Rushing to Opportunity: City Growth and Entrepreneurship*

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January 28, 2025

Abstract

The growth of many cities and industries differs, with some growing slowly and others experiencing rapid change—i.e., rushes. To explain these differences and explore the mechanisms of growth, we develop a model centered on a new trade-off between time-varying fundamentals and time-invariant—but rank-dependent—opportunities. Early population flows depend on the opportunities new entities provide, whether from available land in cities or the accumulation of entrepreneurship human capital in firms. Our model can explain the existence of rushes and their size. We provide suggestive empirical evidence on city- and industry growth consistent with the model's predictions.

Keywords: Opportunities; rapid change; city growth; entrepreneurship.
JEL Classification: J24, R12, R14, L16.

*We thank two thoughtful referees, our supportive editor Gilles Duranton, as well as Jan Brueckner, David Cuberes, James Hines, Sergey Kichko, Edward Lazear, Andreas Park, Lones Smith, Stuart Rosenthal, and Stephen Salant for very helpful comments. We further thank participants at various conferences and many seminars for constructive comments and suggestions. Behrens gratefully acknowledge financial support from the SSHRC Insight Grants program ('Cities in motion', grant #435-2016-1246). Any remaining errors are ours.

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1 Introduction

We provide insights into why some new cities or industries grow at slow and steady rates, whereas others experience periods of rapid change—rushes. One historic example of slow growth versus rushes is the formation of Louisville and Lexington, Kentucky.[1] Louisville grew slowly. In 1790, roughly 10 years after forming, it had a population of 200. In contrast, Lexington boomed and had a population of 18,410 in 1790, despite being formed two years later. While the land around Louisville is heterogeneous—with some closer to the water and some more swampland than others—Lexington's land is homogeneous and in the center of the Bluegrass Region. This implies differences in *opportunities* for newcomers: if early arrivals pick the best land, being first vs second (or tenth or eleventh) means a greater discount in Louisville than in Lexington.

More generally, since land prices increase as new cities grow, because of both increases in congestion costs and rising opportunity costs of land at the urban fringe, being an early arrival in a new city yields higher opportunities—one can 'buy into' the new city at lower prices—than being a late arrival. Ceteris paribus, this pushes towards fast growth or rushes. However, new cities generally also offer lower incomes than larger more established places, which increases the opportunity cost of early arrival. How and at what speed cities grow depends on the trade off between these two factors.

The foregoing mechanism is not limited to cities. A similar logic—involving a trade off between rank-dependent opportunities and time-varying income—can be applied to the growth of new industries. Consider the growth between the 1980s and 2000s in finance (an established industry) and technology (a new industry). Wages were initially higher in finance than in tech. However, workers in new industries such as tech generally have a higher rank in the younger firms and are given a broader portfolio of tasks, which allows them to accumulate more human capital (Liang et al., 2018). Again, the trade off between increased opportunities and foregone income can explain why some new industries grow at slow and steady rates whereas others experience rushes.

We propose a novel model that formalizes the trade off between rank-dependent opportunities and time-varying changes in income to show how these interact to explain why some new entities (cities, industries, or firms) grow at slow and steady rates whereas others experience rushes. As in standard models, income changes with the size

[1] Glaeser (2013) provides several historic examples and a detailed analysis of rapid change—booms and busts—in the US land and housing markets.

of an entity, is initially lower in new entities, and is time-varying. We add to this model opportunities, which depend on when one moves to a new entity: they are time-invariant in the sense that they are locked-in at the time of arrival. They are also rank-dependent because they are determined by how many moved before to the new entity. As an example, early movers can buy in at lower costs in a new city or enter new firms higher in the organizational hierarchy. Without rank-dependent opportunities, individuals are unwilling to move to new entities before the time-varying benefits (e.g., income) in these new entities equal those in established ones. With rank-dependent opportunities, individuals will forgo some time-varying benefits to lock in higher opportunities early on. The creation and growth of cities and industries, therefore, critically depend on the shape of these opportunities.

We show how the model can explain several counterintuitive empirical growth phenomena through a series of comparative statics. First, we can explain why some entities begin slowly and others with a rush, which depends on the characteristics of the rank-dependent opportunities. The entity starts with a rush when the opportunities are initially increasing and eventually decreasing with respect to agents' arrival rank. Many agents rationally 'rush to opportunity'—i.e., move simultaneously from the established entity to the new one—causing sudden growth in the latter. The intuition for this result is that when the opportunities are nonmonotonic in this way there are benefits to being an early arriver—but not being first. Therefore, the entity cannot begin slowly as agents will rationally wait for others to move first. Agents, however, are willing to rush simultaneously as long as the opportunity they receive is greater than the opportunity of being first and greater than moving right after the rush. We show these conditions ensure a unique rush size.

Conversely, the entity starts slowly without a rush when the opportunities are monotonically decreasing in agents' arrival rank. The intuition is similar to that of a game of war of attrition: in equilibrium, the larger benefit of being the kth entrant compared to being the $k+1$th entrant must be offset by the cost of receiving lower benefits in the new entity for a longer duration, thus leading to slow growth. Said differently, if there is a large benefit to being first relative to second, entrants are willing to move earlier to secure that larger benefit. Yet, growth will be slow, and there cannot be a rush because agents would have an incentive to preempt it: being the $k-1$th mover just before the rush allows for discontinuously greater opportunities forever but for lower time-varying income for an infinitesimal period, which cannot be an equilibrium.

Second, and quite surprisingly, a rush occurs earlier and is larger when the opportunities are spread across more agents. The model also predicts that growth will be slower when the difference in opportunities between ranks increases. The result that small differences in opportunities encourage rushes and faster growth may initially seem counterintuitive. The intuition is that, in equilibrium, differences in opportunities must be offset by differences in income an individual receives. Therefore, the benefit in terms of opportunities from moving earlier must be offset by the fact that by moving earlier, an individual foregoes higher income in the established entity for a longer period. If opportunities decrease slowly, the first mover penality increases since moving earlier implies renouncing higher income in the established entity for a longer period, without gaining much in terms of better opportunities. In that case, the larger first-mover penalty entices agents to wait longer until they rush to the new entity.

Finally, our model also solves a persistent coordination problem that arises in standard models of city growth. Standard urban models fail to explain the slow growth of new cities because new cities only form once old cities become grossly oversized (e.g., Henderson and Becker, 2000).[2] Without opportunities there are no gains from being early in the new city since this amounts to renouncing higher wages for a longer period. Without any first-mover advantage, new cities form through large population swings when the time-varying benefits in the new entities equal those in the established ones.

The remainder of the paper is organized as follows. Section 2 lays out the general model, characterizes its solution, and discusses the existence and size of rushes. Sections 3 and 4 present our two examples for city- and industry growth, respectively, as well as suggestive empirical evidence. Section 5 concludes. Formal proofs and details on our data are relegated to the appendix.

[2] Henderson and Venables (2009) provide an urban model to solve the coordination problem using fixed assets such as land and non-malleable housing. Relatedly, there is a large literature on 'social learning' that deals with coordination problems in individual decisions through what may be broadly called information externalities. Rational herding in financial markets (e.g., Devenow and Welch, 1996) may lead to periods of slow movement—where investors follow the herd—followed by periods of sudden change as key investors revise their positions in light of new information. Herding in investment decisions may cause suboptimal investment delays and investment surges (see Chamley and Gale, 1994). The arrival of a new store, specific types of businesses, or affluent residents in a deprived area may reveal information as to the viability of the neighborhood, triggering an influx of other businesses or residents who waited for a signal to move (Caplin and Leahy, 1998; Behrens et al., 2022). Furthermore, firms adopt new technology or enter into a region depending on information revealed by the decisions of others (Conley and Udry, 2010; Ossa, 2013). All these models deal with periods of slow movement and sudden, rapid change.

2 Model

We want to understand how opportunities that arise in new entities shape their creation, rapid transformation, and growth. To this end, we first construct a general model in Section 2.1, provide its solution in Section 2.2, and discuss comparative statics results in Section 2.3. We then apply it to two illustrative examples in Sections 3 and 4.

2.1 General setup

There are two entities, denoted 1 and 2.[3] Time is continuous and indexed by t. There is a potential total population $\overline{N} \equiv \int_0^\infty \widetilde{N}(t)\mathrm{d}t$ of homogenous individuals, where $\widetilde{N}(t)$ denotes the mass at time t. Let $N(t) \equiv \widetilde{N}(t)/\overline{N}$ be the normalized population. We assume its flow $\mathrm{d}N(t)/\mathrm{d}t \equiv \dot{N}(t) \equiv \eta(t) > 0$ is exogenous and known by all individuals. It can be nonmonotonic, i.e., there can be periods of faster or slower flows.

Let $N_1(t)$ and $N_2(t)$ denote the populations in entities 1 and 2 at time t, respectively. All individuals begin in entity 1, defined as the *established entity*, and choose a time $\tau \in [0, \infty)$ to move to entity 2, defined as the *new entity*. Let $\dot{N}_2 \equiv m(t) \geq 0$ denote the population flow in entity 2, which is entirely due to individuals that move from 1 to 2. Since total population flow is $\eta(t)$, it must be that $\dot{N}_1(t) = \eta(t) - m(t)$.

The mass of individuals who have moved to entity 2 before time τ defines the population there at τ: $N_2(\tau) \equiv M(\tau) = \int_0^\tau m(t)\mathrm{d}t$, and $\dot{M}(t) = m(t)$. Individuals who move to entity 2 at time τ are given a rank equal to the mass of individuals $M(\tau)$ who moved before them. We explain later under which assumptions we can index individuals unambiguously by their rank. For simplicity, individuals who move to entity 2 are assumed to subsequently stay there forever. This implies that $m(t) \geq 0$. We derive conditions below to ensure this assumption is satisfied.[4]

Individuals move between entities based on utility differences. Each entity $i = 1, 2$ provides utility from two sources: (i) rank-independent but time-varying benefits $Y_i(t)$, hereafter *income*; and (ii) rank-dependent but time-invariant *opportunities*. We assume income in i depends on i's population but not directly on time: $Y_i(t) = Y_i\big(N_i(t)\big)$. We

[3] This assumption is imposed for convenience and can be relaxed to the more general case with numerous entities. See Seegert (2015) for the context of cities.

[4] This assumption is unnecessary if all individuals who move at time τ receive the average opportunity. In that case, individuals have no incentives to move back to entity 1. It is also unnecessary if there are large fixed moving costs, i.e., if the cost of moving back to entity 1 is sufficiently large.

further assume it is continuously differentiable and subject to first economies and then diseconomies of scale as follows:

$$Y_i(0) = 0, \quad Y_i'(N_i) \gtreqqless 0 \quad \text{for} \quad N_i \lesseqqgtr \widehat{N}_i, \tag{1}$$

where $\widehat{N}_i$ defines the (unique) income-maximizing population of entity i. Assumptions (1) ensure that, as the population grows, there are benefits for individuals to concentrate in one entity but that it is not efficient for all individuals to concentrate indefinitely in the same entity. For ease of exposition, we assume that the total population at time zero is greater than entity 1's income-maximizing population: $N(0) > \widehat{N}_1$ so that $Y_1'(0) < 0$. Put differently, there are decreasing returns to the population in the established entity 1, which provides incentives to leave it and move to the new entity. Yet, since $Y_2(0) = 0$, nobody would make the first move based on income alone, i.e., if there were no opportunities in the new entity, until income in the first entity equaled zero. We further assume that $Y_1(x) < 0$ for some population $x < \infty$. This assumption ensures no equilibrium exists where nobody moves to the second entity.[5]

Without loss of generality, we normalize opportunities in entity 1 to zero. Opportunities in entity 2 depend on individuals' ranks, i.e., on when they moved to that entity. We model them by an *opportunity function*, $R\big(M(\tau)\big) > 0$ for all $M(\tau)$. We assume that R is continuously differentiable and single-peaked. The average opportunity between ranks M and $M + \mathit{\Delta} M$ is defined as $R_M(\mathit{\Delta} M) = (1/\mathit{\Delta} M) \int_M^{M+\mathit{\Delta} M} R(m)\mathrm{d}m$. Opportunities are eventually decreasing in rank but can initially be increasing:

$$R'(M) \gtreqqless 0 \quad \text{for} \quad M \lesseqqgtr \widehat{M}, \quad \text{and} \quad R(M) < R_0(M) \quad \text{for some} \quad M > \widehat{M}, \tag{2}$$

where $\widehat{M} \geq 0$ defines the (unique) opportunity-maximizing population of entity 2. Regularity assumptions (2) exclude opportunity functions that are always increasing, initially decreasing and then increasing, and initially increasing and then decreasing at an insufficient rate (we assume opportunities must eventually fall below those of the first movers to the new entity). Observe that the opportunity an individual secures by moving to entity 2 solely depends on his rank $M(\tau)$, which itself depends on the time τ of the move. After that, the opportunity is locked in and stays constant over time.

[5]Consider toward contradiction an equilibrium where nobody moves to the second entity. In this case, when population flow exceeded x such that the income in the first entity became negative, an individual could move to the second entity and receive a higher income and some positive opportunity.

An individual's lifetime utility depends on his income and—conditional on whether and when he moves from 1 to 2 at time τ—his rank-dependent opportunity profile:

$$\begin{aligned} U\big(\tau, M(\tau)\big) &= \int_0^\tau \mathrm{e}^{-rt} Y_1\big(N(t)-M(t)\big)\mathrm{d}t \\ &\qquad + \int_\tau^\infty \mathrm{e}^{-rt} Y_2\big(M(t)\big)\mathrm{d}t + \int_\tau^\infty \mathrm{e}^{-rt} R\big(M(\tau)\big)\mathrm{d}t \\ &= \int_0^\tau \mathrm{e}^{-rt} Y_1\big(N(t)-M(t)\big)\mathrm{d}t + \int_\tau^\infty \mathrm{e}^{-rt} Y_2\big(M(t)\big)\mathrm{d}t + \frac{R\big(M(\tau)\big)}{r}\mathrm{e}^{-r\tau}, \end{aligned} \tag{3}$$

where r denotes the discount rate. Observe that this utility function is differentiable in the time of the move, τ, for any differentiable profile $M(t)$. We show later that this is the case when opportunities are decreasing in time t, in which case there is a slow flow to the new entity. Yet, in the general case, there may be atoms in the distribution $M(t)$, which precisely occurs when there are periods of rushes.

We define rushes as the case where nobody moves until time τ (i.e., $m(t) = 0$ for $t < \tau$), whereas a mass ΔM of agents move simultaneously at time τ. We show later that there will be, at most, one rush in equilibrium so that agents are either part of that rush at period τ or indifferent in moving sometime after the rush. Assuming that all agents who move during a rush receive the same average opportunity, lifetime utility is:

$$U\big(\tau, M(\tau)\big) = \int_0^\tau \mathrm{e}^{-rt} Y_1\big(N(t)\big)\mathrm{d}t + \int_\tau^\infty \mathrm{e}^{-rt} Y_2\big(M(t)\big)\mathrm{d}t + \frac{R_0\big(\Delta M\big)}{r}\mathrm{e}^{-r\tau}. \tag{4}$$

On top of assumptions (1) and (2), we impose two additional restrictions to focus on equilibria where population flow in entity 2 begins at an interior time ($t > 0$) and is nonnegative ($m(t) \geq 0$). First, we assume that the initial benefit of staying in entity 1 is greater than that of moving to entity 2, irrespective of the mass ΔM of agents that leave entity 1 at $\tau_0 = 0$.[6] Formally, $\lim_{\tau\to\tau_0} \frac{\mathrm{d}U(\tau,M(\tau))}{\mathrm{d}\tau}\Big|_{\tau_0=0, M(\tau_0)=\Delta M} > 0$ for all $\Delta M \leq N(0)$. Using (3), this requires that:

$$Y_1\big(N(0)-\Delta M\big) - Y_2(\Delta M) > R(\Delta M) + \frac{\dot{R}(\Delta M)}{r}, \quad \text{for all} \quad 0 \leq \Delta M \leq N(0). \tag{5}$$

[6]Since agents are homogeneous, this implies that $m(0) = 0$. To derive conditions for this to hold, we consider what would happen if there was a single atom at $t = 0$. The latter implies that $M(0) > 0$, but the function M is smooth afterward (there is no second atom). We can hence write a (right) derivative for lifetime utility in τ and look at the limit as τ goes to zero.

We can view the left-hand side of (5) as the difference in the flow payments (the benefits) in entities 1 and 2, whereas the right-hand side is the continuation benefit from higher opportunities received by moving now versus moving later. Condition (5) states that the initial benefit of staying in entity 1 at $\tau_0 = 0$ is greater than that of moving to entity 2, irrespective of the mass ΔM of moving agents.

Second, we impose a condition that allows us to focus on the fundamental trade-off in the model: waiting in entity 1, which initially offers higher income, or moving to entity 2, thereby forfeiting income in the short run but benefiting from better rank-dependent opportunities in the long run. To this end, we impose two conditions on the utility function—a primitive of the model. We assume utility increases with the timing of the move, not taking into account changes in $M(t)$, whereas utility decreases with the mass of people that have moved before some point in time:

$$\frac{\partial U(\tau, M)}{\partial \tau} > 0, \qquad \text{and} \qquad \frac{\partial U(\tau, M)}{\partial M} < 0. \tag{6}$$

The first condition means that opportunities in entity 2 become more valuable over time, provided nobody moves to exploit them. This entices individuals ceteris paribus to remain in the established entity and not move too early. The second condition means that opportunities in entity 2 become less valuable at each point in time if more agents have moved there before that point and grabbed those opportunities. This entices agents ceteris paribus to not wait too long before moving to the new entity.[7]

2.2 Equilibrium

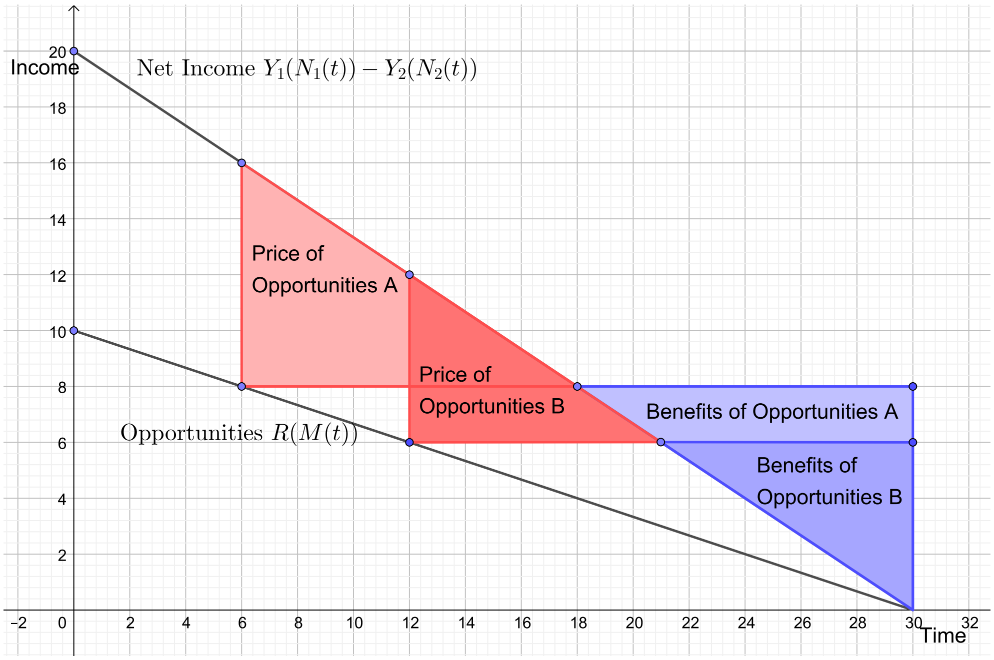
New entities provide opportunities. The price individuals pay for those opportunities is the difference between the higher income they would have received in the original entity and the lower income they accept in the new entity when moving there. In equilibrium, for agents to move the price of the opportunity must equal its benefit.

We start with a graphical illustration to provide the intuition for the trade-offs. The simplified Figure 1 depicts the price and benefit of the opportunities in the new entity.[8]

[7] If conditions (6) do not hold, there may be a time when flow in 2 becomes negative. As stated before, we rule out this case.

[8] Figure 1 has three simplifications. First, it gives income and opportunity functions that eventually hit zero, when in reality they have an asymptote there. Second, it disregards discounting. Last, it assumes time and the benefit of the opportunity stop at 30 instead of continuing forever.

Figure 1: The price of opportunities.



First, consider an individual who moves at time $\tau = 6$ for opportunity A. This individual receives an opportunity equal to 8 forever. The price of this opportunity is the income in the original entity (Y_1) net of the income (Y_2) and opportunity (R) in the new entity. Said differently, the area of the red triangle from (6,8), (6,16), and (18,8), which equals 48, quantifies the price paid by the individual for moving to the new entity. As the difference in incomes decreases between the two entities, eventually, the income plus opportunity (which was locked in) in the new entity exceeds the income in the original entity (starting at $t = 18$ in our example). The benefit of the opportunity is then given by the area of the blue triangle (18,8), (30,8), (30,0), which equals 48 too. Similarly, an individual who moves at time 12 for opportunity B pays the price of 27 for the benefit of 27. Observe that, in equilibrium, individuals are indifferent between moving at $\tau = 6$ or $\tau = 12$, as required by (7). The flow $m(t)$ of entity 2 is the endogenous variable that determines the slopes of the net income and opportunity functions in equilibrium, with respect to time, such that the price equals the benefit.

Let us now more formally analyze the equilibrium. The model defines a game. A player's strategy is the time τ he moves from entity 1 to entity 2 (conditional on being in entity 1). Players receive payoffs according to equation (3), which depend on when the player moves to entity 2 (i.e., τ) and the distribution of when the other players move to entity 2 (i.e., $M(\tau) = \int_0^\tau m(t)\mathrm{d}t$). We now construct an equilibrium toward the goal of proving existence and uniqueness. It is fully characterized by $M(t)$ that implicitly defines a time of creation τ_1 for entity 2 with initial size $\Delta M_1 \geq 0$ at creation.[9] Before creation, $t < \tau_1$, we have by definition $m(t) = 0$, and at time τ_1 we have $m(t) > 0$ for the first time. After creation, we focus on equilibria without *period of inaction*, defined as equilibria in which entity 2 continually grows once created.[10] Formally:

Definition 1. *An equilibrium without periods of inaction is such that $m(t) > 0$ for all $t \geq \tau_1$.*

Entity 2 may be created in a 'smooth' way or suddenly via a rush. Assume a mass ΔM moves at time τ. To uniquely index individuals by their rank, we assume in that case that: (i) ranks are randomly attributed among the simultaneous movers, and (ii) each individual in a rush receives the same average opportunity, which depends on the mass $M(\tau)$ of individuals who moved before the rush. An equilibrium can be constructed using a *no arbitrage condition* and a *boundary condition* for the creation of the new entity. The no-arbitrage condition is obtained by ensuring that individuals are indifferent between moving now or in the 'next period.' The boundary condition is obtained by ensuring that no individual wants to preempt the rush. Formally, $\partial U\big(\tau, M(\tau)\big)/\partial\tau = 0$ whenever $m(t) > 0$. Differentiating (3), using Leibnitz's rule, and rearranging yields:

$$\mathrm{e}^{-r\tau}\left[Y_1\big(N_1(\tau) - M(\tau)\big) - Y_2\big(N_2(\tau)\big) - R\big(M(\tau)\big)\right] - \frac{\dot{R}(M(\tau))}{r} = 0. \tag{7}$$

Recall that (3) is differentiable if M has no atoms, but that it is not generally differentiable at τ_1 when there is an atom. In that case, equation (7) must hold for $t > \tau_1$ (using a right derivative at $t = \tau_1$), whereas $U(\tau) \leq U(\tau_1)$ for all $\tau \in [0, \tau_1)$ in equilibrium (since no agent wants to preempt the rush).

We can prove the following result (see Appendix A.1 for details).

[9] Since agents are homogeneous, we restrict the search to symmetric Nash equilibria. With a continuum of agents, looking for pure strategy Nash equilibria—where each individual deterministically picks a time τ to move—is equivalent to looking for mixed strategy equilibria (Sun, 2006)—where individuals mix across times on when to move to entity 2 according to some probability distribution $q(t)$. With a mass $\overline{N} \equiv 1$ of homogeneous agents over time, $q(t) = m(t)$, and $Q(t) = \int_0^t q(s)\mathrm{d}s = \int_0^t m(s)\mathrm{d}s \equiv M(t)$. The equilibrium is characterized by the cumulative distribution $Q(t)$. Following Anderson, Park and Smith (2017), off-equilibrium behavior can be specified such that all Nash equilibria are also subgame perfect.

[10] The equilibrium without periods of inaction is the only equilibrium that survives a trembling-hand refinement defined as a safe equilibrium (see Anderson, Park and Smith, 2017). It is unique, and an extremal equilibrium such as this has the earliest starting time. Milgrom and Roberts (1994) suggest focusing on such extremal equilibria.

Theorem 1 (Existence and uniqueness of an equilibrium). *There exists a unique ε-safe mixed strategy Nash equilibrium.*

Proof. The proof of Theorem 1 is shown in five steps in Appendix A.1. First, initially moving to the new entity is worse than staying, given the regularity condition in equation (5). Second, the second entity is formed because population flow sufficiently deteriorates income in the first entity. Third, there is a unique starting time where the benefits of moving to the second entity exactly equal the cost. Fourth, the implicit function theorem determines a unique flow pattern after the second entity is formed. Note the indifference condition in equation (7) creates a differential equation $\partial U/\partial\tau = W_M m(t) + W_\tau = 0$ with a known solution. Fifth, this equilibrium has no period of inaction, satisfies the conditions for an $\varepsilon-$safe equilibrium, and no other equilibrium exists without a period of inaction that is not $\varepsilon-$safe. □

Having established the existence of equilibrium, we can analyze how the flow of the population of entity 2 changes with the fundamentals of the opportunity function. Consider first how the flow changes as the opportunity function becomes flatter—said differently, as the difference in opportunities individuals receive shrinks—i.e., $|R'(M)|$ becomes smaller.

Proposition 1 (Flow). *A flatter opportunity function causes population flow to entity 2 to be faster when individuals move to entity 2.*

Proof. Rearranging equation (7), evaluated at $\tau = 0$ using $\dot{R}\big(M(\tau)\big) = -R'\big(M(\tau)\big)m(\tau)$, directly yields

$$m(\tau) = \frac{r\left[Y_1\big(N_1(\tau) - M(\tau)\big) - Y_2\big(N_2(\tau)\big) - R\big(M(\tau)\big)\right]}{-R'\big(M(\tau)\big)}, \tag{8}$$

which is larger for smaller (absolute) values of $R'(M(\tau))$. □

Equation (8) shows that the population flow to entity 2 increases as the opportunity function becomes 'flatter.' The result that small differences in opportunities encourage greater flow may initially seem counterintuitive. To understand it, remember that, in equilibrium, differences in opportunities must be offset by differences in income an individual receives. Therefore, the benefit in terms of opportunities from being mover M rather than mover $M + \varepsilon$ must be offset by the fact that by moving earlier, an individual foregoes more net income for a longer period. Consider the extreme case where everyone would receive the same opportunities, regardless of when they move to entity 2, i.e., a flat opportunity function. In this case, there cannot be any period of prolonged slow flow because individuals who move early will always have an arbitrage opportunity by waiting. In other words, there is a first-mover penalty. The alternative is then that entity 2 'grows infinitely fast' by experiencing one giant rush of agents. This result highlights the importance of the shape of the opportunity function for determining the pattern of equilibrium flow.

The existence of this equilibrium with opportunities reduces the severity of the migration pathology by increasing the payoff of early movers. In particular, this model produces an equilibrium with slow growth and, depending on the opportunity function, only slow growth without rushes (atoms). In contrast, previous models often rely on large rushes or atoms that bifurcate cities when a new city is created (Anas, 1992). We show that rank-related opportunities, our addition to the model, are critical for allowing for growth without rushes. We also show that the rank-related opportunities determine the speed of growth.

While the equilibrium from our model does not require a rush, it may include one. In the following subsection, we characterize the size and timing of rushes.

2.3 Rushes

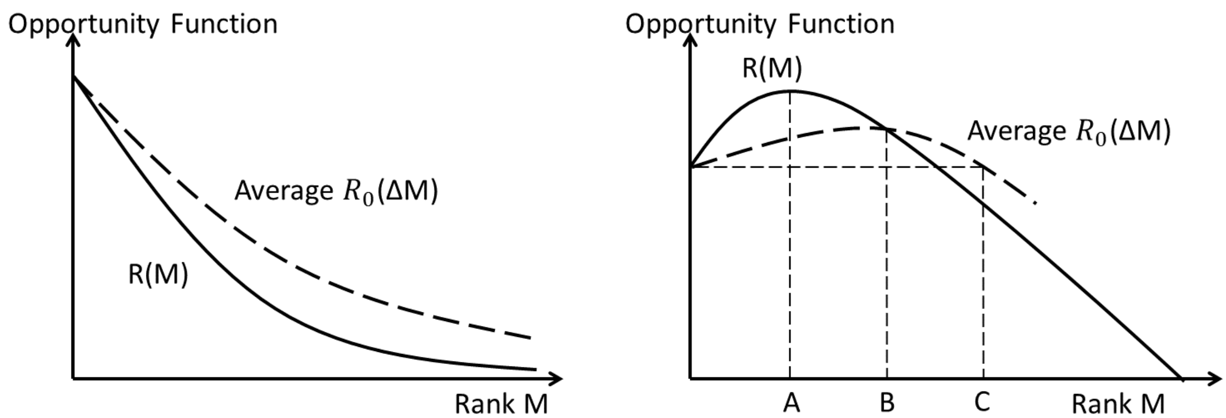
A rush of agents may start an entity. In this case, nobody moves from 1 to 2 before τ_1, while at $t = \tau_1$ there is a sudden movement of a mass ΔM of agents to entity 2, causing an atom in $M(t)$. Afterward, the population flow continues slowly according to the no-arbitrage condition in equation 7. Alternatively, an entity may begin with slow growth, an interior equilibrium characterized by the no-arbitrage condition for all $m(t) > 0$. Whether an entity begins with a rush or slow growth is determined by whether the opportunity function is monotonic. Proposition 2 provides the conditions for entity 2 to be formed by a rush.

Proposition 2 (Existence of rushes)**.** *A necessary and sufficient condition for every equilibrium to involve a rush is that the opportunity function is nonmonotonic and initially increasing.*

Proof. See Appendix A.2. □

Figure 2: Conditions for rushes to occur in equilibrium.

(a) No rush in equilibrum. (b) Rush in equilibrium.



The intuition underlying Proposition 2 is depicted in Figure 2. First, consider a monotonically decreasing opportunity function, as in Figure 2 panel (a).[11] A rush does not exist in this case because there is always an incentive to preempt it. The reason is as follows. Individuals in a rush receive the average opportunity $R_0(\Delta M)$ (recall that entity 2 is still empty, hence $R_{M(t)}(\Delta M) = R_0(\Delta M)$ for all $t \in [0, \tau_1)$), and, when the opportunity function is decreasing, this is less than the initial opportunity $R(0)$. Thus, an individual who preempts the rush receives a discontinuously larger opportunity forever but forfeits some income over an infinitesimal period. The former always dominates the latter, and it follows that there can be no rush at $\tau_1 > 0$ in that case. It follows that when the opportunity function is monotonically decreasing, any equilibrium involves slow growth where $m(t) > 0$ and where the arbitrage condition holds.

Second, consider a nonmonotonic and initially increasing opportunity function, as in panel (b) of Figure 2. The new entity cannot be created by slow growth in this case because there is an incentive to wait. No agent wants to move first because doing so entails both lower opportunities than that of the next mover *and* lower income. Hence, agents wait. As time goes by, income in entity 1 eventually decreases enough so that moving to entity 2 becomes more attractive. Yet, no agent will move individually since

[11]This is a generalization of the illustration in Figure 1 to the case where the opportunity and income functions are not linear and do not hit zero. Whereas we depict Figure 1 as a function of time t, we plot Figure 2 as a function of the mass $M(t)$ of agents who have moved by time t. As explained before, we can alternatively think about this as the agents' rank.

there cannot be slow growth (no agent has an incentive to be first since opportunities are initially increasing). Thus, there must be a rush.

Proposition 3 provides results on the timing and the size of the rush.

Proposition 3 (Timing and size of rushes)**.** *There is at most one rush, which occurs at time τ_1 when entity 2 is created. In an equilibrium with a rush, the size ΔM_1 of a rush is unique and occurs where the marginal opportunity equals the average opportunity, $R(\Delta M_1) = R_0(\Delta M_1)$, i.e., at the maximum average opportunity.*

Proof. We build on the intuition of Proposition 2 to show that there exists a rush at time τ_1. By Proposition 2, we know that, for there to be a rush in equilibrium, the opportunity function must be nonmonotonic and initially increasing. By the reasoning in Proposition 2, entity 2 cannot be created by slow growth if the opportunity function is initially increasing because there will be an incentive for individuals to wait. Therefore entity 2 must be created by a rush.[12]

The size of the equilibrium rush is determined by the conditions that there is no incentive to preempt the rush and there is no incentive to outlast it. Let ΔM_1 denote the mass of individuals who rush at time τ_1 to receive the average opportunity $R_0(\Delta M_1)$, graphed in panel (b) of Figure 2. Define three points: A, B, and C. Point A defines the peak opportunity $\widehat{M}$. Point B defines the point at which the average opportunity equals the marginal opportunity $R(\Delta M_1) = R_0(\Delta M_1)$. Point C defines the point at which the average opportunity equals the opportunity of the first mover: $R_0(M_{\text{max}}) = R(0)$. To ensure there is no incentive to preempt the rush, its size cannot be larger than point C. To ensure there is no incentive to outlast the rush, its size cannot be smaller than point B. In equilibria without periods of inaction (recall Definition 1), individuals must receive the same opportunities being in a rush or moving right after the rush (recall $m(t) > 0$ for $t > \tau_1$; this is only possible if the individual is indifferent between rushing at τ_1 or moving afterward). Otherwise, individuals would be unwilling to move right after the rush. This occurs at the unique point where the marginal opportunity equals the average opportunity, $R(\Delta M_1) = R_0(\Delta M_1)$, point B in Figure 2. The value ΔM_1 is the unique equilibrium rush size for equilibria without periods of inaction.[13]

[12]Thus, in equilibrium $M(t)$ is discontinuous at most once and this occurs at τ_1 when $M(t) = 0$.

[13]For rush sizes larger than point B, the opportunity in the rush is strictly greater than the following opportunity, $R(\Delta M) < R_0(\Delta M_1)$ for $\Delta M > \Delta M_1$, causing there to be a period of inaction after the rush. The period of inaction is costly to the individuals who rush because they receive a lower income in entity 2

To show there cannot be a second rush, we proceed by contradiction. Suppose entity 2 is created at time τ_1 by a rush of size ΔM_1 and that a second rush occurs at time $\tau_2 > \tau_1$. The opportunities for the mass ΔM_2 of individuals in the second rush are given by the running average $R_{M_2}(\Delta M_2)$. These average opportunities during the second rush are strictly less than the ones right after the first rush, i.e., $R_{M_2}(\Delta M_2) < R(M_1) = R(\Delta M_1)$. The reason is that, at the time of the second rush, the average opportunity is decreasing because the equilibrium size of the first rush occurs at the maximum of the average opportunity. Hence, the second rush cannot be an equilibrium because individuals have an incentive to preempt it: by moving just before the rush, they receive discontinuously greater opportunities forever and less income for an infinitesimal period. □

We next consider how the size of a rush, ΔM, changes with the opportunity function. Proposition 3 demonstrates that the equilibrium size of the rush is determined by the peak of the average opportunity function. From this condition, several comparative static results follow. They are summarized in the following proposition.

Proposition 4 (Size of a rush)**.** *The size of a rush is unaffected by a proportional or a level change in the opportunity function; i.e.,* $R_\phi(M) = \phi R(M)$ *or* $R_\phi(M) = R(M) + \phi$*. The size of a rush decreases as the domain is compressed; i.e.,* $R_\phi(M) = R(\phi M)$*, with* $\phi > 1$*; and it increases as the domain is stretched; i.e.,* $R_\phi(M) = R(\phi M)$*, with* $0 < \phi < 1$.

Proof. See Appendix A.3. □

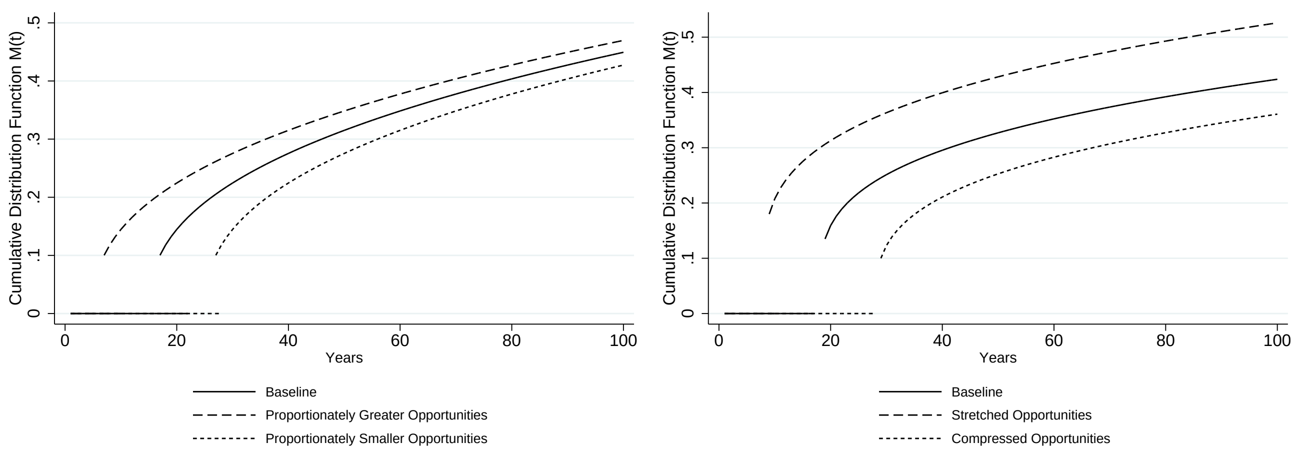
The first two comparative static results show that policies or events that cause a proportional or level shift of the opportunity function—such as a proportional tax or subsidy—will not affect the size of a rush. The third comparative static result shows that policies that spread the opportunities over fewer people will cause rushes to be smaller.

Figure 3 provides examples of growth in entity 2 with proportional shifts and stretches (or compression) of the opportunity function. The size of the rush is given by the upward jump on the y axis, which begins at some time given on the x axis. Both panels depict growth with non-monotone opportunity functions that cause entity 2 to be formed by a rush. Panel (a) of Figure 3 depicts growth with a rush and two comparative growth paths with proportional shifts in the opportunity function. The rush size does not change, but

for a longer period. In equilibrium, the period of inaction is uniquely determined for a given rush size, such that the benefits individuals receive from a larger opportunity equals the cost incurred by receiving lower income for a longer period.

Figure 3: Effects of changes in the opportunity function.

(a) Proportional shift. (b) Skewed and compressed.



the timing of the rush and the growth after the rush are both affected by a proportional shift. For example, the long-dashed line depicts growth with a proportional increase in opportunities, which in comparison to the baseline growth path, begins earlier and has slower equilibrium growth. Panel (b) of Figure 3 depicts growth with a rush and two comparative growth paths with a stretched or a compressed opportunity function, respectively. The stretched opportunity function (long-dashed line) has a larger rush that occurs earlier because the income in entity 2, after the rush, is larger due to the larger rush. After creation, the stretched opportunity function also leads to faster growth in the new entity because it is flatter (see Proposition 1).

We now illustrate our general framework using two examples. The first example considers the decision by individuals to stay in an established city or move to a new one. The second example considers the decision to stay in an established industry or leave to start a new firm or join a startup in an emerging industry.

3 City growth

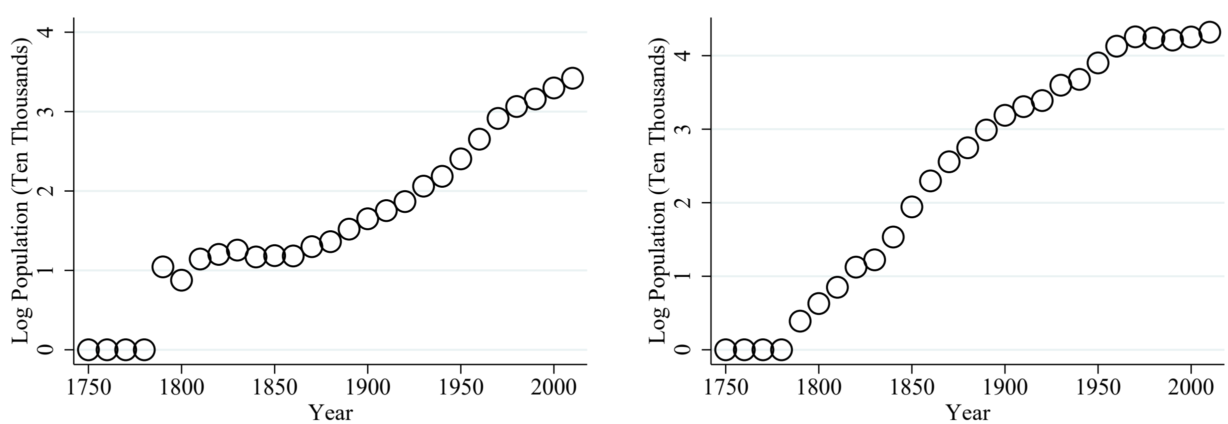
We investigate the growth of cities to highlight the predictions of Proposition 1 and the existence of rushes in Proposition 2.

3.1 Motivating example

We start with a motivating example that compares the creation and growth of Lexington and Louisville, Kentucky. It highlights the potential for rushes and differences in the rate of growth. Louisville and Lexington, Kentucky, provide an interesting comparison of the creation and growth of cities because they are only seventy-five miles apart and were chartered within two years of each other (1780 and 1782, respectively). Anecdotally, land was an important determinant for the growth of these cities, as it is in many contexts of urban growth (Wade, 1996).[14] Louisville is located next to the falls of the Ohio River, which were the only navigational barrier on the river at the time. The falls created a stopping point—a portage site—that disrupted the flow of traffic on the river. Like many other portage sites, this provided a natural place to develop a city (Bleakley and Lin, 2012). The land surrounding the falls is heterogeneous, both in distance from the river and in terms of suitability to build due to excessive swampland. This suggests Louisville's opportunity function was relatively steep. In contrast, Lexington is not on a navigable river. However, Lexington is located in the center of the inner Bluegrass Region, which provides vast amounts of fertile and homogeneous land. This suggests Lexington's opportunity function was relatively flat, i.e., $|R'(M(\tau))_{Lexington}| < |R'(M(\tau))_{Louisville}|$.

Figure 4: Population growth in Lexington and Louisville.

(a) Lexington. (b) Louisville.



Notes: This figure uses data from the decennial census. The model predicts rapid growth or a rush in Lexington and slow initial growth in Louisville based on the differences in the heterogeneity of land in both cities.

[14] Land was an important feature in the 1889 land rush in Oklahoma. Brown and Cuberes (2022) use the land rush and oil boom in Oklahoma to separate the first- and second-nature forces of urban growth.

Given the differences in slopes, Propositions 1 and 4 predict that Lexington will experience faster growth initially, even possibly being created by a rush. The model predicts, however, that Louisville will eventually become larger, due to its transport cost advantage. These predictions are corroborated by history, as shown in Figure 4. Lexington experienced rapid growth, reaching a population of 18,410 by 1790, only eight years after being chartered. In the same year, Louisville's population was 200, despite being chartered two years earlier than Lexington. It took Louisville roughly 60 years for it to surpass Lexington in population.

3.2 A formal model of city growth and opportunities

We now develop a formal model of city growth building on Albouy et al. (2019) and adding micro-founded opportunities.[15]

Total population grows exogenously according to $N(t) = N_0 + t^{\alpha}$ where $N_0 > 0$ and $\alpha > 0$ are exogenous constants. There are two cities, one being an established city of size $N_1(t)$ and the other a potential new city of size $M(t)$. We subscript variables pertaining to the established city with 1, and the new city with 2. Initially, everyone is located in the established city: $N_1(0) = N_0$ and $M(0) = 0$. We model the net wage (payoff) for a resident in the existing city using the following reduced-form expression:

$$Y_1\big(N_1(t)\big) = N_1(t)^{\varepsilon_1} - kN_1(t)^{\gamma_1}, \tag{9}$$

where $\varepsilon_1 > 0$ and $\gamma_1 > 0$ are the agglomeration elasticity and the elasticity of urban costs, respectively, and $k > 0$ is a constant. In what follows, we do not analyze the established city explicitly. We only use it to model the supply of workers to the new city, which depends on the differences between the time-varying gains in the established city and the new city, as well as the opportunities provided by the new city.

There is a single homogeneous consumption good produced everywhere under constant returns to scale and perfect competition. For simplicity, there are no trade costs, i.e., the good is available at the same price everywhere. We choose it as the numeraire. Workers are homogeneous and benefit from urban agglomeration economies. At time t,

[15] An alternative model, with different microfoundations but similar tradeoffs, is given in Appendix D.

the marginal productivity—hence wage—of a worker in the new city of size $M(t)$ is

$$w\big(a, M(t)\big) = aM(t)^{\varepsilon_2}, \tag{10}$$

where $a > 0$ is a TFP shifter and $\varepsilon_2 > 0$ is the strength of agglomeration economies in the new city. We assume the new city is monocentric, linear, and symmetric around the central business district (CBD), which is located at $x = 0$. All production takes place in the CBD and workers have to commute there. Commuting is costly. A commuting round trip of distance d to the CBD costs θd^{γ_2} in terms of the numeraire, where $\gamma_2 > 0$ is the distance-elasticity of commuting costs, and where $\theta > 0$ is a parameter.

The new city starts to develop at time τ. Let $M(t)$ and $m(t) = \dot{M}(t)$ denote the new city's population and population flow. By construction, $M(t) = m(t) = 0$ for all $t < \tau$. New residents arriving at time t to the city must purchase unoccupied parcels to live on, which are assumed to be located at the urban fringe. All new parcels are adjacent to the previously occupied ones, i.e., no holes in the urban structure exist. Land can be used either for agriculture or for urban development. Agricultural land at distance x from the CDB commands agricultural land rent $A(x)$, which varies across space with agricultural productivity. There is a competitive land market with a continuum of atomistic absentee parcel owners who can use their land for agriculture or urban development. The first parcel—which we henceforth consider as the CBD—is assumed to be located where the opportunity cost of land is the lowest, i.e., where agricultural land rent is minimal. The city extends from the CBD at $x = 0$ to the urban fringe F. The latter changes over time as the city expands.

Since there is a one-to-one mapping between parcels and their distance from the CBD, x, we henceforth refer to parcels as x for short, i.e., we index parcels by the continuous variable x. In line with suggestive empirical evidence (see Table 1 in Appendix C), we assume that parcels get larger as the city expands and the urban fringe moves outwards. To fix ideas, assume that $S(x) = x^{\kappa}$ denotes the size of parcel x, where $\kappa > 0$ is a constant term that captures the elasticity of parcel size with respect to distance from the city center.[16] Because parcels grow in size, parcel x is not located at a distance x from

[16] We assume that S is symmetric. Strictly speaking, since the CBD corresponds to the parcel $x = 0$, we should write $S(|x|)$. To alleviate notation, we only look at one side of the city ($x > 0$) since the other is assumed to be symmetric.

the CBD but rather at a distance

$$d(x) = \int_0^x S(z)\mathrm{d}z = x^{1+\kappa}\frac{1}{1+\kappa}. \tag{11}$$

The distance of parcels from the city center depends on the parcel size elasticity κ.

3.2.1 Parcel prices and spatial equilibrium

Let $r(x,t)$ denote the unit price of land (e.g., the price per square meter) of parcel x at time t. Hence, the *price* of parcel x—the land rent—is $p(x,t) = r(x,t)S(x)$. We assume that to buy a parcel at time t requires borrowing the value of the parcel at that time. Hence, to buy parcel x requires borrowing the amount $p(x,t)$. For simplicity, there is no down payment, i.e., agents need to borrow the full amount of the parcel.

In what follows, we denote by ρ and T the interest rate and the term of the loan, which we assume for simplicity to both be constant over time. Hence, the (continuous-repayment mortgage) payment 'per period'—the continuous cash flow—and the total cost of the loan are given by

$$\ell(t) = \rho\frac{p(x,t)}{1-\mathrm{e}^{-\rho T}} \quad \text{and} \quad L(\tau) = T\ell(\tau), \tag{12}$$

respectively, which naturally depend on the timing t of the move to the city.

Utility u of an agent in the new city depends on income net of commuting costs and the flow payment to land rent, as well as the size of the parcel the agent occupies. For simplicity, was assume that the instantaneous utility provided by parcel x is given by:

$$u(x,t) = aM(t)^{\varepsilon_2} - \theta d(x)^{\gamma_2} - \ell(x,t) + \zeta S(x), \tag{13}$$

where parcel size and distance are time invariant, whereas wage is location invariant. Only the flow payment $\ell(x,t)$ for the parcel depends on both its location and time. In the above equation, ζ is a utility parameter that captures the agent's willingness-to-pay for parcel size.

Spatial equilibrium in the new city at any point in time requires that agents are indifferent across all locations in the city. We assume that agents stay in the city once they moved there. Hence, if an agent sells his parcel x, since the agent has to live somewhere, he needs to purchase another parcel y. With homogeneous agents, spatial equilibrium requires that agents are indifferent between parcels. We show in Appendix B.1 that utility equalization across locations implies

$$p(y,t) - p(x,t) = \frac{1-\mathrm{e}^{-\rho T}}{\rho}\left\{\zeta(y^{\kappa}-x^{\kappa}) + \theta\,(1+\kappa)^{-\gamma_2}\left[x^{\gamma_2(1+\kappa)} - y^{\gamma_2(1+\kappa)}\right]\right\}. \tag{14}$$

Condition (14) states that no agent can profitably sell his parcel and buy another one that would make him strictly better off: any *hypothetical gain in land value from selling the current parcel and buying a new one* must reflect differences in parcel size and distance to the CBD.[17]

The foregoing indifference condition also holds at the urban fringe, i.e., for the newcomers to the city: they compete with the incumbents for the new parcels available at the urban fringe, and hence land rents adjust so that everyone—incumbents plus newcomers—is indifferent across locations.[18] Evaluating condition (14) at the urban fringe parcel $y = F(t)$ at time t implies that the parcel price gradient is pinned down as follows:

$$\begin{aligned} p(x,t) - p(F,t) \quad &= \quad \frac{1-\mathrm{e}^{-\rho T}}{\rho}\left\{\zeta\left[x^{\kappa} - F(t)^{\kappa}\right]\right. \\ &\qquad \left. +\theta(1+\kappa)^{-\gamma_2}\left[F(t)^{(1+\kappa)\gamma_2} - x^{(1+\kappa)\gamma_2}\right]\right\}, \end{aligned} \tag{15}$$

which is a standard Alonso-Muth condition. Then, given a total city population $M(t)$ at time t, and since each of the $M(t)$ inhabitants owns one parcel and because the city is symmetric about the origin, from (11) the urban fringe parcel $F(t) = M(t)/2$ is located at the following distance from the CBD:

$$d\big(F(t)\big) = F(t)^{1+\kappa}\frac{1}{1+\kappa} = \left[\frac{M(t)}{2}\right]^{1+\kappa}\frac{1}{1+\kappa}. \tag{16}$$

To analyze the model and understand the role played by opportunities, we need to determine the price of land at the urban fringe. Since parcels differ along three dimensions—their distance from the CDB, their size, and the productivity of the under-

[17]This result hinges on our assumption that interest rates and loan terms are constant. Observe also that since instantaneous utilities are equalized across the city, life-time utilities are equalized too.

[18]Since land rents adjust at all times to make agents indifferent across locations, we can assume that once agents move to the city they stay on their parcels and do not move (there is nothing to be gained from moving to another location in equilibrium).

lying agricultural land—we need to analyze how $p(F,t)$ is determined. We can show the following:

Proposition 5. *Assume that the willingness to pay for the parcel characteristics $S(x)$ and $d(x)$ are a continuous function $\mathcal{O}(x) \equiv \mathcal{O}(S(x), d(x))$ of x. If the agricultural land rent $A\big(F(t)\big)$ is continuous in F, we have $p(F,t) = A\big(F(t)\big)$ for all t. In words, the land at the urban fringe sells for its opportunity cost at any point in time.*

Proof. See Appendix B.2. □

Proposition 5 is important since it establishes that even if there are opportunities that the owners of land may want to price into the sales of land, they cannot do so because of competition among very similar parcels at the urban fringe. Thus, the presence of opportunities that shape the movement of agents towards the city in this model does not rely on there being a 'missing market'.[19]

3.2.2 Lifetime utility and opportunities

Following Proposition 5, let $p(F,t) = A\big(F(t)\big)$ denote the price of parcels at the urban fringe. It then follows from (15), and from $F(t) = M(t)/2$ for the urban fringe parcel in a symmetric city, that

$$\begin{aligned} p(x,t) &= \frac{1-\mathrm{e}^{-\rho T}}{\rho}\left\{\zeta\left\{x^{\kappa}-\left[\frac{M(t)}{2}\right]^{\kappa}\right\}\right. \\ &\left.+\theta(1+\kappa)^{-\gamma_2}\left\{\left[\frac{M(t)}{2}\right]^{(1+\kappa)\gamma_2}-x^{(1+\kappa)\gamma_2}\right\}\right\}+A\left(\frac{M(t)}{2}\right). \end{aligned} \tag{17}$$

Equation (17) shows that the prices of parcels change over time for two reasons: first, the city expands and parcels increase in size; and, second, because of changes in the opportunity cost of land at the urban fringe. Fast growing cities that increasingly expand on valuable agricultural land (or recreational green space that people value highly) will thus see their prices increase more rapidly.[20]

[19] Our argument depends, of course, on a continuum of parcels and on the continuity of the opportunities and the opportunity cost of land. Furthermore, sellers cannot hold out on land sales. Developing a model with holdout on the seller side or where there is some 'market power' to price the opportunities would be an interesting future extension but is beyond the scope of this paper.

[20] Alternatively, there could be increases in the cost of housing supply as the city expands progressively into land that is harder to build on (e.g., steep slopes, wetland, or other unfavorable soil characteristics).

At period t, the size of the new city is by definition $M(t)$ and the available new parcels are at the city fringe $F(t)$. Hence, using (17), the total amount of the loan (12) required to buy at the urban fringe is as follows:

$$L\big(F(t), t\big) = \rho T \frac{A\big(M(t)/2\big)}{1 - \mathrm{e}^{-\rho T}}, \tag{18}$$

where we have used $F(t) = M(t)/2$. As is the case for the price of parcels, the value of the loan increases more in fast growing cities that increasingly expand on valuable land as captured by $A(M(t)/2)$.

In line with suggestive empirical evidence (see Table 2 in Appendix C), we assume that $A'(F) > 0$, i.e., the opportunity cost of land at the urban fringe increases as the city gets larger. This can be driven by at least two different mechanisms. First, construction progressively infringes on land that has a higher agricultural productivity and thus a higher opportunity cost. This would be predicted by standard models where developers successively buy the cheapest land at the urban fringe and develop land with a higher opportunity cost later. Second, the city fringe likely provides green amenities (think, e.g., of greenbelts around cities such as London, UK, or Toronto, Canada) that are valued by urban residents. Hence, city expansion puts a premium on those amenities (see, e.g., Koster, 2023), which increases the cost of land at the urban fringe as the city expands.

In Appendix B.3, we derive the expression for life-time utility—for an agent who moves to the city and buys a parcel at period τ—and show that the opportunity function in our model can be expressed as

$$R\big(M(\tau)\big) = \zeta \left[\frac{M(\tau)}{2}\right]^{\kappa} - \mathrm{e}^{T(\rho-\xi)} \rho A \left(\frac{M(\tau)}{2}\right) \frac{\mathrm{e}^{\xi T} - 1}{\mathrm{e}^{\rho T} - 1} - \theta \left[\frac{M(\tau)^{1+\kappa}}{2^{1+\kappa}(1+\kappa)}\right]^{\gamma_2}. \tag{19}$$

Overall life-time utility, starting at $t = 0$, takes into account that the agent earns income in the established city before moving to the new city. Hence, given the size $N_1(t)$ of the existing city at time t and payoffs $Y_1\big(N_1(t)\big)$ there, we have

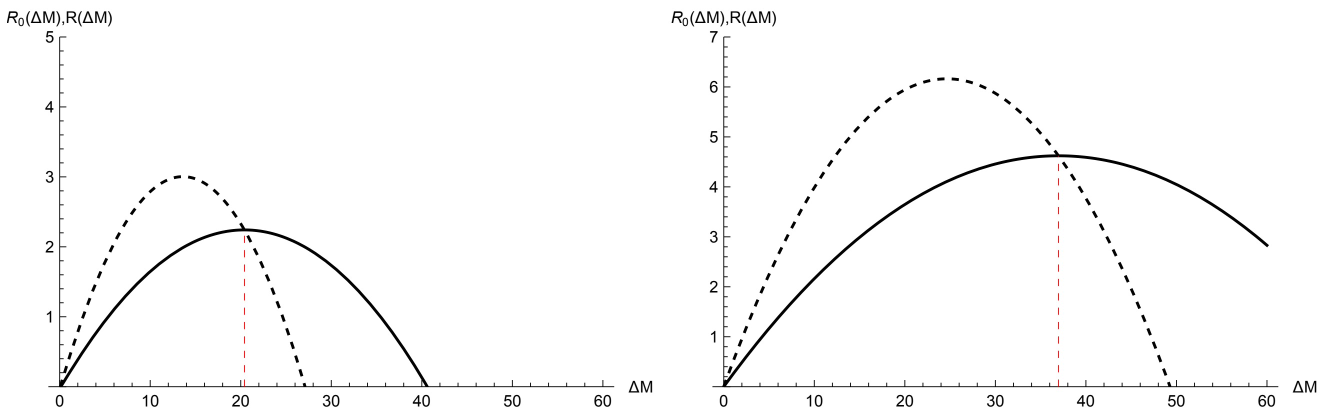
$$\begin{aligned} U(\tau) &= \int_0^{\tau} \mathrm{e}^{-\xi t} Y_1\big(N_1(t)\big) \mathrm{d}t + u(\tau) \\ &= \int_0^{\tau} \mathrm{e}^{-\xi t} Y_1\big(N_1(t)\big) \mathrm{d}t + a \int_{\tau}^{\infty} \mathrm{e}^{-\xi t} M(t)^{\varepsilon} \mathrm{d}t + R\big(M(\tau)\big) \frac{\mathrm{e}^{-\xi \tau}}{\xi}. \end{aligned} \tag{20}$$

3.2.3 The size of rushes

We can simulate the model numerically to illustrate its behavior and to investigate the size of rushes. Since $N(t) = N_0 + t^\alpha$, we have $N_1(t) = N_0 + t^\alpha - M(t)$. The payoff function for the existing city is given by (9).

Figure 5: Marginal and average opportunities for different interest rates.

(a) Opportunities ($\rho = 0.08$) (b) Opportunities ($\rho = 0.03$)

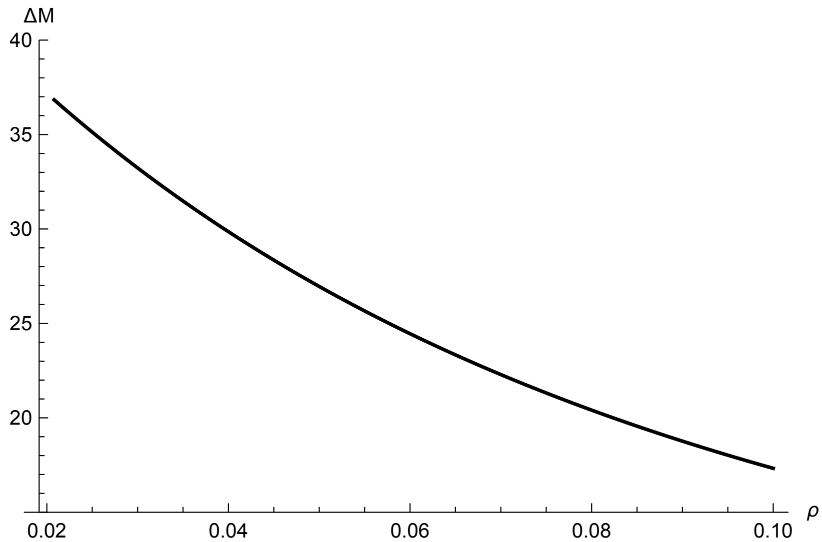


Notes: Marginal (dashed) and average (solid) opportunities. We set the agricultural rent as $A(M) = M^\beta$, with $\beta = 2$. The other parameter values for the figures are set as follows: $\varepsilon_1 = 0.2$, $\gamma_1 = 0.3$, $\varepsilon_2 = 0.04$, $\gamma_2 = 0.08$, $N_0 = 100$, $k = 0.4$, $\alpha = 2$, $\kappa = 1$, $\theta = 0.05$, $\xi = 0.05$, $T = 25$, and $\zeta = 0.9$.

To simulate the model we need the expressions for the average and marginal opportunities. We provide them in Appendix B.4. Figure 5 depict these average (solid) and marginal (dashed) opportunities for a larger value of mortgage rates ($\rho = 0.08$; panel (a)) and for a smaller value ($\rho = 0.03$; panel (b)). Panel (a) of Figure 5 shows that the marginal opportunity exceeds the average opportunity for $\Delta M \leq 20.41$: hence, the size ΔM of the rush cannot be smaller that 20.41 since an agent marginally outlasting the rush would obtain a discontinuously higher (marginal) opportunity from the move than the (average) opportunity of the movers. Conversely, the size ΔM of the rush cannot be larger that 20.41 since an agent marginally preempting the rush would obtain a discontinuously higher (marginal) opportunity from the move than the (average) opportunity of the movers. The size of the rush thus must be $\Delta M = 20.41$ as indicated by the dashed red line in panel (a) of Figure 5.

Panel (b) shows that the rush is larger ($\Delta M = 33.23$) when interest rates are lower. This is an illustration of our general result that a flatter opportunity function leads to larger rushes. We plot the size of the rush, ΔM, as a function of the mortgage rate ρ in Figure 6. As shown, higher rates progressively lead to smaller rushes as the opportunity function gradually becomes steeper.

Figure 6: Interest rates and size of the rush.



Notes: We set the agricultural rent as $A(M) = M^{\beta}$, with $\beta = 2$. In panel (a), ρ varies from 0.02 to 0.1. In panel (b), we plot the case where the foregone income in the established city is low ($\varepsilon_1 = 0.2$, solid line), and the case where the foregone income in the established city is high ($\varepsilon_1 = 0.5$, dashed line). The other parameter values for the figures are set as follows: $\gamma_1 = 0.3$, $\varepsilon_2 = 0.04$, $\gamma_2 = 0.08$, $N_0 = 100$, $k = 0.4$, $\alpha = 2$, $\kappa = 1$, $\theta = 0.05$, $\xi = 0.05$, $T = 25$, and $\zeta = 0.9$.

4 Entrepreneurship

In our second application, we investigate the flows of entrepreneurship, where opportunities allow us to acquire entrepreneurship human capital. First, we provide a motivating example by looking at industries in the United States. Second, we build a model of entrepreneurship in the context of two industries. In both cases, opportunities to gain entrepreneurship human capital depend on whether an industry is relatively young. Newer industries provide more opportunities than older established industries.

4.1 Motivating examples

We posit that workers trade off pay for experiences that help them gain entrepreneurship human capital (Becker, 1962). Further, we suggest that the experiences that build entrepreneurship human capital depend on a worker's rank within an industry or firm. Early workers are given a broader portfolio of tasks to do that may more quickly build entrepreneurship human capital. In addition, early workers gain a better understanding of how the firm and industry operate and may be able to use that knowledge to start their own businesses. We consider differences in flows between industries to investigate whether rank-dependent opportunities may help explain these differences.

Consider the growth between the 1980s and 2000s in finance (an established industry) and technology, or 'tech' for short (a new industry). Several stylized facts match our model's predictions. First, wages are initially higher in finance than in tech. Second, an individual could be of higher rank in the tech industry than in finance—but there was a risk in tech due to fewer jobs. This characterization of the opportunities suggests that they were nonmonotonic—you wanted to be early but not first. Third, as the model predicts with a nonmonotonic opportunity function, there was a tech boom (a rush), where the industry grew rapidly.[21]

4.2 A formal model of industry growth and entrepreneurship

Our entrepreneurship model builds on the idea that workers trade off earnings and other benefits such as human capital accumulation (Becker, 1962). There are opportunities to being early to an industry because early entrants have a higher rank or seniority and gain more business acumen. We also allow for there to be costs to entering an industry too early. For instance, there is increased displacement risk in young industries and that risk is especially high in smaller cities. In a nutshell, there are opportunities to being early but not necessarily first.

Let us start with the income function. Assume that entities 1 and 2 are industries—one established with $N_1(t)$ workers and one new with $N_2(t)$ workers. Each worker receives industry-specific earnings that combine agglomeration and rivalry externalities, $A_i(N_i)$ and $P_i(N_i)$, respectively, as follows. $Y_i\big(N_i(t)\big) = A_i\big(N_i(t)\big) + P\big(N_i(t)\big)$. As an industry grows, firms can share suppliers and other fixed costs and better match with workers (Duranton and Puga, 2004). Initially, these agglomeration benefits are large. Formally, we model this as $A\big(N_i(t)\big) = -a_i[(1 + N_i(t))^{-1}) - 1]$, following Buchanan (1965).[22] As an industry grows, competition increases, which bids up the cost of inputs.

[21]In Appendix C.2, we provide suggestive empirical evidence consistent with our mechanism of opportunities due to higher human capital accumulation in the younger industry. For example, we find the flow of entrepreneurs is positively correlated with the tech industry's growth and negatively correlated with the finance industry's growth.

[22]Buchanan (1965) used a similar specification to discuss the agglomeration benefits and congestion costs of public goods. We refer to the assumption that income increases with industry size as an agglomeration externality. Since the model is dynamic and since industry size $N_i(t)$ reflects some accretion of

This rivalry externality is likely small initially and eventually grows large. Formally, we model this as $P\big(N_i(t)\big) = (N_i(t)+1)^{-\gamma} - 1$, where $\gamma < 1$.

With these agglomeration and rivalry externalities, income is: (i) initially increasing with the workforce in the industry and eventually decreasing; and (ii) initially higher in industry 1 (the established industry) than in industry 2 (the new industry). Note that when industry 2 becomes sufficiently developed, it could provide higher incomes. However, until industry 2 has reached a critical mass, workers who choose to work there forgo the higher income in industry 1. Hence, for workers to move to industry 2, that industry must provide them with opportunities.

We model opportunities as gains in human capital. Workers gain this human capital depending on their rank within the industry, which depends on the timing of the move. For individuals to start a successful business, they need what is typically called "business acumen" (Liang et al., 2018). Not all job experiences, however, lead to business acumen. Individuals that are given low-level tasks likely gain less business acumen than those who have a broader portfolio of tasks and decision-making authority. The amount of human capital an individual acquires therefore depends on that person's rank within the industry. Individuals selecting an industry trade off the initially lower incomes in the newer industry with the opportunity to accumulate the human capital that these industries provide by allowing them to move up more quickly.

Working in a new industry is not without risk, however. Individuals are much more likely to be displaced in a new industry, due to increased demand and cost uncertainty. The cost of displacement is not only the lost wages during the transition but a loss of firm- and industry-specific human capital, which can lead to depressed future earnings (a 'scarring effect'; Topel, 1990; Neal, 1995). The cost of this risk, however, decreases as an industry grows, because the probability that a worker can find a job within the industry increases with industry size.

Formally, assume there exists a rank $\overline{M}$ such that ranks greater than $\overline{M}$ do not benefit from being early. Let an individual's human capital accumulation $m_i(\cdot)$ be an increasing function of the share of the workforce $s_i(\cdot)$ below their rank $M(\tau)$. The share of the

physical and human capital over the past, these terms can also be viewed as capturing industry maturity.

workforce below rank $M(\tau)$ who benefits from being early in an industry is given by

$$s(M(\tau)) = \begin{cases} [e^{-\rho_i(\overline{M}-M(\tau))/\overline{M}} - 1]/(e^{-\rho_i} - 1), & \text{if } M(\tau) < \overline{M}. \\ 0, & \text{otherwise,} \end{cases}$$

where ρ_i is the expected steady-state growth rate of industry i. We capture the risk to workers in new industries as $-\rho_i\zeta_i M(\tau)e^{-\rho_i\zeta_i M(\tau)}$, where a high ζ_i denotes a risky industry, and we maintain that $\rho_i\zeta_i < 1$. In this formulation, the risks are decreasing and convex for industries with positive flows and increasing and convex for industries with negative flows. The opportunities a worker receives from moving to a new industry combine the benefits of human capital accumulation and the risk due to displacement as follows: $R_i(M(\tau)) = m_i(s(M(\tau)) + \rho_i\zeta_i M(\tau)e^{-\rho_i\zeta_i M(\tau)}$. If a new industry were perfectly safe, $\zeta_i = 0$, then the opportunity function would be monotonically decreasing. Riskier industries, however, may cause the opportunity function to be hump-shaped—indicating the benefit of being early but not first.

5 Conclusions

Economic change in cities and industries is often characterized by rushes—the rapid and simultaneous movement of new workers and firms from established cities or industries to new ones. We have proposed a model that generates such rushes. Contrary to the literature that explains rapid change and correlations in individual decisions through what may be broadly called 'information externalities,' our model does not need the presence of such externalities. In our model, rushes occur because agents trade off time-varying incomes against time-invariant but rank-dependent opportunities. Being early is good, but being too early is not. When payoffs generated by opportunities are non-monotonic, agents are enticed to wait—nobody wants to preempt the rush and be first—but not wait too long—since opportunities become less valuable after the rush has occurred. Put simply, "I'd rather be second than first, but not third."

We have derived general results and illustrated them using simple models drawn from urban economics and industrial organization. The applicability of our ideas is broader than that because many economic phenomena offer payoffs that are a mixture of size-dependent—but rank-independent—incomes and size-independent—but rank-dependent—opportunities. The simple examples we have developed suggest that our mechanism is relevant. Devising empirical tests that identify it, disentangle it from alternative explanations—such as herding or information cascades—and quantify its magnitude goes beyond this paper and is left for future research.

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Appendix material

Appendix A Proofs

Appendix A.1 Proof of Theorem 1

To show the existence and uniqueness of the equilibrium, we use a natural refinement of mixed strategy Nash equilibria in timing games, called ε-safe (Anderson, Park and Smith, 2017). This refinement resembles a trembling hand refinement, and, in our context, it ensures there is only one equilibrium rush size. It excludes equilibria where tiny timing mistakes $\varepsilon > 0$ on both sides at time τ cause a significant payoff loss. This refinement does not exclude equilibria if only early movement or only late movement causes a significant payoff loss in equilibrium. In these cases, an individual could guard against these losses by ensuring they were never early or never late. This provides the intuition for the ε-safe payoff:

$$U_{\varepsilon}(\tau, M(\tau)) = \max\left\{\inf_{\max\{t-\varepsilon,\ 0\}\leq s<t} U(s, M(s)),\ \inf_{s\in[t,t+\varepsilon)} U(s, M(s))\right\}.$$

A Nash equilibrium is safe if there exists $\bar{\varepsilon} > 0$ such that $U_{\varepsilon}(\tau, M(\tau)) = U(\tau, M(\tau))$ for all $t \in [\tau_1, \infty)$ and for all $\varepsilon \in (0,\ \bar{\varepsilon})$. Anderson, Park and Smith (2017) prove that this refinement excludes equilibria with periods of inaction.

The following lemma will be useful for proving the theorem.

Lemma 1 (Regularity assumption)**.** *Assume the opportunity function $R(M(\tau))$ is: (i) monotonically increasing; or (ii) initially decreasing and then increasing. Then there does not exist an equilibrium.*

Proof. (i) Suppose toward contradiction there is an equilibrium for an entity with an opportunity function that is monotonically increasing. Consider a deviation from this equilibrium where an individual moves to entity 2 at time $t+\varepsilon$ instead of their equilibrium prescribed time $t > \tau_1$, where τ_1 is the time where $M(\tau) > 0$ for the first time. This is a profitable deviation because the individual receives more income and more opportunities. The income in entity 1 is higher than in entity 2. Therefore, there is no equilibrium in which the opportunity function $R(M(\tau))$ is monotonically increasing.

(ii) Suppose toward contradiction there is an equilibrium for an entity with an opportunity function that is initially decreasing and then increasing. Consider a deviation from this equilibrium where an individual moves to entity 2 at time $t + \varepsilon$ instead of their equilibrium prescribed time $t > \tau_1$. Further, assume at time t and $t + \varepsilon$, the opportunity function is increasing (we are beyond the minimum of the average opportunity). Then, by the same reasoning as above, the deviation is profitable as the individual will receive higher income and better opportunities. □

We can now prove Theorem 1.

Proof. The proof of the theorem proceeds by construction and uses the regularity assumption in equation (4), the Implicit Function Theorem, and the equilibrium refinement $\varepsilon-$safe equilibrium (Anderson, Park and Smith, 2017).

Step 1: Initial period with no growth in entity 2. By condition (5), initially moving is worse than staying (given any feasible size of movement).

Step 2: Eventual growth in entity 2. Given that population is growing and $Y_1'(N(0)) < 0$, there exists a time $\tau < \infty$ and a mass of agents ΔM such that it is strictly better to move: $Y_1(N(\tau) - \Delta M) < Y_2(\Delta M) + R(\Delta M) + \dot{R}(\Delta M)/r$.

Step 3: Unique starting time in entity 2. Proposition 3 establishes that there exists a unique rush size ΔM_1. Then, by the smoothness of the income and opportunity functions, there exists a unique point τ_1 where entity 2 begins to grow, determined by $Y_1(N(\tau_1) - \Delta M_1) = Y_2(\Delta M_1) + R(\Delta M_1) + \dot{R}(\Delta M_1)/r$.

Step 4: Implicit Function Theorem. Since utility is continuously differentiable after τ_1 (recall from Proposition 3 that there are no more atoms after τ_1) $m(t)$ solves the equilibrium condition $\partial U/\partial\tau = 0$, for $\tau \in [\tau_1, \infty)$. The Implicit Function Theorem implies, $\partial_\tau U(\tau, M(\tau))\mathrm{d}\tau + \partial_M U(\tau, M(\tau))\mathrm{d}M = 0$ which reduces to

$$\frac{\mathrm{d}M(\tau)}{\mathrm{d}\tau} = -\frac{\partial_\tau U(\tau, M)}{\partial_M U(\tau, M)} > 0$$

by condition (6). Therefore, there exists an equilibrium with $m(\tau) \equiv \frac{\mathrm{d}M(\tau)}{\mathrm{d}\tau} > 0$ for all $\tau \in [\tau_1, \infty)$.

Step 5: $\varepsilon-$**safe equilibria** The constructed equilibrium is the only one without periods of inaction and is a $\varepsilon-$safe equilibrium. Other potential equilibria with inaction are not $\varepsilon-$safe equilibria (Anderson, Park and Smith, 2017). □

Appendix A.2 Proof of Proposition 2

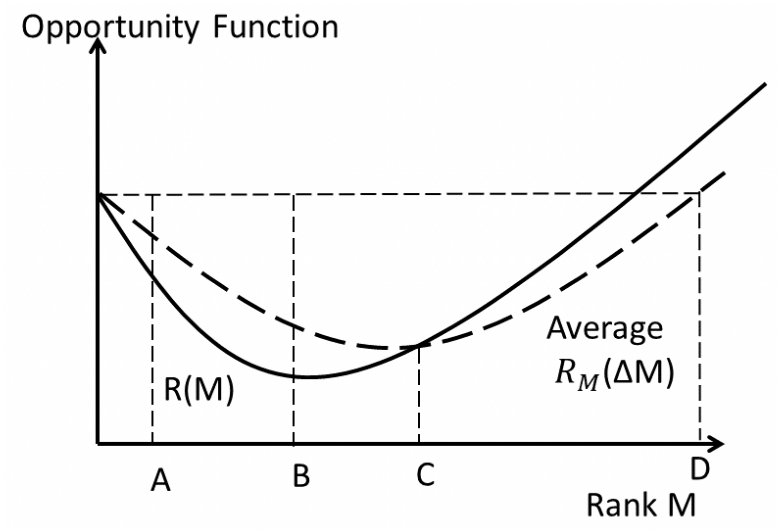
Proposition 1 is proved in the main text. We now prove Proposition 2.

Proof. The proof involves four successive steps.

Step 1 (No equilibrium with slow creation). Assume (1.a) that the opportunity function is non-monotonic and initially increasing, as depicted in panel (b) of Figure 2, but that there exists an equilibrium without a rush. For this to be an equilibrium there must be no profitable deviation from the equilibrium strategy. Consider the first individual to move to entity 2. Income there is lower than in the existing entity but increases with time as more individuals move to entity 2. The opportunity function is also initially increasing by assumption (1.a). This shows that the first individual has an incentive to delay his move, thereby avoiding some time with lower income and receiving greater opportunities associated with being of a lower rank. Therefore, there does not exist an equilibrium where the entity is created with gradual migration when the opportunity function is non-monotonic and initially increasing.

Step 2 (No rush with monotonic opportunities). To demonstrate that the creation of an entity by a rush is possible only when the opportunity function is non-monotonic and initially increasing, consider the cases with a monotonic opportunity function and a non-monotonic opportunity function that is initially decreasing. Begin with a monotonic opportunity function. The size of the rush is determined by the point at which the average opportunity function intersects the (marginal) opportunity function. For monotonic opportunity functions, the average opportunity function and the opportunity function intersect only for the first mover ($R_0(0) = R(0)$). For a rush of size ΔM, we have either $R_0(\Delta M) < R(\Delta M)$ (when the opportunity function is increasing) or $R_0(\Delta M) > R(\Delta M)$ (when the opportunity function is decreasing). Hence, the arbitrage condition is violated and individuals pre-empt the rush if the opportunity function is decreasing or outlast the rush if the opportunity function is increasing.

Figure 7: Potential Opportunity Function



Step 3 (No rush with initially decreasing non-monotonic opportunities). Consider a non-monotonic opportunity function that is initially decreasing, as depicted in Figure 7. In this case, although there exists a ΔM such that $R(\Delta M_1) = R_0(\Delta M_1)$ it is the minimum value for $R_0(\Delta M)$, as depicted in Figure7. Hence, there is room for arbitrage and individuals pre-empt the rush because $R(0) > R_0(\Delta M_1)$. Hence, no equilibrium with a rush exists.

Step 4 (Existence of equilibrium with a rush). To demonstrate that an equilibrium exists where entity 2 is formed by a rush, consider the payoffs of individuals who create that entity and who receive the average opportunity during the rush. The entity is created where $R_0(\Delta M_1) = (1/\Delta M_1) \int_0^{\Delta M_1} R(M(t))\mathrm{d}t$, where the size of the rush is ΔM_1. This condition states that the average opportunity must equal the (marginal) opportunity at the rank equal to size of the rush. If this condition did not hold, there would be arbitrage opportunities: (i) for individuals in the rush to wait and move right after the rush (if $R_0(\Delta M_1) < R(\Delta M_1)$); or (ii) for individuals after the rush to join the rush (if $R_0(\Delta M_1) > R(\Delta M_1)$). Finally, there could be opportunities for individuals to pre-empt the rush because $R_0(\Delta M_1) > R(0)$. The latter inequality holds because the average opportunity is maximized where it intersects the marginal opportunity. Therefore, there exists an equilibrium where entity 2 is formed by a rush when the opportunity function is non-monotonic and initially increasing. □

Appendix A.3 Proof of Proposition 4

Proposition 3 is proved in the main text. We now prove Proposition 4.

Proof. The size of the rush is determined by

$$\frac{1}{\Delta M}\int_0^{\Delta M} R(M(t))\mathrm{d}t = R(\Delta M),$$

Clearly, if $R_\phi(M) = \phi R(M)$ or $R_\phi(M) = R(M) + \phi$, we have

$$\frac{1}{\Delta M}\int_0^{\Delta M} R_\phi(M(t))\mathrm{d}t = R_\phi(\Delta M) \quad \Rightarrow \quad \frac{\phi}{\Delta M}\int_0^{\Delta M} R(M(t))\mathrm{d}t = \phi R(\Delta M)$$

and

$$\frac{1}{\Delta M}\int_0^{\Delta M} R_\phi(M(t))\mathrm{d}t = R_\phi(\Delta M) \quad \Rightarrow \quad \frac{1}{\Delta M}\int_0^{\Delta M} R(M(t))\mathrm{d}t + \phi = R(\Delta M) + \phi,$$

so that ΔM is the same in both cases. Any multiplicative or additive scaling does not affect the point where the average equals the marginal.

Last, if $R_\phi(M) = R(\phi M)$ we have

$$\frac{1}{\Delta M}\int_0^{\Delta M} R_\phi(M(t))\mathrm{d}t = R_\phi(\Delta M) \quad \Rightarrow \quad \frac{1}{\Delta M}\int_0^{\Delta M} R(\phi M(t))\mathrm{d}t = R(\phi\Delta M).$$

We know that the equality holds for $\phi = 1$. Assume that $\phi > 1$. The foregoing equation continues to hold true for $y = M/\phi < M$ since $\phi > 1$. Hence, the intersection between the marginal and the average shifts to the left, and the size of the rush decreases. Conversely, when $0 < \phi < 1$, we let $y = M/\phi > M$ since $\phi < 1$. The intersection between the marginal and the average shifts to the right and the size of the rush increases. □

Appendix B Expressions and proofs for the urban model

Appendix B.1 Derivation of (14)

Utility must be equalized across locations, which requires that

$$aM(t)^{\varepsilon_2} - \theta d(x)^{\gamma_2} - \ell(x,t) + \zeta S(x) = aM(t)^{\varepsilon_2} - \theta d(y)^{\gamma_2} - \ell(y,t) + \zeta S(y)$$

for any pair of parcels (x, y) in the city. The foregoing condition can be rewritten as follows:

$$\begin{aligned} \ell(y,t) - \ell(x,t) &= \frac{\rho}{1 - \mathrm{e}^{-\rho T}} \left[p(y,t) - p(x,t)\right] \\ &= \zeta(y^{\kappa} - x^{\kappa}) + \theta\,(1+\kappa)^{-\gamma_2} \left[x^{\gamma_2(1+\kappa)} - y^{\gamma_2(1+\kappa)}\right], \end{aligned}$$

which finally implies that

$$p(y,t) - p(x,t) = \frac{1 - \mathrm{e}^{-\rho T}}{\rho} \left\{\zeta(y^{\kappa} - x^{\kappa}) + \theta\,(1+\kappa)^{-\gamma_2} \left[x^{\gamma_2(1+\kappa)} - y^{\gamma_2(1+\kappa)}\right]\right\}.$$

Appendix B.2 Proof of Proposition 5

Proof. Moving to the city and buying at time t a parcel at the urban fringe $F(t)$ offers opportunities that depend on t. Notably, the characteristics of available parcels changes with time. For simplicity, we only consider two such characteristics: their distance $d(x(t))$ to the city center, and their sizes $S(x(t))$. These depend on how many other agents $M(t)$ moved to the city before. Furthermore, the opportunity cost of the seller, equal to the underlying productivity of the agricultural land $A\big(F(t)\big)$, also changes with the timing of the move.

Let $\mathcal{O}(x)$ denote the willingness to pay (WTP) for the opportunities offered by parcel x. We assume that absentee parcel owners do not consider living themselves on the parcels, i.e., the opportunities offered by the parcel will not be directly 'consumed' by the absentee owners. However, if possible, the absentee owners want to extract the rents associated with these opportunities via the sales price of their land.

Consider the owners of parcels F and $F + \mathrm{d}F$ at time t. The minimal prices they are willing to quote are $A(F, t)$ and $A(F + \mathrm{d}F, t)$, respectively, i.e., the opportunity cost of their land. The maximum prices the owners can quote are restricted by the WTP, $\mathcal{O}(F)$ and $\mathcal{O}(F + \mathrm{d}F)$. Hence, we have $A(F, t) \leq p_F \leq \mathcal{O}(F)$ and $A(F + \mathrm{d}F, t) \leq p_{F+\mathrm{d}F} \leq \mathcal{O}(F_0 + \mathrm{d}F)$. If the agricultural land rent A and the opportunities $\mathcal{O}$ are continuous, then for $\mathrm{d}F \to 0$ we have $\lim_{\mathrm{d}F\to 0} \mathcal{O}(F + \mathrm{d}F) = \mathcal{O}(F)$ and $\lim_{\mathrm{d}F\to 0} A(F + \mathrm{d}F) = A(F)$. Since opportunities are essentially the same on two close parcels, the agent will buy at the lowest price. Hence, the standard Bertrand price competition undercutting argument with equal marginal cost applies and p_F goes to $A(F)$, i.e., the sales price equals the opportunity cost of land. □

Appendix B.3 Derivation of (19)

Utility at time t, conditional on moving at time τ, is given by

$$u\big(F(\tau), M(t)\big) = \begin{cases} aM(t)^{\varepsilon_2} - \theta d\big(F(\tau)\big)^{\gamma_2} - \ell\big(F(\tau)\big) + \zeta S\big(F(\tau)\big) & \text{if} \quad \tau \leq t \leq \tau + T \\ aM(t)^{\varepsilon_2} - \theta d\big(F(\tau)\big)^{\gamma_2} + \zeta S\big(F(\tau)\big) & \text{if} \quad t > \tau + T, \end{cases}$$

where the first line applies to the period where the mortgage is amortized, and the second line applies to the period where the mortgate has been fully repaid. Since we assume constant mortgage payments, the indirect utility changes discontinuously in period $\tau + T$. The terms $d(\cdot)$, $\ell(\cdot)$, and $S(\cdot)$ depend on the timing of the move, τ, only, and are hence the continuous value of the opportunity that the agent locks in when moving.

Life-time utility in the new city of an agent who moves there and buys a parcel at period τ is given by

$$u(\tau) = \int_{\tau}^{\infty} \mathrm{e}^{-\xi t} u\big(F(\tau), M(t)\big) \mathrm{d}t,$$

where $\xi > 0$ is the discount factor. Using the size of the parcel at time τ, given by $S\big(F(\tau)\big) = [M(\tau)/2]^{\kappa}$ and computing

$$\int_{\tau}^{\tau+T} \mathrm{e}^{-\xi t} \ell\big(F(\tau)\big) \mathrm{d}t = \mathrm{e}^{T(\rho-\xi)} \rho A\big(F(\tau)\big) \frac{\mathrm{e}^{\xi T} - 1}{\mathrm{e}^{\rho T} - 1} \frac{\mathrm{e}^{-\xi\tau}}{\xi}$$

we can finally express the life-time utility in the new city as follows:

$$\begin{aligned} u(\tau) &= a \int_{\tau}^{\infty} \mathrm{e}^{-\xi t} M(t)^{\varepsilon_2} \mathrm{d}t \\ &\quad + \underbrace{\left\{ \zeta \left[\frac{M(\tau)}{2} \right]^{\kappa} - \mathrm{e}^{T(\rho-\xi)} \rho A \left(\frac{M(\tau)}{2} \right) \frac{\mathrm{e}^{\xi T} - 1}{\mathrm{e}^{\rho T} - 1} - \theta \left[\frac{M(\tau)^{1+\kappa}}{2^{1+\kappa}(1+\kappa)} \right]^{\gamma_2} \right\} \frac{\mathrm{e}^{-\xi\tau}}{\xi}}_{R(M(\tau))\frac{\mathrm{e}^{-\xi\tau}}{\xi}}, \end{aligned}$$

where we have used $d(F(\tau))$ from (16). The first term in the foregoing expression is the discounted sum of (time-varying) life-time income in the new city and the second term is the opportunities that are locked in at time τ when the agent moves to the new city.

The opportunity function in our model is thus given by

$$R\big(M(\tau)\big) = \zeta \left[\frac{M(\tau)}{2}\right]^{\kappa} - \mathrm{e}^{T(\rho-\xi)}\rho A \left(\frac{M(\tau)}{2}\right) \frac{\mathrm{e}^{\xi T}-1}{\mathrm{e}^{\rho T}-1} - \theta \left[\frac{M(\tau)^{1+\kappa}}{2^{1+\kappa}(1+\kappa)}\right]^{\gamma_2}.$$

Appendix B.4 Expressions for simulation

The average and marginal opportunities for the formation of a new city following a rush of size ΔM, at the formation period τ when $m(t) = 0$ for all $t < \tau$ (so that $M(t) = 0$ for all $t < \tau$ and $M(\tau + \mathrm{d}\tau) = \Delta M$ for $\mathrm{d}\tau \geq 0$), are given by:

$$\begin{aligned} R_0(\Delta M) &= \frac{1}{\Delta M}\int_0^{\Delta M} \left\{\zeta \left(\frac{M}{2}\right)^{\kappa} - \mathrm{e}^{T(\rho-\xi)}\rho A \left(\frac{M}{2}\right) \frac{\mathrm{e}^{\xi T}-1}{\mathrm{e}^{\rho T}-1} - \theta \left[\frac{M^{1+\kappa}}{2^{1+\kappa}(1+\kappa)}\right]^{\gamma_2}\right\} \mathrm{d}M \\ R(\Delta M) &= \zeta \left(\frac{\Delta M}{2}\right)^{\kappa} - \mathrm{e}^{T(\rho-\xi)}\rho A \left(\frac{\Delta M}{2}\right) \frac{\mathrm{e}^{\xi T}-1}{\mathrm{e}^{\rho T}-1} - \theta \left[\frac{(\Delta M)^{1+\kappa}}{2^{1+\kappa}(1+\kappa)}\right]^{\gamma_2}. \end{aligned}$$

Appendix C Details on the empirical illustrations

Appendix C.1 City model

Our model makes two assumptions. First, parcel sizes increase as we move away from the CDB (i.e., as the city grows). Second, the value of the opportunity cost of land increases as the urban fringe moves outwards. This appendix provides some tentative illustrative evidence for these two assumptions.

Table 1: Elasticity of parcel size with respect to distance from the CDB, Montreal, 2021.

	(1) log parcel size	(2) log parcel size	(3) log parcel size	(4) log parcel size
log distance CDB	0.9795[a]	0.9997[a]	1.3339[a]	1.2296[a]
	(0.002)	(0.002)	(0.013)	(0.014)
log distance CDB × residential use			-0.3680[a]	-0.3151[a]
			(0.013)	(0.013)
residential use			-2.4825[a]	0.1581[a]
			(0.078)	(0.035)
Other use-category fixed effects			✓	✓
Municipality fixed effects				✓
Observations	499,119	499,119	499,119	499,119
R-squared	0.3059	0.3798	0.3834	0.4086

Notes: The dependent variable is the log of the land surface of the parcel. All distances are measured with respect to the KPMG tower in downtown Montréal, which we take as our proxy the the CBD. The use categories are defined by Québec's Ministry of Municipal Affairs and Housing (`https://www.mamh.gouv.qc.ca/evaluation-fonciere/manuel-devaluation-fonciere-du-quebec/ codes-dutilisation-des-biens-fonds/`). Our residential dummy takes value one for use category 1 (residential) and zero otherwise. The parcel data is available from the Open Data portal of the city of Montréal (`https://donnees.montreal.ca/dataset/unites-evaluation-fonciere`). What we call 'municipality' fixed effects are fixed effects for the different boroughs (arrondissements) of Montréal.

Table 1 shows regressions of log parcel size on log distance from the CBD for the city of Montreal. As shown, the residential parcel size has a distance elasticity of close to -1 and is precisely estimated, even when controlling for parcel use-type and borough fixed effects. Clearly, residential plots become larger as we move away from the city center.

Table 2: Agricultural land rents in the US by county and type, cross sections in 2010 and 2020.

	(1) log irrigate	(2) log non-irrigate	(3) log pasture	(4) log irrigate	(5) log non-irrigate	(6) log pasture
log county population	0.1107^{a}	0.0360^{a}	0.1216^{a}	0.0865^{a}	0.0464^{a}	0.0624^{a}
	(0.015)	(0.012)	(0.011)	(0.014)	(0.013)	(0.012)
State fixed effects	✓	✓	✓	✓	✓	✓
Observations	606	2,166	2,024	675	2,258	1,949
R-squared	0.0853	0.0043	0.0536	0.0512	0.0058	0.0133

Notes: The dependent variable is the rent per acre of agricultural land. It is measured in $ of expenses per acre. Source: https://quickstats.nass.usda.gov/results/E0F5EB36-3313-3D7B-9E7F-E56A3365CF2B#9A9F55D7-E267-38C6-ACB9-DF106291B5A7. The population data by county and year is from https://seer.cancer.gov/stdpopulations/ for 2008-2023 and for the total population (all ages).

Table 2 uses US data to provide suggestive evidence that agricultural rents are higher in more populous places, which increases the opportunity cost of agricultural land at the urban fringe. This would be predicted by any standard model where developers progressively buy up the cheapest land at the urban fringe and progressively develop on land with a higher opportunity cost later. Table 2 shows that—irrespective of the type of agricultural land—more populous counties have higher agricultural land rents, i.e., higher opportunity costs of converting that land to urban use. Note that we include state fixed effects, i.e., we identify the effects from within-state variation in countr ies' population sizes.

Of course, we do not provide any attempt at causal identification and the foregoing relationships are pure correlations. Yet, we believe that they would hold up in more detailed analyses.

Appendix C.2 Industry model

We use census data on the growth of the finance and technology (tech) industries between 1980 and 2000 to investigate the size of rushes following Proposition 4. These two industries represent important industries with similar levels of skill but different relative ages. The finance industry is old and established, while the technology industry is relatively new (and was especially so in 1980). These two industries, therefore, provide an ideal comparison for growth.

The model states that initially, wages in the technology industry are lower than those in finance but that there are opportunities to accumulate entrepreneurship human capital. This suggests places with faster tech growth will also have high growth in self-employed people. While there are opportunities in the technology industry, it is also risky. There are fewer jobs in the technology industry than in finance, and this is magnified in some smaller geographic areas. This risk can create a nonmonotonic opportunity function—inducing a rush into the tech industry. Further, following Proposition 4, the rush will be larger in cities with fewer initial tech jobs, where the risk of entering the tech industry is larger.

We use data from the Integrated Public Use Microdata Series (IPUMS) of the US Census and the County Business Patterns (CBP). These data include individual-level data on occupation (NAICS), worker class (e.g., self-employed or private wage earner), wages, and demographic information (e.g., age, sex, race, and education). We report descriptive evidence from the finance and technology industries between 1980 and 2000 in Table 3. Columns (1) and (2) show that (mincerized) wages in 1980 are 2.6% lower in the technology industry than in finance, but by 2000, they are 2.7% higher in the technology industry. Given this discrepancy in wages, we would predict that, for individuals to be willing to enter the tech industry, it must provide entrepreneurship human capital. We report evidence in support of this in columns (3) through (6).

The growth rate of self-employed individuals in MSAs is positively correlated with the MSA's growth rate in the technology industry and negatively correlated with its growth rate in the finance industry. This correlation is robust to controlling for the total growth rate (column 3) and including state- and industry-level controls (column 4). We report placebo tests in columns (5) and (6), using the growth rate of private sector wage earners and government employees. For both of these worker classes, their growth rate is not positively correlated with the growth rate in the technology industry. This evidence is consistent with our model, where new industries provide entrepreneurship human capital in lieu of the higher wages paid in more established industries.

As a result of these opportunities, the technology industry boomed between 1980 and 2000. Column (7) shows that the growth rate in the technology industry between 1980 and 2000 was substantially larger than in finance. Consistent with Proposition 4, we find that technology growth was higher in cities with smaller initial amounts of technology employment (column 8). Entering the industry in these cities is riskier because there are fewer potential jobs in case of job displacement. This risk creates an initially flatter opportunity function and therefore faster growth or larger rushes. For example, we find the technology growth rate was higher in Provo, UT (low initial employment), than in Ann Arbor, MI (high initial employment). This evidence is supported by the resulting wage growth rates in the technology industry, relative to finance, and in small initial computer industry MSAs, relative to large initial tech industry MSAs. Columns (9) and (10) show that wages grew faster in technology than in finance and the wage grew even faster in small initial tech MSAs, though the last result is not precisely estimated.

To summarize, rank-dependent opportunities may explain rushes into and growth differences between industries. We find evidence that job and wage growth were faster in technology than in finance—and even faster in MSAs with relatively low tech in 1980, which made the industry geographically riskier. We also find that growth in technology jobs, and not finance jobs, is correlated with more entrepreneurship.

Table 3: Finance and tech industries growth, 1980 to 2000.

Dependent variable	Wage		Growth rate worker class				Growth rate industries		Growth rate wages	
	1980	2000	Self-employed		Private	Gov.	Both	Tech	Both	Tech
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Tech industry	-0.026^a	0.027^a					1.584^a		1.228^a	
	(0.000)	(0.000)					(0.569)		(0.227)	
MSA small initial tech industry								2.154^b		0.476
								(1.028)		(0.469)
Growth rate tech industry			0.411^a	0.474^a	-0.010^c	-0.036				
			(0.077)	(0.087)	(0.005)	(0.055)				
Growth rate finance industry			-1.227^b	-0.909	0.165^a	0.585				
			(0.501)	(0.553)	(0.034)	(0.357)				
Total growth rate			2.044^a	1.629^b	0.875^a	0.310				
			(0.581)	(0.642)	(0.039)	(0.414)				
MSA fixed effects	Yes	Yes	No	No	No	No	No	No	No	No
State fixed effects	No	No	No	Yes	No	No	No	No	No	No
Level controls	No	No	No	Yes	No	No	No	No	No	No
Adj. R-Square	0.601	0.714	0.898	0.901	0.999	0.804	0.015	0.015	0.060	0.000
Observations	270,096	384,092	223	223	223	223	446	223	446	223

Notes: The dependent variable in columns (1) and (2) are the mincerized wages, where the sample is restricted to 1980 or 2000 respectively. The dependent variable in columns (3)–(6) is the growth rate in worker class; self-employed incorporated in (3) and (4), private wage earners in (5), and government in (6). The dependent variable in columns (7) and (8) is the growth rate in the finance and tech industries in (7) and the tech industry in (8). The dependent variable in columns (9) and (10) is the wage growth rate in the finance and technology industries in (9) and the technology industry in (10). An observation in columns (1) and (2) is a person from the IPUMS sample. An observation in columns (3)–(6), (8), and (10) is an MSA. An observation in columns (7) and (9) is an MSA industry. The independent variable *Tech industry* is an indicator for wage earner or growth rate being in the technology industry. The independent variable *MSA small initial tech industry* is an indicator variable equal to one if an MSA initially has fewer technology employees than the median in 1980 and zero otherwise. Additional controls in column (4) are the aggregate number of workers in the technology and finance industries. Standard errors in parentheses. ^{a} $p < 0.01$; ^{b} $p < 0.05$; ^{c} $p < 0.1$

Supplemental online appendix

Appendix D Alternative model of city growth

We build on the canonical model of cities of Henderson (1974), where city size is determined by the trade-off between agglomeration and congestion externalities. The benefit of living in a city is initially increasing with population but eventually decreases when the city becomes too large. How new cities form as the population in established cities grows is a thorny issue in that literature because of coordination failure in migration decisions (the so-called 'migration pathology'): no one wants to be the only one to move to a new place—which provides very low utility initially—so cities teem and become grossly oversized before new ones can form.

The inclusion of the opportunity function solves this coordination failure. In the context of cities, it is natural to think of land as providing opportunities to early residents, though other examples surely exist. In particular, our opportunity function builds on the insights from monocentric city models (Alonso, 1964; Mills, 1967; Muth, 1969).[23] Early residents can acquire parcels of land closer to the center, which they value because of commuting costs or their option value. Individuals who arrive later must acquire parcels farther from the center and incur larger commuting costs but have larger parcels—reflecting the fact that density decreases from the center of the city. The growth pattern of cities, therefore, depends on the trade-off between higher incomes in established cities and opportunities in the form of more centrally located land in new cities.

Income function. Following Henderson (1974), we assume the aggregate production function for some composite good in city i is $F_i(N_i(t)) = A_i(t)N_i(t)$, where $N_i(t)$ denotes the population at time t (which is also the labor input to production). Production amenities (TFP) $A_i(t) = a_i N_i(t)^{\varepsilon_i}$ in cities are captured by the parameter a_i, and the agglomeration economies are subsumed by the term $N_i(t)^{\varepsilon_i}$, where $\varepsilon_i > 0$ captures matching, sharing, and learning externalities (see, e.g., Duranton and Puga 2004, and

[23] Recent work has extended Henderson's (1974) model to consider the creation and growth of new cities. For example, Henderson and Venables (2009) build a dynamic model with durable housing capital that avoids population swings in cities that arise in other urban models. Our model can be viewed as a version that replaces the role of housing capital with opportunities that exist in cities.

Abdel-Rahman and Anas 2004, for extensive reviews of micro-foundations of external economies of scale).

Given perfect competition, a worker's wage w_i equals the value of that person's marginal product of labor: $w_i(N_i(t)) = A_i(t) = a_i N_i(t)^{\varepsilon_i}$. Production of the composite good also generates pollution, $P(N_i(t)) = F_i(N_i(t))^{\gamma_i}$ where $\gamma_i > 0$ is a parameter.[24] Each individual is assumed to bear the average cost of the pollution produced within the city. Hence every individual receives (net) income—'consumption-based utility'—as follows: $Y_i(N_i(t)) = w_i(N_i(t)) - P(N_i(t))/N_i(t)$. This formulation ensures income is zero with no inhabitants and then strictly rises and falls with population, consistent with the inverted-∪ shape of Henderson (1974) and with our assumptions (1).

Opportunity function. In the canonical urban model, there are no rank-dependent opportunities. Hence, since initially $Y_1(N(0)) > Y_2(0) = 0$, an individual who moves to the new city 2 forgoes the higher income in the established city 1. Why would anyone be willing to move to the new city 2? In the canonical urban model of Henderson (1974), the short answer is: 'They won't.'[25] For a new city to form, it must offer the same income as the existing city. This requires city 1 to become grossly oversized so that income there falls to zero; only then will a new city form in a catastrophic manner.

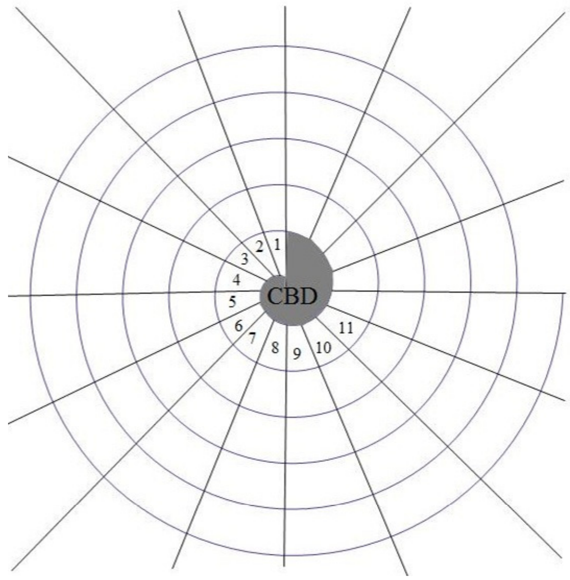
In our framework, individuals move because the new city provides opportunities that the established one does not, and those compensate for foregone income. These opportunities depend on how many people have already moved to the new city. Specifically, we model the opportunities in city 2 as parcels of land that individuals claim as they enter the city. The first person who claims a parcel of land defines the center of the city. All subsequent entrants claim a parcel that is adjacent and next in order to the previously claimed parcel.

We assume the city grows according to an Archimedean spiral, as Figure 8 shows. The use of an Archimedean spiral to model city growth builds on the monocentric city model pioneered by Alonso (1964), Mills (1967), and Muth (1969). Cities are often as-

[24]This parametric example of pollution is consistent with Tolley (1974)'s description on p. 334: "The nature of pollution and congestion is that extra pollutants and vehicles do not shift production functions at all at low amounts, and extra amounts have increasingly severe effects as levels are raised until ultimately fumes kill and there are so many vehicles that traffic cannot move."

[25]Anas (1992) provides an early example of a model of city growth in the absence of opportunities. The dynamics of Anas (1992) suggest large swings in population when a new city is formed. This model sparked research by Helsley and Strange (1994), Brueckner (2000), Cuberes (2009), and Henderson and Venables (2009), which included land developers or durable housing to solve this collective action problem.

Figure 8: Parcels of land in a city.



sumed to grow as a disc from their centers, sometimes as concentric circles. The use of a spiral allows the parcels to differ continuously in distance and area. Formally, assume that the radius is given by the angle θ and production in the central business district use $2\pi + 1$ parcels of land, where π is the mathematical constant. Each parcel of land is assumed to be formed by two lines radiating from the spiral's pole with an angle of one between them.[26] Parcels differ in their area and distance from the center. Those who arrive later have to progressively claim parcels that are farther away from the center but are larger (see Figure 8).

The distance and area of a parcel can be derived as a function of rank $M(\tau)$, using a discrete analog depicted in Figure 8. The distance of the $M(\tau)$th parcel from the center of the city is given by the radius, which, given our assumptions, is simply $M(\tau)$. Individuals value the distance of their parcel to the center of the city because they incur a cost of commuting, $M(\tau)^{\phi_i}$, where $\phi_i > 0$ is the elasticity of commuting costs with respect to rank (and, implicitly, distance). The cost of commuting is allowed to be nonlinear with respect to distance, capturing possible fixed costs and other deteriorating factors.[27]

Let $\alpha_i(M(\tau))$ denote the area of the $M(\tau)$th parcel, which is found by integrating

[26] The assumption that the angle between the lines radiating from the spiral's poles is one implicitly assumes there are 2 π parcels of land for a given rotation.

[27] The model is general enough to allow for a host of factors that may deteriorate from the center, where the best land is. Historically, this could be that land farther from the center is swampier or less suited for building. The presence of rivers that have to be crossed as the city expands may also play a role.

between the two curves θ and $\theta - 2\pi$ in polar coordinates between the angles $M(\tau)$ and $M(\tau) + 1$.[28] This yields

$$\alpha_i(M(\tau)) = \frac{1}{2}\int_{M(\tau)}^{M(\tau)+1} \left[\theta^2 - (\theta - 2\pi)^2\right] d\theta = 2\pi M(\tau) - 2\pi^2 + \pi.$$

The area of the parcel an individual receives increases with rank, $\alpha'(\cdot) > 0$. Individuals receive increasing value from the area of their parcel, according to the function $\rho_i(\alpha_i(M(\tau)))$, such that $\rho_i'(\cdot) > 0$. The opportunity an individual receives in the new city is given by the (net) value of that person's parcel:

$$R_i(M(t)) = \rho_i(\alpha_i(M(\tau))) - c_i M(\tau)^{\phi_i}.$$

City 2 is formed by a rush when, according to Propositions 2 and 3, the opportunity function is nonmonotonic and initially increasing. Consider an area value function of the form $\rho_i = (\delta_i(\nu_i + \alpha_i(M(t))))^\zeta$. Further, let $\delta_i = 1/(2\pi)$ and $\nu_i = 2\pi^2 - \pi$, such that the value of land is normalized to zero for the person of rank 0, i.e., $\rho_i(0) = 0$. In this case, the size of the rush is given by

$$\Delta M = \left[\frac{1}{c_i}\frac{\zeta}{\zeta + 1}\frac{\phi_i + 1}{\phi_i}\right]^{\frac{1}{\phi_i - \zeta}}.$$

The size of the rush increases with the value of the area of a parcel, specifically ζ, and decreases with the costs of being farther from the center, specifically c_i and ϕ_i. This result suggests that cities with more heterogeneous land, where the cost of being further from the center is greater, will experience smaller rushes. The growth patterns of cities depend on the shape of the opportunity function. In this example, the parameter c_i characterizes how heterogeneous land is in city i. Consider two potential new cities that differ only in that one has relatively more heterogeneous land. In the city where land is heterogeneous, the value of being close is relatively greater, given by a high value of c_i. In this case, the equilibrium size of the rush is larger, the opportunity function is steeper and, from the general results, we know that growth will be slower.

[28] The area between two curves, r_1 and r_2, in between the angles a and b is given by $\frac{1}{2}\int_b^a (r_1^2 - r_2^2) d\theta$.