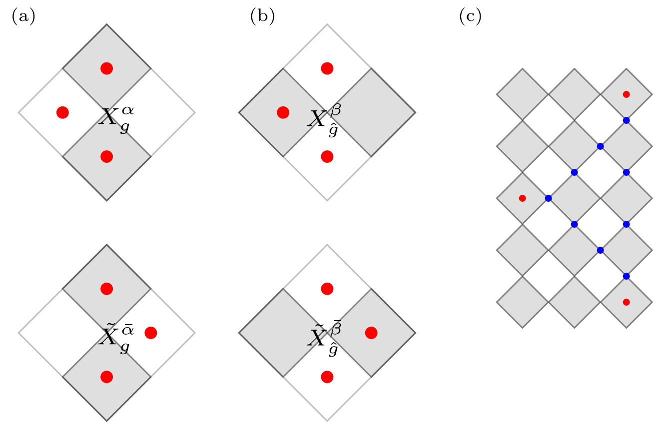


Fig. 3 | Twisted operators and their excitations. a G-twisted operators violating three plaquette terms depicted in red. **b** $\hat{G}$-twisted operators violating three plaquette terms depicted in red. **c** Combining twisted operators with the shape of the Sierpiński triangle, acting on the blue vertices, excitations on the red plaquettes are created.

Therefore, we have constructed the generalization of the *XZZX*-code for any Abelian group G by using the emergent symmetries of the concatenation of (1 + 1)D gauging operators.

$H^{\text{Emerg.}}_{G,\text{bulk}}$ commutes with $\{\otimes_i Z^{i,j}_{\hat{g}}\}^{\hat{g}}_{j\,\text{odd}}$ and $\{\otimes_i Z^{i,j}_{g}\}^{g}_{j\,\text{even}}$, the emergent global symmetries of the ground state $\mathcal{G}$, which correspond to the horizontal logical operators. The vertical logical operators are $\{\otimes_j X^{\hat{i},\hat{j}}_{\hat{g}}\}^{\hat{g}}_{\hat{i}}$ and $\{\otimes_j X^{i,j}_{g}\}^{g}_{i}$, that applied to $\mathcal{G}$ generate the $|G|^2$ ground states of $H^{\text{Emerg.}}_{G,\text{bulk}}$.

Twisting the bulk

The gauging map can be twisted, as introduced in ref. 21, by a 2-cocycle $\alpha \in \mathcal{H}^2[G, U(1)]$. To do so we introduce the α and $\bar{\alpha}$ projective representations X^α_g and $\tilde{X}^{\bar{\alpha}}_g$ defined by $X^\alpha_g|h\rangle = \alpha(g,h)|gh\rangle$ and $\tilde{X}^{\bar{\alpha}}_g|h\rangle = \bar{\alpha}(hg^{-1},g)|hg^{-1}\rangle$. These two representations commute so we construct $\mathcal{P}_\alpha = |G|^{-1}\sum_{g\in G}\tilde{X}^{\bar{\alpha}}_g \otimes Z_g \otimes X^\alpha_g$ and define $\mathcal{G}_\alpha = \prod_i P^i_\alpha \otimes_{\hat{i}} |e\rangle^{\hat{i}}$. Importantly the twisted gauging operator also realizes the same emergent dual symmetry of $\hat{G}$: $\otimes_i Z^{\hat{i}}_{\hat{g}} \cdot \mathcal{G}_\alpha = \mathcal{G}_\alpha$ since the operators satisfy $Z_{\hat{g}} \cdot X^\alpha_g = \hat{g}(g) \cdot X^\alpha_g \cdot Z_{\hat{g}}$.

For untwisted gauging maps on odd layers, the emergent Hamiltonian resulting from concatenating $\mathcal{G}_0\mathcal{G}_\alpha$, shares the $\hat{G}$-plaquette terms of (6). However, G-plaquette terms are now:

$$B^\alpha_g = \begin{array}{ccc} & Z^\dagger_g & \\ \tilde{X}^{\bar{\alpha}}_g & & X^\alpha_g \\ & Z_g & \end{array} \Rightarrow \prod B^\alpha_g = \bigotimes_{i,j} Z^{i,j}_{\imath_g\alpha}\ , \tag{7}$$

where $\imath_g\alpha$ (the so-called slant product) belongs to $\hat{G}$ since $\imath_g\alpha(h) = \frac{\alpha(g,h)}{\alpha(h,g)}$ and where we have used $X^\alpha_g \cdot \tilde{X}^{\bar{\alpha}}_g = Z_{\imath_g\alpha}$. The fact that the product of all G-plaquette terms is the product of the horizontal $\hat{G}$-logical operators, Eq. (7), and not the identity has several consequences. First, the only logical vertical operators are $\{\otimes_j X^{\hat{i},\hat{j}}_{\hat{g}}\}^{\hat{g}}_{\hat{i}}$, since $\{\otimes_j X^{i,j}_{g}\}^{g}_{i}$ do not commute with B^α_g. Then the former generate just a $|G|$-fold ground space: the topological order has changed. Second, there are local operators, X^α_g and $\tilde{X}^{\bar{\alpha}}_g$, that violate (depicted as a red dot) an odd number of plaquettes, see Fig. 3a.

These excitations are confined creating strings whose energy grow with their length. But gluing them together $\tilde{X}^{\bar{\alpha}}_g \otimes X^\alpha_g$, a dipole excitation is created that moves free vertically. One can bend the dipole excitation by acting with Z_g horizontally, leaving an excitation on the corner or splitting the excitation on to right and left. Notice that the dipole commutes with the horizontal $Z_{\hat{g}}$-string excitations: so these two kind of excitations braid trivially. The only remaining anyons are vertical $X_{\hat{g}}$-strings and horizontal Z_g-strings that braid non-trivially.

We can also twist the odd layers by $\beta \in \mathcal{H}^2[\hat{G}, U(1)]$ where the local excitations are created by $X^\beta_{\hat{g}}$ and $\tilde{X}^{\bar{\beta}}_{\hat{g}}$, depicted in Fig. 3b. The combine action of $X^\beta_{\hat{g}}$ and X^α_g with the shape of the Sierpiński fractal generate excitations at its corners, see Fig. 3c. We note that twisting all the layers reduces drastically the topological order (depending on α and β).

Boundary conditions

In this section we consider periodic boundary conditions (PBC) in the horizontal direction by using PBC gauging operators—see Supplementary Note 3 on how to define the gauging map with open boundary conditions (OBC). We also take $u_g = Z_g$ for simplicity. The vertical boundaries of $\mathcal{G}$, see Eq. (5), correspond to the input Hilbert space $\mathcal{H}_0$ and the last gauge fields introduced $\mathcal{H}_M$.

The case of PBC on the vertical boundaries correspond to the state $\text{Tr}_{\mathcal{H}_0=\mathcal{H}_M}[\mathcal{G}] \in \otimes^{M-1}_{j=1}\mathcal{H}_j$, whenever M is even, which results in a square rotated lattice in a torus.

For vertical OBC, we close the boundaries with the states $|\psi\rangle \in \mathcal{H}_0$ and $|\psi'\rangle \in \mathcal{H}_M$ which are globally symmetric under G (or $\hat{G}$ for the case of $|\psi'\rangle$ if M is odd). Therefore, the resulting state is $\langle\psi'|\mathcal{G}|\psi\rangle \in \otimes^{M-1}_{j=1}\mathcal{H}_j$.

There are two kind of boundary terms that commute with the bulk stabilizers. The first one can be chosen to be (see Fig. 2)

$$\tilde{X}^{\bar{\beta}}_{\hat{g}} \otimes Z^\dagger_{\hat{g}} \otimes X^\beta_{\hat{g}}, \quad \beta \in \mathcal{H}^2[\hat{G}, U(1)], \tag{8}$$

and the second one corresponds to single Z_g acting on the first layer. The first type affects the anyons labeled by G and the second influences the anyons indexed by $\hat{G}$.

The appearance of the boundary terms as symmetries of $\langle\psi'|\mathcal{G}|\psi\rangle$ will be determined by the quantum phases of $|\psi\rangle$ and $|\psi'\rangle$. Let us first describe the situation for the first type of boundary terms. If we concatenate ℓ of these terms its action on $|\psi\rangle$ through $\mathcal{G}$ is $\tilde{X}^{\bar{\beta}}_{\hat{g}} \otimes Z^{\otimes\ell}_{\imath_{\hat{g}}\beta} \otimes X^\beta_{\hat{g}}$. So the term of Eq. (8) is a symmetry of $\mathcal{G}|\psi\rangle$ only if

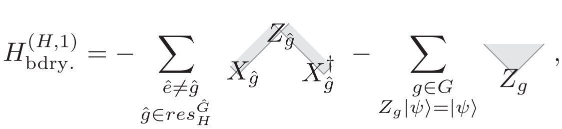
$\langle\psi|\tilde{X}^{\bar{\beta}}_{\hat{g}} \otimes Z^{\otimes \ell}_{\imath_{\hat{g}}\beta} \otimes X^{\beta}_{\hat{g}}|\psi\rangle = 1$. This corresponds to the expectation value of this string order parameter for the global symmetry $\otimes_i Z^i_g$, whose value depends on the quantum phase of $|\psi\rangle$ viewed as one of the ground states of a 1D symmetric Hamiltonian H^{1D}_{ψ}. Therefore, the quantum phase of the 1D boundary determines which boundary stabilizer are present and then which anyons condense at the boundary (by commuting with all Hamiltonian terms in the boundary).

Remarkably our construction unifies the fact that the mathematical object that classifies both gapped boundaries of quantum double models of G[27–29] and globally G symmetric 1D systems[30,31] is the same: module categories over Vec_G[32] given by pairs $(H \subseteq G, \beta \in \mathcal{H}^2[H, U(1)])$.

Incorporating the second type of boundaries, the condition to be a symmetry of $\langle\psi'|\mathcal{G}|\psi\rangle$ is that Z_g is also a local symmetry for $|\psi\rangle$ (provided that it generates its global symmetry $\otimes_i Z^i_g$).

Let us analyze in detail the case of $\beta = 1$ where the boundary terms are $X^{\dagger}_{\hat{g}} \otimes Z^{\dagger}_{\hat{g}} \otimes X_{\hat{g}}$ so we evaluate $\langle\psi|X^{\dagger,i}_{\hat{g}} \otimes X^{i+\ell}_{\hat{g}}|\psi\rangle$, a two point symmetric correlation function that characterizes the pattern of symmetry breaking. If $H \subseteq G$ is the unbroken symmetry group characterizing the quantum phase of H^{1D}_{ψ}, generically $\langle\psi|(X^i_{\hat{g}} \otimes X^{\dagger i'}_{\hat{g}})|\psi\rangle \neq 0$ if $\hat{g}(h) = 1$ for all $h \in H$. We further impose that $\langle\psi|(X^i_{\hat{g}} \otimes X^{\dagger i'}_{\hat{g}})|\psi\rangle = 1$ which is achieved at the RG fixed point, see the Supplementary Note 2 for an explicit construction. So the boundary symmetries of the first type correspond to the elements $\hat{g}$ in the subgroup $res^{\hat{G}}_H \subseteq \hat{G}$, where $res^{\hat{G}}_H = \{\hat{g} \in \hat{G} \,|\, \hat{g}(H) = 1\}$.

Finally, we can construct the associated boundary Hamiltonian as:



(9)

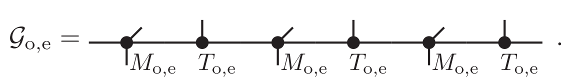
whose first kind of terms are violated only by anyons created by string operators $\otimes_{j\in\ell} X^{i,j}_{\hat{g}}$ ending in the boundary with $g \in G\backslash H$, see Fig. 2c. The second kind of terms are violated by anyons indexed by $\hat{g} \in \hat{G}$ such that $\hat{g}(g) \neq 1$ for all $g \in G$ satisfying that $Z_g|\psi\rangle = |\psi\rangle$.

As an example we can take $|\psi\rangle = \otimes_i |\hat{e}\rangle^i$ which belongs to the trivial symmetric phase: $(H = G, \beta = 1)$ and it is also locally symmetric under Z_g for all $g \in G$. In this case only G-anyons condense at the boundary.

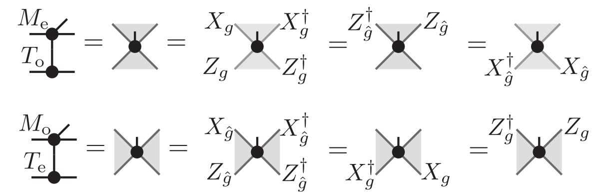
The same discussion could have been applied if we would have started from a $\hat{G}$ global symmetry and also to last layer $|\psi'\rangle$ of $\langle\psi'|\mathcal{G}|\psi\rangle$.

Tensor network description

The gauging operators, $\mathcal{G}_{\mathrm{o,e}}$ in odd and even layers, are matrix product operators constructed from two tensors (see Supplementary Note 3) of bond dimension $|G|$:



Then, the state $\mathcal{G}$ in (5) is a projected entangled pair state (PEPS)[22] emerging from the concatenation of 1D gauging operators. Subsequently, we dub this subfamily of PEPS as projected entangled pair emergent states (PEPES). The two different tensors, corresponding to the two types of vertices in the rotated squared checkboard lattice of the PEPES (see Fig. 2) have the following symmetries:



The first three relations of each tensor correspond to the virtual G and $\hat{G}$-invariance characterizing 2D topological order in PEPS[23,24] and the last relations makes compatible those symmetries. Note that the virtual loop symmetries propagates only in the horizontal direction.

Another view on our construction

We can interpret $\mathcal{G}$ as the projection of stacked layers of 1D product states. By using that $\mathcal{G}_j = \mathcal{P}_j \otimes_i |e\rangle^i_j$ we can write

$$\mathcal{G} = \left(\prod_{j=1} \mathcal{P}_{2j} \cdot \mathcal{P}_{2j-1}\right) \mathcal{P}_0 \left(|\psi\rangle \otimes_j \otimes_i |e\rangle^{\hat{i}}_{2j} \otimes_i |\hat{e}\rangle^{i}_{2j-1}\right),$$

where $\otimes_i |e\rangle^{\hat{i}}$ is locally invariant under any $Z^{\hat{i}}_g$ and the projectors P_{2j} and $\mathcal{P}_{2j-1}$ do not commute. The two previous properties differ from the common approach of creating topologically ordered models from stacking lower dimensional ones (where the coupling generally commutes)[33–38].

Discussion

In this work we have established a bulk–boundary correspondence between 1D global symmetric systems and 2D topologically ordered models. We do so by sequentially gauging the emergent 1D global symmetries that maps the local 1D symmetries after gauging to 2D plaquette operators. As a result we obtain a family of 2D Hamiltonians:

$$H^{G}_{\mathrm{Emerg.}} = H^{\alpha,\gamma}_{\mathrm{bulk}} + H^{[|\psi\rangle]}_{\mathrm{bdry.}}.$$

This family covers the generalization of the *XZZX*-code for any Abelian group G. Also, these Hamiltonians are able to realize interesting anyon confinement phenomena where there are local excitations violating 3 plaquette terms. Moreover, the boundary terms are given by the quantum phase of the 1D Hamiltonian of $|\psi\rangle$ and determines which anyons condense at the boundary. Such connection illuminates the fact that both settings are classified by the same mathematical object.

The question of how our construction can be generalized to non-Abelian topological orders remains open. We left for future work the emergence of non-trivial (3+1)D phases from the gauging of (2+1)D symmetries.

Methods

The main relations used in the analytical calculations are $Z_g \cdot X_{\hat{g}} = \hat{g}(g) \cdot X_{\hat{g}} \cdot Z_g$ and $X_g \cdot Z_{\hat{g}} = \hat{g}(g^{-1}) \cdot Z_{\hat{g}} \cdot X_g$ for any $g \in G$ and $\hat{g} \in \hat{G}$. These relations allow us to compute how the different operators translate through the gauging maps.

Data availability

The author declares that the data supporting the findings of this study are available within the paper.

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