INTRODUCTION TO CONFIGURATION SPACES AND THEIR APPLICATIONS

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1. Introduction

These notes, an introduction to the subject, develop basic, classical properties of configuration spaces as well as pointing out several natural connections between these spaces and other subjects. The main topics here arise from classical fibrations, homogeneous spaces, configuration spaces of surfaces, mapping class groups and loop spaces of configuration spaces, together with the relationships of these objects to simplicial groups and homotopy groups. Properties of the simplicial setting of homotopy groups are analogous to features of the Borromean rings or 'Brunnian' links and braids.

The confluence of structures encountered here is within low dimensional topology, as well as homotopy theory. These structures appear in a variety of contexts given by knots, links, homotopy groups and simplicial groups. Thus some homological consequences are developed, together with a description of how these results fit with linking phenomena in Section 20.

The structure of a simplicial group and Δ-group, which arise in the context of configuration spaces, are also described below. Connections to homotopy groups show how classical congruence subgroups arise in this context and coincide with certain natural subgroups of braid groups occurring in geometric group theory. These structures date back to the 1800's [48].

These notes are intended as a short introduction to a few basic properties and applications of configuration spaces. Much excellent as well as beautiful work of many people on this subject has been deliberately omitted because of space and time restrictions. The author apologizes to many friends and colleagues for these omissions.

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A more thorough development of configuration spaces is a book in preparation with Sam Gitler and Larry Taylor. One final remark: Sections 16 through 21 give a revised version of notes in [21].

Contents

2. Basic Definitions

The first definition is that of the configuration space.

Definition 2.1. Let M denote a topological space. Define the *configuration space of ordered k-tuples of distinct points in M* as the subspace of M^k given by

$$\mathrm{Conf}(M,k) = \{(m_1, m_2, \dots, m_k) \mid m_i \neq m_j \text{ for all } i \neq j\}.$$

The symmetric group on k-letters, Σ_k, acts on $\mathrm{Conf}(M,k)$ from the left by

$$\sigma(m_1, \dots, m_k) = (m_{\sigma(1)}, \dots, m_{\sigma(k)}).$$

One basic example is given next.

Example 2.2. This example gives classical properties of the configuration space of points in the plane $\mathbb{R}^2$ which is also regarded as the complex numbers $\mathbb{C}$. In this case, Artin's braid group with k strands, B_k, as well as Artin's pure braid group with k strands, P_k, defined in Section 8 below, arise naturally.

If $M = \mathbb{R}^2$, then $\mathrm{Conf}(\mathbb{R}^2, k)$ is a $K(P_k, 1)$ and $\mathrm{Conf}(\mathbb{R}^2, k)/\Sigma_k$ is a $K(B_k, 1)$ with proof first given in [32, 37] as well as sketched as Theorem 12.2 below. This example arises in the context of classical polynomials in one complex variable. Consider the space of unordered k-tuples of points in the complex numbers $\mathbb{C}^k/\Sigma_k$, a space well-known as the k-fold symmetric product. A point in $\mathbb{C}^k/\Sigma_k$ may be regarded as the set of roots $\{r_1, \dots, r_k\}$, possibly repeated, of any monic, complex polynomial of degree k in one indeterminate z over $\mathbb{C}$.

There is a homeomorphism

$$\mathrm{Root} : \mathbb{C}^k/\Sigma_k \to \mathbb{C}^k$$

for which

$$\mathrm{Root}(\{r_1, \dots, r_k\}) = p(z)$$

where

$$p(z) = \prod_{1 \leq i \leq k} (z - r_i).$$

Thus the space of complex polynomials $p(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0$ is identified by this homeomorphism which sends the roots of $p(z)$, $\{r_1, \dots, r_k\}$, to the point in $\mathbb{C}^k$ with coordinates $(a_{k-1}, \dots, a_0)$, the coefficients of $p(z)$, where the a_j are given, up to sign, by the elementary symmetric functions in the r_i.

The subspace $\mathrm{Conf}(\mathbb{C}, k)/\Sigma_k$ of $\mathbb{C}^k/\Sigma_k$ is homeomorphic to the space of monic, complex polynomials $p(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0$ for which $p(z)$ has exactly k distinct roots. The classical homeomorphism sends an equivalence class $[r_1, \dots, r_k] \in \mathrm{Conf}(\mathbb{C}, k)/\Sigma_k$ to the polynomial

$$p(z) = \prod_{1 \leq i \leq k} (z - r_i).$$

Features of the inverse of this homeomorphism are one of the main topics in classical Galois theory. Further features of $\mathrm{Conf}(\mathbb{C}, k)/\Sigma_k$ concerning the homotopy groups of the 2-sphere will be addressed in Section 20.

Example 2.3. The configuration space $\mathrm{Conf}(\mathbb{R}^n, k)$ is homeomorphic to

$$\mathbb{R}^n \times \mathrm{Conf}(\mathbb{R}^n - Q_1, k-1)$$

where Q_1 is the set with a single point given by the origin in $\mathbb{R}^n$. An extension of this fact is given in Example 2.6 below.

Notice that $\mathrm{Conf}(\mathbb{R}^n, 2)$ has S^{n-1} as a strong deformation retract with one choice of equivalence given by

$$A : S^{n-1} \to \mathrm{Conf}(\mathbb{R}^n, 2),$$

the map defined on points by the formula $A(z) = (z, -z)$ for z in S^{n-1}, where S^{n-1} is regarded as the points of unit norm in $\mathbb{R}^n$. A map

$$B : \mathrm{Conf}(\mathbb{R}^n, 2) \to S^{n-1}$$

is defined by the formula

$$B((x, y)) = \frac{(x-y)}{|x-y|}.$$

Observe that $B \circ A$ is the identity, and $A \circ B$ is homotopic to the identity via a homotopy leaving S^{n-1} point-wise fixed.

Example 2.4. Some features of configuration spaces for a sphere are listed next.

(1) The configuration space $\mathrm{Conf}(S^n, 2)$ is homotopy equivalent to S^n with one choice of equivalence given by the map

$$g : S^n \to \mathrm{Conf}(S^n, 2)$$

for which $g(z) = (z, -z)$.

(2) Let $\tau(S^n)$ denote the unit sphere bundle in the tangent bundle for S^n. There is a homotopy equivalence $E : \tau(S^n) \to \mathrm{Conf}(S^n, 3)$ defined on points (z, v) by

$$E(z, v) = (z, \exp(v), \exp(-v)).$$

Furthermore, this map is a fibre homotopy equivalance as implied by Theorem 3.2 below [32, 31].

(3) Properties of the tangent bundle and normal bundle for a smooth manifold M arise repeatedly in [20] where cofibre sequences are developed for configuration spaces given in terms of Thom spaces of associated normal bundles. Similar features arise in Totaro's spectral sequence [77].

Example 2.5. A classical fact is that the configuration space $\mathrm{Conf}(S^2, 3)$ is homeomorphic to $PGL(2, \mathbb{C})$, as for example, stated as Lemma 9.3 below. It follows that the configuration space $\mathrm{Conf}(S^2, k+3)$ is homeomorphic to the product $PGL(2, \mathbb{C}) \times \mathrm{Conf}(S^2 - Q_3, k)$ for all $k \geq 0$, for which Q_3 denotes a set of three distinct points in S^2. The group $PGL(2, \mathbb{C})$ has $SO(3)$ as a maximal compact subgroup and is thus homotopy equivalent to the real projective space $\mathbb{RP}^3$. This case was basic in [5, 11].

Example 2.6. If G is a topological group, then there is a homeomorphism

$$h : \mathrm{Conf}(G, k) \to G \times \mathrm{Conf}(G - \{1_G\}, k-1)$$

where $h(g_1, \ldots, g_k) = (g_1, ({g_1}^{-1}g_2, \ldots, {g_1}^{-1}g_k))$. Thus if $k \geq 2$, there are homeomorphisms

$$\mathrm{Conf}(\mathbb{R}^2, k) \to \mathbb{R}^2 \times (\mathbb{R}^2 - \{0\}) \times \mathrm{Conf}(\mathbb{R}^2 - Q_2, k-2)$$

where $Q_2 = \{0, 1\} \subset \mathbb{R}^2$ [32, 31, 15].

Natural variations are listed next. The first, a fibre-wise analogue of $\mathrm{Conf}(M, k)$ has been used to give certain natural $K(\pi, 1)$'s and to provide an application of a classical 'incidence bundle' [11] to produce computations of the cohomology of certain discrete groups.

Definition 2.7. Let $\alpha\colon M \to B$ be any continuous map. The *fibre-wise configuration space*

$$\mathrm{Conf}_\alpha(M, k)$$

is the subspace of $\mathrm{Conf}(M, k)$ given by

$$\mathrm{Conf}_\alpha(M, k) = \{(m_1, \ldots, m_k) \mid m_i \neq m_j \text{ for all } i \neq j \text{ and } \alpha(m_i) = \alpha(m_1) \text{ for all } i, j\}.$$

Given the projection map for a fibre bundle $\beta\colon E \to B$ with fibre X, define the *incidence bundle of k points in E* to be

$$\mathrm{Conf}_\beta(E, k).$$

There is a natural projection

$$\pi : \mathrm{Conf}_\beta(E, k) \to B$$

which, with mild restrictions given in the next example, is fibration with fibre $\mathrm{Conf}(X, k)$.

One example is listed next.

Example 2.8. Let G be a topological group which acts on the left of a space M and thus diagonally on $\mathrm{Conf}(M, k)$. Consider the Borel construction for $\mathrm{Conf}(M, k)$ given by

$$EG \times_G \mathrm{Conf}(M, k)$$

together with the natural projection maps

$$\gamma : EG \times_G M \to BG$$

and

$$\gamma_k : EG \times_G \mathrm{Conf}(M,k) \to BG.$$

Then the natural map

$$EG \times_G \mathrm{Conf}(M,k) \to \mathrm{Conf}_\gamma(EG \times_G M, k)$$

is a homeomorphism. If G is a compact Lie group, then the natural projection map

$$EG \times_G \mathrm{Conf}(M,k) \to BG$$

is a fibration with fibre $\mathrm{Conf}(M,k)$, as implied by [64]. A more general setting arises with the proof of Theorem 3.2 in Section 6.

In Sections 8 through 11, we will use this example in the special case of

$$\eta : BSO(2) = ESO(3) \times_{SO(3)} S^2 \to BSO(3)$$

to obtain $K(\pi,1)$'s closely connected to mapping class groups. In this case, the group

$$\pi = \pi_1(\mathrm{Conf}_\eta(BSO(2),k)/\Sigma_k)$$

is isomorphic to the group of path-components of the orientation preserving group of diffeomorphisms of S^2 which preserve a given set of k points [11], i.e., the mapping class group for a punctured 2-sphere.

A small modification to principal $U(2)$-bundles gives a $K(\pi,1)$ for which π is the mapping class for genus two surfaces, see Example 9.10 below.

Example 2.9. Consider the natural action of $O(n)$ on S^{n-1}. Regard $O(k)$ as the subgroup of $O(n+k)$ given by $O(k) \times 1^n$, and $O(n)$ as the subgroup of $O(n+k)$ given by $1^k \times O(n)$. Let $V(n+k,k)$ denote the Stiefel manifold $O(n+k)/O(k)$ and $Gr(n+k,k)$ denote the Grassmann manifold $O(n+k)/O(k) \times O(n)$.

Consider

$$V(n+k,k) \times_{O(n)} S^{n-1}$$

as the total space of a fibre bundle with projection

$$\gamma : V(n+k,k) \times_{O(n)} S^{n-1} \to Gr(n+k,k)$$

and with fibre S^{n-1}. The associated incidence bundle is

$$\gamma_q : V(n+k,k) \times_{O(n)} \mathrm{Conf}(S^{n-1},q) \to Gr(n+k,k)$$

with fibre $\mathrm{Conf}(S^{n-1}, q)$.

One result from [11] is that the 'incidence bundle'

$$\cup_{k\to\infty} V(3+k,k) \times_{O(3)} \mathrm{Conf}(S^2, q)/\Sigma_q$$

is a $K(\pi, 1)$ where π is a $\mathbb{Z}/2Z$ extension of the mapping class group for a 2-sphere which has been punctured q times, for $q \geq 3$. The mapping class group for the punctured 2-sphere is the fundamental group of an analogous bundle where $Gr(3+k,k)$ is replaced by $O(3+k)/(O(k) \times SO(3))$, the subject of Section 9 here.

A third construction is given next.

Definition 2.10. Let Γ be a group which acts freely and properly discontinuously on the space M so that the projection map

$$M \to M/\Gamma$$

is the projection map in a principal Γ-bundle. Define the *orbit configuration space*

$$\mathrm{Conf}^{\Gamma}(M, k) = \{(m_1, \ldots, m_k) \mid m_i\Gamma \cap m_j\Gamma = \emptyset \text{ for all } i \neq j\}.$$

The group Γ^k acts on $\mathrm{Conf}^{\Gamma}(M, k)$ (from the left) by the formula

$$(\gamma_1, \ldots, \gamma_k)(m_1, \ldots, m_k) = (\gamma_1 \cdot m_1, \ldots, \gamma_k \cdot m_k).$$

Example 2.11. If S is a surface homeomorphic to $S^1 \times S^1$, then

$$S = \mathbb{R}^2/\Gamma$$

where $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ or any parameterized lattice in $\mathbb{C}$. If S is an orientable surface of genus g with $g > 1$, then S is the quotient of the upper half-plane $\mathbb{H}^2$ by the fundamental group of the surface. The spaces $\mathrm{Conf}^{\mathbb{Z}\oplus\mathbb{Z}}(\mathbb{R}^2, k)$ and $\mathrm{Conf}^{\Gamma}(\mathbb{H}^2, k)$ were studied in [85, 63, 14] where Γ is a Fuchsian group. In addition, further useful properties of $\mathrm{Conf}^{\Gamma}(M, k)$ were developed in [85].

3. Fibrations

Throughout this section M denotes a topological manifold with $\mathrm{Top}(M)$ the group of homeomorphisms of M, topologized with the compact-open topology. The purpose of this section is to introduce two theorems first proven in [32] with a small modification below in terms of homogeneous spaces as well as classifying spaces of certain homeomorphism groups.

In the special case for which M is a closed orientable surface, the theorems in this section are used in sections 8 through 11 below to construct certain natural $K(\pi, 1)$'s associated to the mapping class group of a surface.

Definition 3.1. The group $\mathrm{Top}(M)$ acts diagonally on the configuration space $\mathrm{Conf}(M,k)$. Thus if $f : M \to M$ is an element in $\mathrm{Top}(M)$, then the action

$$\theta : \mathrm{Top}(M) \times \mathrm{Conf}(M,k) \to \mathrm{Conf}(M,k)$$

is defined by the formula

$$\theta((f,(m_1,\ldots,m_k))) = (f(m_1), f(m_2), \ldots, f(m_k)).$$

Let $Q_k = \{q_1, \ldots, q_k\}$ denote the underlying set of points obtained from a fixed point $\vec{q} = (q_1, \ldots, q_k)$ in $\mathrm{Conf}(M,k)$. The subgroup of elements in $\mathrm{Top}(M)$ which point-wise fixes the set Q_k is denoted $\mathrm{Top}(M,k)$ here. There is an induced map

$$\rho_{\vec{q}} : \mathrm{Top}(M)/\mathrm{Top}(M,k) \to \ \mathrm{Conf}(M,k)$$

defined by

$$\rho_{\vec{q}}(f) = (f(q_1), \ldots, f(q_k)).$$

The action specified by θ gives rise to natural fibre bundles as given in the next two Theorems which are proven in Sections 7 and 6, with original sources [32, 31].

Theorem 3.2. *Assume that M is a topological manifold without boundary.*

(1) *The group* $\mathrm{Top}(M,k)$ *acts on* $\mathrm{Top}(M)$ *by composition (from the left). The natural quotient map*

$$\mathrm{Top}(M) \to \ \mathrm{Top}(M)/\mathrm{Top}(M,k)$$

is the projection map for the principal $\mathrm{Top}(M,k)$*-bundle*

$$\mathrm{Top}(M,k) \to \ \mathrm{Top}(M) \to \ \mathrm{Top}(M)/\mathrm{Top}(M,k).$$

(2) *The induced map* $\rho_{\vec{q}} : \mathrm{Top}(M)/\mathrm{Top}(M,k) \to \ \mathrm{Conf}(M,k)$ *is a homeomorphism.*

(3) *The homotopy theoretic fibre of the natural map*

$$B\mathrm{Top}(M,k) \to \ B\mathrm{Top}(M)$$

is $\mathrm{Conf}(M,k)$*, and, if* $G = \mathrm{Top}(M)$*, then* $EG \times_G \mathrm{Conf}(M,k)$ *is homeomorphic to* $B\mathrm{Top}(M,k)$*.*

Observe that the natural projection maps

$$p_i : M^k \to M^{k-1}$$

which delete the i-th coordinate restrict to maps on the level of configuration spaces

$$p_i : \mathrm{Conf}(M,k) \to \mathrm{Conf}(M,k-1).$$

The second theorem is as follows [32, 31].

Theorem 3.3. *If M is a manifold without boundary, the natural projection map*

$$p_i : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1)$$

is a fibration with fibre

$$M - Q_{k-1}.$$

Corollary 3.4. *Assume that M is a manifold without boundary, and $I = (i_1, \ldots, i_r)$ is a sequence of integers with $1 \leq i_1 < i_2 < \cdots < i_r \leq k$. Then the natural composite of projection maps $p_I = p_{i_1} \circ \cdots \circ p_{i_r}$ is a fibration*

$$p_I : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-r)$$

with fibre

$$\mathrm{Conf}(M - Q_{k-r}, r).$$

Remark 3.5. Several related remarks concerning features of Theorems 3.3 and 3.2 and their proofs are given next.

(1) Theorem 3.3 was stated and proven in a classical paper by Fadell-Neuwirth [32]. The result also follows from earlier work of R. Palais who was addressing a different question analogous to Theorem 3.2 [62]. Elegant further developments are in the book [31].

(2) The proofs below are those of [32, 31] with a small addition concerning principal fibrations.

(3) Additional hypothesis on M, such as $M = \mathbb{R}^n \times N$, imply that the collection of groups $\pi_1(\mathrm{Conf}(\mathbb{R}^n \times N, k))$, $k \geq 1$, form a simplicial group (see Section 16 for the concept of simplicial group). The group $\pi_1(\mathrm{Conf}(\mathbb{R}^n \times N, k))$ is regarded as the $(k-1)$-st group in the simplicial group, see [5] and Section 16 below.

One basic example is given by the collection $\pi_1(\mathrm{Conf}(\mathbb{R}^2, k)) = P_k$, $k \geq 1$, a simplicial group denoted $AP_\bullet$ in section 19. This simplicial group is closely tied to the homotopy groups of the 2-sphere [21].

For general M, the collection of fundamental groups $\pi_1(\mathrm{Conf}(M, k))$, $k \geq 1$, admit the structure of a Δ-group, with the Δ-structure induced by the projection maps $p_i : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1)$. This idea is developed in [5]; in the special case of $M = S^2$ there is a connection with the homotopy groups of the 2-sphere. These structures are also described in section 16 below, as well as [83].

(4) In the case of the simplicial group $\{\pi_1(\mathrm{Conf}(\mathbb{R}^n \times N, k))\}$ for $M = \mathbb{R}^n \times N$, consider the kernel of the induced map on fundamental groups

$$\pi_1(p_k) : \pi_1(\mathrm{Conf}(M, k)) \to \pi_1(\mathrm{Conf}(M, k-1)).$$

These kernels inherit the structure of a simplicial group isomorphic to Moore's simplicial loop space construction [61] applied to the simplicial group $\{\pi_1(\mathrm{Conf}(\mathbb{R}^n \times N, k))\}$. Thus, the natural projection maps in Theorem 3.3 also naturally give Moore's simplicial loop space for these simplicial groups, a point described in Sections 19, 20, and 21 below.

The main feature here is that the fibres of the projection maps

$$p_i : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1)$$

give a precise topological analogue for the group-theoretic process of forming a simplicial loop space from a simplicial group. Thus these projection maps are informative for other subjects.

4. On cross-sections for configuration spaces

The purpose of this section is to describe certain cross-sections for the projections maps $p_i : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1)$ when they exist for direct reasons. These maps are useful in what follows below.

Example 4.1. The first natural case is given by cross-sections for the projection maps

$$p_k : \mathrm{Conf}(M \times \mathbb{R}^n, k) \to \mathrm{Conf}(M \times \mathbb{R}^n, k-1).$$

A section is specified by

$$\sigma_k : \mathrm{Conf}(M \times \mathbb{R}^n, k-1) \to \mathrm{Conf}(M \times \mathbb{R}^n, k)$$

with

$$\sigma_k((m_1, r_1), \ldots, (m_{k-1}, r_{k-1})) = ((m_1, r_1), \ldots, (m_{k-1}, r_{k-1}), (m_1, L\vec{e}_1))$$

where

$$L = 1 + \mathrm{max}_i ||r_i||$$

and $\vec{e}_1 = (1, 0, \ldots, 0)$.

Notice that a direct variation of this map applies to give sections for the projection maps

$$p_k : \mathrm{Conf}(\zeta, k) \to \mathrm{Conf}(\zeta, k-1)$$

where ζ is an n-plane bundle over M which supports a nowhere vanishing cross-section $\sigma : M \to \zeta$ for the bundle projection $p : \zeta \to M$. See the next example.

Example 4.2. Define

$$\sigma_k : \mathrm{Conf}(\zeta, k-1) \to \mathrm{Conf}(\zeta, k)$$

by the formula

$$\sigma_k((z_1, \ldots, z_{k-1})) = (z_1, \ldots, z_{k-1}, \lambda\widehat{\sigma(z_1)})$$

where

$$\widehat{\sigma(z_1)} = \sigma(z_1)/|\sigma(z_1)|$$

and $\lambda = 1 + \max_i|\sigma(z_i)|$.

Next consider a manifold without boundary M, together with a fixed subset $Q \in M$ consisting of a single point.

Example 4.3. The projection maps

$$p_k : \mathrm{Conf}(M - Q, k) \to \mathrm{Conf}(M - Q, k-1)$$

admit cross-sections up to homotopy [32].

Example 4.4. An example for which the projection map

$$p_3 : \mathrm{Conf}(M, 3) \to \mathrm{Conf}(M, 2)$$

does not admit a cross-section is given by $M = S^2$. Observe that $\mathrm{Conf}(S^2, 3)$ is homeomorphic to $PGL(2, \mathbb{C})$, a classical fact stated as Lemma 9.3 below. Since $PGL(2, \mathbb{C})$, $SO(3)$, and $\mathbb{RP}^3$ are homotopy equivalent, there does not exist a section for $p_3 : \mathrm{Conf}(S^2, 3) \to \mathrm{Conf}(S^2, 2)$ as $H_2(S^2) = H_2(\mathrm{Conf}(S^2, 2)) = \mathbb{Z}$, but $H_2(\mathrm{Conf}(S^2, 3)) = H_2(\mathbb{RP}^3) = \{0\}$.

Similarly, the projection maps $p_3 : \mathrm{Conf}(S^{2n}, 3) \to \mathrm{Conf}(S^{2n}, 2)$ do not admit sections for all $n > 0$ as $\mathrm{Conf}(S^{2n}, 3)$ is homotopy equivalent to the unit sphere bundle in the tangent bundle of S^{2n} with $H_{2n}(\mathrm{Conf}(S^{2n}, 3)) = \{0\}$, and $H_{2n}(\mathrm{Conf}(S^{2n}, 2)) = \mathbb{Z}$.

5. Preparation for Theorems 3.2 and 3.3

Lemma 5.1. *Assume that M is a non-empty manifold without boundary, of dimension at least 1. Then* $\mathrm{Top}(M, k)$ *is a closed subgroup of* $\mathrm{Top}(M)$.

Proof. Given any point f in the complement of $\mathrm{Top}(M, k)$ in $\mathrm{Top}(M)$, there is at least one point q_i in Q_k that is not fixed by f. Since M is Hausdorff, there is a non-empty open set U in M that does not contain q_i. An open set in the complement of $\mathrm{Top}(M, k)$ in $\mathrm{Top}(M)$ containing f is given by the set of continuous functions that carry the point q_i into U. The lemma follows. □

The next definition is given in [76] on page 30.

Definition 5.2. Let H be a closed subgroup of a topological group G with natural quotient map $p : G \to\ G/H$. A *local cross-section of G in H* is a continuous function

$$f : U \to G$$

for U an open set in G/H such that

$$pf(x) = x$$

for every $x \in U$. To be more precise, such a continuous function f is also called a *local cross-section of G in H over the open set U*. A *local cross-section over a point $x \in G/H$* is a local cross-section of G in H over some open set U with $x \in U \subset G/H$.

The next theorem is given in Steenrod's book 'The topology of fibre bundles' [76], page 30.

Theorem 5.3. *Let H be a closed subgroup of G and assume that the map $p : G \to\ G/H$ admits local cross-sections over every point $x \in G/H$. Then the projection map*

$$BH \to BG$$

is the projection map in a fibre bundle with fibre given by the space of left cosets G/H.

Remark 5.4. Steenrod's proof gives a homeomorphism

$$EG \times_G G/H \to BH$$

under the conditions of the theorem. A statement and proof of this fact is recorded next for completeness.

Theorem 5.5. *Let H be a closed subgroup of G and assume that the map $p : G \to\ G/H$ admits local cross-sections over every point $x \in G/H$. Then the natural map*

$$\pi : EG \times_G G/H \to BH$$

is a homeomorphism.

Proof. Let G/H denote the space of left cosets $\cup_{g \in G} gH$. Let X be any space that has a right G-action that is free and properly discontinuous, thus the projection $p : X \to X/G$ is a principal G-bundle. Notice that G acts on the product $X \times G/H$ via the formula

$$\gamma \cdot (x, g) = (x \cdot \gamma^{-1}, \gamma \cdot g).$$

Furthermore there is a natural map

$$q : X \times G/H \to X/H$$

defined by

$$q(x, gH) = [x\bar{g}],$$

the class of $x\bar{g}$ in X/H which is independent of the choice of $\bar{g} \in gH$. Since

$$q(x \cdot \gamma^{-1}, \gamma \cdot g) = q(x, g),$$

there is an induced map

$$\pi : X \times_G G/H \to X/H$$

defined by the equation $\pi([x, gH]) = q(x, gH)$.

A second map $\alpha : X \to X \times_G G/H$, defined by the equation

$$\alpha(x) = [x, 1H],$$

passes to quotients

$$\beta : X/H \to X \times_G G/H.$$

The composites $\pi \circ \beta$ and $\beta \circ \pi$ are both the identity. Thus the map

$$q : (X \times_G G/H) \to X/H$$

is a homeomorphism. Steenrod's theorem follows by setting $X = EG$ with the identification $BH = EG/H$. □

Remark 5.6. Steenrod showed that the natural quotient map $\pi : G \to G/H$ is the projection in a principal fibre bundle in case H is a closed subgroup of G and the map has local sections [74]. Similarly, if the map π is the projection in a bundle, then it has local sections. Thus local sections are necessary as well as sufficient in order that $\pi : G \to G/H$ be the projection in a principal fibre bundle, in case H is a closed subgroup of G.

The proofs of Theorems 3.2 and 3.3 depend on the next lemma. Here, let D^n denote the n-disk, the points in $\mathbb{R}^n$ of Euclidean norm at most 1, with interior denoted $\overset{o}{D^n}$ and with the origin in D^n denoted $(0, 0, \ldots, 0)$. The map θ in the next lemma was useful in [32], the article by Fadell and Neuwirth, while the formula here is given explicitly in [85].

Lemma 5.7. (1) *There is a continuous map*

$$\theta : \overset{o}{D^n} \times D^n \to D^n$$

such that $\theta(x, -)$ *fixes the boundary of* D^n *point-wise and*

$$\theta(x, x) = (0, 0, \ldots, 0)$$

for every x *in* $\overset{o}{D^n}$.

(2) *If M is a topological manifold without boundary, then there exists a basis of open sets U for the topology of* $\mathrm{Conf}(M,k)$ *together with local sections* $\phi : U \to \mathrm{Top}(M)$ *such that the composite* $U \to \mathrm{Top}(M) \to \mathrm{Top}(M)/\mathrm{Top}(M,k) \to \mathrm{Conf}(M,k)$ *is a homeomorphism onto U.*

(3) *The natural map* $\rho_{\vec{q}} : \mathrm{Top}(M)/\mathrm{Top}(M,k) \to \mathrm{Conf}(M,k)$ *given by evaluation at a point* $\vec{q} = (q_1, \ldots, q_k) \in \mathrm{Conf}(M,k)$ *is a homeomorphism.*

Proof. Define

$$\alpha : \overset{o}{D^n} \to \mathbb{R}^n$$

by the formula $\alpha(x) = x/(1-|x|)$, and so $\alpha^{-1}(z) = z/(1+|z|)$. Let $\partial(D^n)$ denote the boundary of D^n.

For a fixed element q in $\overset{o}{D^n}$, define

$$\gamma_q : D^n \to D^n$$

by the formula

$$\gamma_q(y) = \begin{cases} y & \text{if} \quad y \in \partial(D^n), \\ \alpha^{-1}\left(\frac{y}{1-|y|} - \frac{q}{1-|q|}\right) & \text{if} \quad y \in \overset{o}{D^n}. \end{cases}$$

Define $\theta : \overset{o}{D^n} \times D^n \to D^n$ by the formula

$$\theta(q,y) = \gamma_q(y).$$

Notice that θ is continuous, and $\theta(q,q) = (0,0,\ldots,0)$. Thus part (1) of the lemma follows.

To prove part (2), consider a point $(q_1, q_2, \ldots, q_k)$ in $\mathrm{Conf}(M,k)$ together with disjoint open discs $\overset{o}{D^n}(q_1), \overset{o}{D^n}(q_2), \ldots, \overset{o}{D^n}(q_k)$ where $\overset{o}{D^n}(q_i)$ is a disc with center q_i. (There is a choice of homeomorphism in the identification of each such open disc with an open coordinate patch of M; this choice is suppressed here.) Let

$$U = \overset{o}{D^n}(q_1) \times \overset{o}{D^n}(q_2) \times \cdots \times \overset{o}{D^n}(q_k)$$

Observe that U is an open set in $\mathrm{Conf}(M,k)$ and that the sets U give a basis for the topology of $\mathrm{Conf}(M,k)$ as the $(q_1, q_2, \ldots, q_k)$ range over the points in $\mathrm{Conf}(M,k)$.

Define

$$\phi : U \to \mathrm{Top}(M)$$

by the formula

$$\phi((y_1, y_2, \ldots, y_k)) = H$$

for H in $\mathrm{Top}(M)$ where $(y_1, y_2, \ldots, y_k)$ is in $U = \overset{o}{D^n}(q_1) \times \overset{o}{D^n}(q_2) \times \cdots \times \overset{o}{D^n}(q_k)$, and H is the homeomorphism of M given as follows.

(1) $H(x) = x$ if x is in the complement in the disjoint union $\coprod_{1\leq i\leq k} \overset{o}{D}^n(q_i)$, and
(2) $H(x) = \theta(q_i, x)$ if x is in $D^n(q_i)$.

Clearly H is in $\mathrm{Top}(M)$ as the two parts of the definition for H agree on the boundary of $D^n(q_i)$. To finish part (2) of the lemma, it suffices to check that ϕ is continuous. Notice that all spaces here are locally compact, and Hausdorff. Thus it follows that ϕ is continuous if and only if the adjoint

$$\mathrm{adj}(\phi) : U \times M \to M$$

defined by the formula

$$\mathrm{adj}(\phi)(u, m) = H(u, m)$$

is continuous. Then continuity of ϕ follows at once from the continuity of H. The second part of the lemma follows.

To finish the third part of the lemma, it must be checked that the natural map

$$\rho : \mathrm{Top}(M)/\mathrm{Top}(M, k) \to \mathrm{Conf}(M, k)$$

is a homeomorphism. Notice that part 2 of the lemma gives local sections

$$\phi : U \to \mathrm{Top}(M).$$

Thus consider the composite $\lambda : U \to \mathrm{Top}(M)/\mathrm{Top}(M, k)$ given by the composite $p \circ \phi$ where $p : G \to G/H$ is the natural quotient map. Notice that $\lambda : U \to \lambda(U)$ is a continuous bijection, and $\lambda(U) = \phi^{-1}(U)$. Thus $\lambda(U)$ is open, and the map ϕ is open. Thus ρ is open, and hence a homeomorphism. The lemma follows. □

6. Proof of Theorem 3.2

By Lemma 5.1, $\mathrm{Top}(M, k)$ is a closed subgroup of $\mathrm{Top}(M)$. Furthermore, local sections exist for $\mathrm{Top}(M) \to \mathrm{Top}(M)/\mathrm{Top}(M, k)$ by Lemma 5.7. Thus there is a principal fibration

$$\mathrm{Top}(M, k) \to \mathrm{Top}(M) \to \mathrm{Top}(M)/\mathrm{Top}(M, k).$$

The first part of the theorem follows.

In addition, the natural evaluation map $\mathrm{Top}(M) \to \mathrm{Conf}(M, k)$ factors through the quotient map $\mathrm{Top}(M) \to \mathrm{Top}(M)/\mathrm{Top}(M, k)$. Thus, the induced map

$$\rho_{\vec{q}} : \mathrm{Top}(M)/\mathrm{Top}(M, k) \to \mathrm{Conf}(M, k)$$

is a homeomorphism by Lemma 5.7. Part 2 of the theorem follows.

The third statement in the theorem follows at once from Theorems 5.3, and 5.5.

7. Proof of Theorem 3.3

The statement to be proven is that if M is a manifold without boundary, then the natural projection map

$$p_i : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1)$$

is a fibration with fibre homeomorphic to

$$M - Q_{k-1}$$

where p_i denotes the projection map which deletes the i-th coordinate. To prove that p_i is a fibration, it suffices to check that the map is locally trivial, by a theorem of A. Dold [73]. It suffices to check the result in case $i = k$ by applying the permutation which swaps i and k.

Consider the projection which deletes the last coordinate

$$p_k : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1).$$

The inverse image of the point $(q_1, \ldots, q_{k-1}) \in \mathrm{Conf}(M, k-1)$, $p_k^{-1}((q_1, \ldots, q_{k-1}))$, is homeomorphic to $M - Q_{k-1}$ with $Q_{k-1} = \{q_1, \ldots, q_{k-1}\}$ because the natural inclusion

$$\iota_k : M - Q_{k-1} \to p_k^{-1}((q_1, \ldots, q_{k-1})),$$

defined by the equation $\iota_k(m) = (q_1, \ldots, q_{k-1}, m)$, is a homeomorphism.

Consider the point $(q_1, \ldots, q_{k-1})$ in $\mathrm{Conf}(M, k-1)$ together with disjoint open discs $\overset{o}{D}{}^n(q_1), \ldots, \overset{o}{D}{}^n(q_{k-1})$ with centers q_i. Then consider the open set

$$V = \overset{o}{D}{}^n(q_1) \times \cdots \times \overset{o}{D}{}^n(q_{k-1}).$$

To show that p_k is locally trivial, we need to show that the following holds. Given any point $\vec{q} = (q_1, \ldots, q_{k-1})$ in $\mathrm{Conf}(M, k-1)$, there is an open set V, containing $\vec{q}$, together with a homeomorphism

$$\Phi : V \times (M - Q_{k-1}) \to p_k^{-1}(V)$$

for which there is a commutative diagram

$$\begin{array}{ccc} V \times (M - Q_{k-1}) & \xrightarrow{\Phi} & p_k^{-1}(V) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_k}\downarrow \\ V & \xrightarrow{1} & V \end{array}$$

where p is the natural projection map.

We will now define Φ. The definition given is exactly that of [32] or [31] and is given by the formula

$$\Phi((m_1, \dots, m_{k-1}), m_k) = \begin{cases} (m_1, \dots, m_{k-1}, m_k) & \text{if } m_k \notin \cup_{1 \le i \le k-1} D^n(q_i), \text{ and} \\ (m_1, \dots, m_{k-1}, \theta(m_i, m_k)) & \text{if } m_k \in D^n(q_i) \end{cases}$$

where the map θ is that of Lemma 5.7. That Φ is a homeomorphism and the theorem then follow.

8. Surfaces, braid groups and connections to mapping class groups

The subject of this section is basic properties of braid groups of surfaces as well as their connections to mapping class groups. Throughout this section S denotes a surface, possibly open or possibly non-orientable. The definition of the braid group of a surface is given next.

Definition 8.1. Let S denote a surface.

(1) The *k-stranded braid group for S* is

$$B_k(S) = \pi_1(\mathrm{Conf}(S, k)/\Sigma_k).$$

(2) The *k-stranded pure braid group for S* is

$$P_k(S) = \pi_1(\mathrm{Conf}(S, k)).$$

(3) In case $S = \mathbb{R}^2$, let B_k, respectively P_k, denote $B_k(\mathbb{R}^2)$, respectively $P_k(\mathbb{R}^2)$.

Remark 8.2. Useful consequences of the definition of the (pure) braid groups rely heavily on the fact that S is a surface. In this case, the natural inclusion

$$i : \mathrm{Conf}(S, k) \to S^k$$

does not induce an isomorphism on the level of fundamental groups, a feature which has been proven to be quite useful. Vershinin has written an informative survey of braid groups of surfaces in [78].

However, in case N is a manifold of dimension at least 3, the natural inclusion

$$i : \mathrm{Conf}(N, k) \to N^k$$

does induce an isomorphism on fundamental groups. Thus to obtain interesting structures which are analogous to braid groups of surfaces in the case of manifolds of dimension at least 3, new constructions are required.

Constructions which provide non-trivial analogues of braid groups for any space M are defined in [19]. The definition arises by considering the structure of the space of 'suitably compatible maps' $(S^1)^n \to \mathrm{Conf}(M, k)$ which provides an alternative definition of the braid group of a surface, and which extends in a natural way to give analogues of braid groups for any manifold M. These analogues of braid groups have their own version of Vassiliev invariants as well as other natural properties, and are reminiscent of Fox's torus homotopy groups [36], but with further global structure. One version is the group of pointed homotopy classes of pointed maps $[\Omega(S^2), \Omega(\mathrm{Conf}(M,k))]$, a group with tractable structure in case all spaces have been localized at the rational numbers. In the case of $[\Omega(S^2), \Omega(\mathrm{Conf}(\mathbb{R}^{2m},k))]$ for $m > 1$, the 'rationalization' of this group is isomorphic to the Malĉev completion of P_k [19].

Theorem 8.3. *If S is a surface not equal to either S^2 or $\mathbb{RP}^2$, and $Q_i = \{q_1, \ldots, q_i\}$ is a sub-set of S having cardinality i, possibly zero, then $\mathrm{Conf}(S - Q_i, k)$ and consequently $\mathrm{Conf}(S - Q_i, k)/\Sigma_k$ are $K(\pi, 1)$'s.*

Proof. Notice that S as well as $S - Q_i$ are both $K(\pi, 1)$'s. An induction on k using the fibrations in Theorem 3.3 implies that $\mathrm{Conf}(S - Q_i, k)$ is a $K(\pi, 1)$, as follows.

Since $S - Q_i$ is a surface not equal to either S^2 or $\mathbb{RP}^2$, it follows that $S - Q_i$ is a $K(\pi, 1)$. The inductive step is to observe that there is a fibration

$$\mathrm{Conf}(S - Q_i, k) \to S - Q_i$$

with fibre given by $\mathrm{Conf}(S - Q_{i+1}, k-1)$, by Theorem 3.3. Since $\mathrm{Conf}(S - Q_{i+1}, k-1)$ and $S - Q_i$ may be assumed to be $K(\pi, 1)$'s, it follows that $\mathrm{Conf}(S - Q_i, k)$ is also a $K(\pi, 1)$.

Furthermore, the natural quotient maps $\mathrm{Conf}(S, k) \to \mathrm{Conf}(S, k)/\Sigma_k$ are projections in a covering space. Thus, $\mathrm{Conf}(S, k)/\Sigma_k$ is also a $K(\pi, 1)$ and the theorem follows. □

Remark 8.4. In case S is the surface S^2 or $\mathbb{RP}^2$, then constructions of $K(B_k(S^2), 1)$ and $K(B_k(\mathbb{RP}^2), 1)$ are derived in Section 9. These spaces are given by total spaces of various natural choices of fibre bundles obtained from the natural $SO(3)$-actions on either S^2 or $\mathbb{RP}^2$.

One (classical) definition of the mapping class group for a closed, orientable Riemann surface S with fundamental group $\pi_1(S)$ is the group of outer-automorphisms $\mathrm{Out}(\pi_1(S))$ [55], page 175. An alternative definition is given by the group of path-components of the orientation preserving homeomorphisms of the surface. Useful variations are given next which are obtained by restricting to homeomorphisms which leave certain subspaces of S fixed.

Definition 8.5. Let S be a closed orientable surface of genus g with a given point $*$ in S.

(1) The mapping class group Γ_g is the group of path-components of the orientation preserving homeomorphisms of S, $\mathrm{Top}^+(S)$.

(2) The mapping class group Γ_g^k is the group of path-components of the orientation preserving homeomorphisms of S which leave a set of k distinct points Q_k in S invariant, and is equal to $\pi_0(\mathrm{Top}^+(S,k))$. The pure mapping class group $P\Gamma_g^k$ is the kernel of the natural homomorphism $\Gamma_g^k \to \Sigma_k$.

(3) The pointed mapping class group $\Gamma_g^{k,*}$ is the group of path components of the orientation preserving homeomorphisms which (i) preserve the point $*$, and (ii) leave a set of k distinct points in $S - *$, invariant, $\mathrm{Top}^+(S,\{*\},k)$. The pure pointed mapping class group $P\Gamma_g^{k,*}$ is the kernel of the natural homomorphism $\Gamma_g^{k,*} \to \Sigma_k$. We use $\mathrm{Top}^+(S,\{*\})$ to denote the group of orientation preserving homeomorphisms which leaves the point $*$ fixed.

(4) If, in addition, m disjoint disks are given in S together with k distinct points in the complement of the union of these disks, then define $\Gamma_{g,m}^k$ as the group of path-components of the orientation preserving homeomorphisms of S which leave the set of k distinct points invariant, as well as the boundaries of all m disks fixed point-wise.

Remark 8.6. Definition 8.5 provides a definition of the 'pure pointed mapping class group' $P\Gamma_g^{k,*}$ and the 'pointed mapping class group' $\Gamma_g^{k,*}$. The remarks here give some motivation for this variation as there are several natural applications as illustrated next.

First, consider the function spaces of all continuous maps $\mathrm{Map}(S,X)$ together with the subspace of pointed continuous maps $\mathrm{Map}_*(S,X)$ for a pointed space X. The homology of the space $\mathrm{Map}_*(S,X)$ is sometimes much more accessible than that of the 'free mapping space' $\mathrm{Map}(S,X)$. One case in point is where S is S_g a closed, orientable surface of genus g, and X is S^{2L}, an even dimensional sphere. The homology of the function space $\mathrm{Map}_*(S_g,S^{2L})$ is easily accessible while that of $\mathrm{Map}(S_g,S^{2L})$ is complicated.

In these cases, the group $\mathrm{Top}^+(S_g)$ acts naturally on the function space $\mathrm{Map}(S_g,S^{2L})$. Consider the homotopy orbit space $E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Map}(S_g,S^{2L})$. There is a 'pointed-version' given by

$$E\mathrm{Top}^+(S_g,\{*\}) \times_{\mathrm{Top}^+(S_g,\{*\})} \mathrm{Map}_*(S_g,S^{2L}).$$

One application is that the homology of all of the groups Γ_g^k are given at once by the homology of the space $E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Map}(S_g,S^{2L})$ with a degree shift depending on the choices of L and k [12]. The actual computations do not appear to be accessible even in the cases $g = 0,1$.

An analogous result is satisfied for the homology of the 'pointed version' given by the space $E\mathrm{Top}^+(S_g,\{*\}) \times_{\mathrm{Top}^+(S_g,\{*\})} \mathrm{Map}_*(S_g,S^{2L})$. In this case, the homology of all of the groups $\Gamma_g^{k,*}$ are given at once by the homology of this space with a degree shift depending on the choices of L and k [12]. However, these homology groups are much more accessible in the 'pointed' case, as given in [12].

In the case of $g = 1$, the cohomology of this space was worked out and gives the cohomology of $\Gamma_1^{k,*}$ in terms of classical modular forms. This case is also addressed in Section 10.

A classical theorem concerning the homeomorphism group and diffeomorphism group of a closed orientable Riemann surface S_g of genus g is stated next [27, 28]. Let $\mathrm{Diff}^+(S_g)$ denote the group of orientation preserving diffeomorphisms. The next theorem follows from results proven in [27, 28]. That is, the group of path-components for $\mathrm{Diff}^+(S_g)$ and $\mathrm{Top}^+(S_g)$ are isomorphic, and the components of the identity have the same homotopy type. The author would like to thank Benson Farb for a late Saturday night e-mail conversation regarding this point.

Theorem 8.7. *The natural inclusion*

$$\mathrm{Diff}^+(S_g) \to \mathrm{Top}^+(S_g)$$

is a homotopy equivalence.

In what follows, the groups $\mathrm{Diff}^+(S_g)$ and $\mathrm{Top}^+(S_g)$ will be used in different ways in which these differences are stated explicitly. The groups $\mathrm{Top}(S_g)$ and $\mathrm{Top}^+(S_g)$ act on the configuration space of points in S_g, $\mathrm{Conf}(S_g, k)$, diagonally. A useful 'folk theorem' gives (1) there are natural $K(\pi, 1)'s$ obtained from the associated Borel construction (homotopy orbit spaces) for groups $\mathrm{Top}(M)$ acting on configuration spaces, and (2) these configuration spaces are analogous to homogeneous spaces in the sense that they are frequently homeomorphic to a quotient of a topological group by a closed subgroup.

Namely, let G be a subgroup of $\mathrm{Top}(M)$, and consider the diagonal action of G on $\mathrm{Conf}(M, k)$ together with the homotopy orbit spaces

$$EG \times_G \mathrm{Conf}(M, k)$$

and

$$EG \times_G \mathrm{Conf}(M, k)/\Sigma_k.$$

In case M is a surface, these constructions frequently give $K(\pi, 1)$'s where the group π is given by certain mapping class groups. Three different cases which depend on the genus of the surface are given in the next three sections.

A remark concerning orientations is listed next. Observe that S_g has an orientation reversing involution which leaves k points invariant. Consider the natural map

$$h : \mathrm{Top}(S_g) \to \mathrm{Aut}(H_2(S_g))$$

defined by

$$h(f) = f_* : H_2(S_g) \to H_2(S_g).$$

This map satisfies the following properties.

(1) The map h surjects to $\mathrm{Aut}(H_2(S_g)) = \mathbb{Z}/2\mathbb{Z}$.
(2) The map h restricts to a surjection $h|_{\mathrm{Top}(S_g,k)} : \mathrm{Top}(S_g, k) \to \mathrm{Aut}(H_2(S_g))$.
(3) The kernel of h is $\mathrm{Top}^+(S_g)$ while the kernel of $h|_{\mathrm{Top}(S_g,k)}$ is $\mathrm{Top}^+(S_g, k)$.
(4) Thus the natural map

$$\mathrm{Top}^+(S_g)/\mathrm{Top}^+(S_g, k) \to \mathrm{Top}(S_g)/\mathrm{Top}(S_g, k)$$

is a homeomorphism.

This point is recorded in the following lemma (which of course is a special case).

Lemma 8.8. *Assume that S_g is an orientable surface of genus g. The natural map*

$$\mathrm{Top}^+(S_g)/\mathrm{Top}^+(S_g, k) \to \mathrm{Top}(S_g)/\mathrm{Top}(S_g, k)$$

is a homeomorphism.

9. On configurations in S^2

The natural actions of $SO(3)$ on S^2 by rotations, as well as on $\mathbb{RP}^2$ by rotation of a line through the origin in $\mathbb{R}^3$, are applied in this section to give $K(\pi, 1)$'s where the groups π are certain mapping class groups. We will also make use of the group S^3, the connected double cover of $SO(3)$, and its action on $\mathrm{Conf}(S^2, q)$ through the diagonal action of $SO(3)$ on products of S^2.

The purpose of this section is to derive properties of the configuration spaces $\mathrm{Conf}(S^2, k)$ as well as the spaces

$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, k)/\Sigma_k.$$

The first theorem in this direction is due to Smale who proved the following result [72].

Theorem 9.1. *The natural inclusions*

$$SO(3) \subset \mathrm{Diff}^+(S^2) \subset \mathrm{Top}^+(S^2)$$

are homotopy equivalances.

Smale's theorem has the following consequences as pointed out in [11, 6].

Theorem 9.2. *If $q \geq 3$, then*

(1) *the space*
$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q)/\Sigma_q$$
is a $K(\Gamma_0^q, 1)$, *and*
(2) *the space*
$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q)$$
is a $K(P\Gamma_0^q, 1)$.

A slightly stronger version of Theorem 9.2 is obtained from the next classical lemma, proven by considering cross-ratios, for which S^2 is regarded as the space of complex lines though the origin in $\mathbb{C}^2$, namely $\mathbb{CP}^1$. Consider the evaluation map
$$e : PGL(2, \mathbb{C}) \times \mathrm{Conf}(\mathbb{CP}^1, 3) \to \mathrm{Conf}(\mathbb{CP}^1, 3)$$
defined by the equation
$$e(\alpha, (L_1, L_2, L_3)) = (\alpha(L_1), \alpha(L_2), \alpha(L_3))$$
where L_i are 3 fixed, distinct lines through the origin in $\mathbb{C}^2$ with (L_1, L_2, L_3) a point in $\mathrm{Conf}(\mathbb{CP}^1, 3)$ and α is an element in $PGL(2, \mathbb{C})$.

Lemma 9.3. *The restriction of the map*
$$e : PGL(2, \mathbb{C}) \times \mathrm{Conf}(\mathbb{CP}^1, 3) \to \mathrm{Conf}(\mathbb{CP}^1, 3),$$
to the subspace
$$PGL(2, \mathbb{C}) \times \{(L_1, L_2, L_3)\},$$
for any point (L_1, L_2, L_3) *in* $\mathrm{Conf}(\mathbb{CP}^1, 3)$, *is a homeomorphism. Thus* $PGL(2, \mathbb{C})$ *is homeomorphic to* $\mathrm{Conf}(\mathbb{CP}^1, 3)$.

A cruder version of Lemma 9.3 which exhibits a homotopy equivalence rather than a homeomorphism follows directly from Theorem 3.3 as follows.

Lemma 9.4. *Restrict the map*
$$e : SO(3) \times \mathrm{Conf}(\mathbb{CP}^1, 3) \to \mathrm{Conf}(\mathbb{CP}^1, 3)$$
defined by the equation
$$e(\alpha, (L_1, L_2, L_3)) = (\alpha(L_1), \alpha(L_2), \alpha(L_3))$$
to the subspace
$$SO(3) \times \{(L_1, L_2, L_3)\}$$
for any point (L_1, L_2, L_3) *in* $\mathrm{Conf}(\mathbb{CP}^1, 3)$. *This restriction is a homotopy equivalence.*

Proof. Since $SO(3)$ is path-connected, it suffices to check the lemma for the points in S^2 given by

(1) $L_1 = (1, 0, 0)$,
(2) $L_2 = (0, 1, 0)$, and
(3) $L_3 = (0, 0, 1)$.

The map

$$E : SO(3) \to \mathrm{Conf}(S^2, 3)$$

defined by the equation

$$E(\alpha) = (\alpha(L_1), \alpha(L_2), \alpha(L_3))$$

gives a morphism of fibrations

$$\begin{array}{ccc} SO(2) & \xrightarrow{E|SO(2)} & S^2 - Q_2 \\ \downarrow & & \downarrow \\ SO(3) & \xrightarrow{E} & \mathrm{Conf}(S^2, 3) \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{\beta} & \mathrm{Conf}(S^2, 2) \end{array}$$

where $Q_2 = \{L_1, L_2\}$ with $\beta : S^2 \to \mathrm{Conf}(S^2, 2)$ defined by

$$\beta(\alpha(L_1)) = (\alpha(L_1), \alpha(L_2)).$$

The induced maps on the fibre $E|SO(2) : SO(2) \to S^2 - Q_2$, as well as on the base, $\beta : S^2 \to \mathrm{Conf}(S^2, 2)$, are homotopy equivalences by inspection. Since the right-hand side is a fibration by Theorem 3.3, the lemma follows.

□

Lemma 9.5. *The spaces*

$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, 3),$$

and

$$EPGL(2, \mathbb{C}) \times_{PGL(2,\mathbb{C})} \mathrm{Conf}(S^2, 3)$$

are contractible.

Remark 9.6. The preceding lemma gives a special case of Theorem 9.2 for $q = 3$ in which the group π is the trivial group.

Proof. It suffices to check that the space $ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, 3)$ is contractible as $SO(3)$ is the maximal compact subgroup of $PGL(2, \mathbb{C})$ and the inclusion $SO(3) \subset PGL(2, \mathbb{C})$ is a homotopy equivalance.

Since the natural map

$$E : SO(3) \to \mathrm{Conf}(S^2, 3)$$

is a homotopy equivalence by the proof of Lemma 9.4, and is also $SO(3)$-equivariant by construction, the result follows as $ESO(3) \times_{SO(3)} SO(3)$ is contractible. □

The proof of Theorem 9.2 is given next with an application to the cohomology of the associated mapping class groups [11].

Proof of Theorem 9.2. There are two steps to the proof of this theorem. The first is that if $q \geq 3$, then the resulting space $ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q)$ is a $K(\pi, 1)$. The second step is to work out the fundamental group π of the space in question.

Assume that $q \geq 3$ and observe that the natural projection map to the first three coordinates

$$p(1, 2, 3) : \mathrm{Conf}(S^2, q) \to \mathrm{Conf}(S^2, 3)$$

is $SO(3)$-equivariant by inspection of the definitions. In addition, the map $p(1, 2, 3)$ is a fibration with fibre $\mathrm{Conf}(S^2 - Q_3, q - 3)$, by Theorem 3.3. Thus the induced projection map

$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q) \to ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, 3)$$

is a fibration with fibre $\mathrm{Conf}(S^2 - Q_3, q - 3)$.

On the other-hand, the space $ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, 3)$ is contractible by Lemma 9.5. Thus the inclusion $\mathrm{Conf}(S^2 - Q_3, q - 3) \subset ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q)$ is a homotopy equivalence. Furthermore, the space $\mathrm{Conf}(S^2 - Q_3, q - 3)$ is a $K(\pi, 1)$ by Theorem 8.3 as $S^2 - Q_3$ is a surface which is not S^2 or $\mathbb{RP}^2$. Thus $\mathrm{Conf}(S^2 - Q_3, q - 3)$ and consequently $ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q)$ are $K(\pi, 1)$'s.

The final step is to show that if $q \geq 3$, the spaces

$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q),$$

and

$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q)/\Sigma_q$$

are respectively $K(P\Gamma_0^q, 1)$, and $K(\Gamma_0^q, 1)$. Thus it suffices to work out their fundamental groups.

To carry out this step, it is useful to note that by Theorem 9.1, the natural inclusions

$$SO(3) \subset \mathrm{Diff}^+(S^2) \subset \mathrm{Top}^+(S^2)$$

are homotopy equivalances. Furthermore, by Theorem 3.2, there is a fibration sequence

$$\mathrm{Top}(S^2,q) \to \mathrm{Top}(S^2) \to \mathrm{Conf}(S^2,q) \to B\mathrm{Top}(S^2,q) \to B\mathrm{Top}(S^2).$$

Since the induced map

$$\mathrm{Top}^+(S^2)/\mathrm{Top}^+(S^2,q) \to \mathrm{Top}(S^2)/\mathrm{Top}(S^2,q)$$

is a homeomorphism by Lemma 8.8, it follows that there is a homotopy equivalence

$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2,q) \to B\mathrm{Top}^+(S^2,q).$$

□

Remark 9.7. A similar argument, given in the thesis of J. Wong [80], gives the contruction of $K(\pi,1)$'s where π is the braid group of either S^2 or $\mathbb{RP}^2$. Furthermore, Wong used these spaces to work out the cohomology of the associated braid groups.

Theorem 9.8. *Assume that $q \geq 3$.*

1. *The space $ES^3 \times_{S^3} \mathrm{Conf}(S^2,q)/\Sigma_q$ is a $K(B_q(S^2),1)$. Furthermore, the space $ES^3 \times_{S^3} \mathrm{Conf}(S^2,q)$ is a $K(P_q(S^2),1)$.*
2. *The space $ES^3 \times_{S^3} \mathrm{Conf}(\mathbb{RP}^2,q)/\Sigma_q$ is a $K(B_q(\mathbb{RP}^2),1)$. Furthermore, the space $ES^3 \times_{S^3} \mathrm{Conf}(\mathbb{RP}^2,q)$ is a $K(P_q(\mathbb{RP}^2),1)$.*

In addition, an attractive presentation of the braid group $B_k(S^2)$, the fundamental group of $\mathrm{Conf}(S^2,k)/\Sigma_k$, is given in an elegant paper by Fadell and van Buskirk [33, 3].

Remark 9.9. The spaces $ES^3 \times_{S^3} \mathrm{Conf}(S^2,q)/\Sigma_q$ and their fundamental groups are intimately connected with mapping class groups. One connection is to a group $\Delta_{2g+2} \subset \Gamma_g^0$ called the hyper-elliptic mapping class group, which is the centralizer of a certain choice of involution of Γ_g^0. There is a central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Delta_{2g+2} \to B_{2g+2}(S^2) \to 1.$$

The group Δ_{2g+2} is the fundamental group of a bundle arising from complex 2-plane bundles and associated configuration space bundles, a point described in the next example.

Geometrically, a $K(\Gamma_2^0,1)$ can be obtained by 'twisting together' actions of $S^3 \times S^1$ to give an action of $U(2)$ on $\mathrm{Conf}(S^2,k) \times_{\Sigma_k} S^1$. This is addressed in the next example.

Example 9.10. The variation of the above construction for Δ_{2g+2} is described in this example as developed in [11]. This construction gives models for certain mapping class groups.

There are three ingredients here:

(1) The group $\pi = \mathbb{Z}/2\mathbb{Z}$ is the center of $SU(2)$ while $SO(3)$ is the quotient $SU(2)/\pi$. Furthermore $U(2)$ is a quotient of $SU(2) \times S^1$ obtained from this action as follows. Regard $\mathbb{Z}/2\mathbb{Z}$ as the central subgroup of $S^3 \times S^1$ generated by $(-1, -1)$. Then form the central quotient $SU(2) \times_{\mathbb{Z}/2\mathbb{Z}} S^1$. There is an isomorphism of groups

$$SU(2) \times_{\mathbb{Z}/2\mathbb{Z}} S^1 \to U(2),$$

a construction given in work of Atiyah, Bott, and Shapiro known as $Spin^c(3)$.

(2) Define an action of the product $S^3 \times S^1$ on $S^2 \times S^1$ by requiring that (i) S^3 act on S^2 through the natural action of rotations by $SO(3)$, and (ii) the group S^1 acts on itself as follows:

$$(\alpha, \beta) = \alpha^2 \cdot \beta.$$

(3) The diagonal action of $S^3 \times S^1$ on $\mathrm{Conf}(S^2, k) \times_{\Sigma_k} S^1$ descends to an action of $U(2)$.

Consider the Borel construction of this action. One result of is that

$$EU(2) \times_{U(2)} \mathrm{Conf}(S^2, k) \times_{\Sigma_k} S^1$$

is a $K(\pi, 1)$. If $k = 6$, then this space is a $K(\Gamma_2, 1)$ [11]. Furthermore, if $k = 2g + 2$ for g even, then the fundamental group of $EU(2) \times_{U(2)} \mathrm{Conf}(S^2, k) \times_{\Sigma_k} S^1$ is the centralizer of the class of a hyper-elliptic involution in Γ_g which is denoted Δ_g.

Consider the fibration

$$ESO(3) \times_{SO(3)} \mathrm{Conf}(S^2, q)/\Sigma_q \to BSO(3)$$

with fibre $\mathrm{Conf}(S^2, q)/\Sigma_q$. Then consider the long exact homotopy sequence for this fibration together with the fact that $\pi_1(BSO(3))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ to prove the following theorem [3].

Theorem 9.11. *If $q \geq 3$, then there is a central extension*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to B_q(S^2) \to \Gamma_0^q \to 1.$$

A similar result applies to the spaces

$$EU(2) \times_{U(2)} \mathrm{Conf}(S^2, k) \times_{\Sigma_k} S^1.$$

Theorem 9.12. *If g is even with $g \geq 2$, there are central extensions*

(1) $1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma_2 \to \Gamma_0^6 \to 1$, *and*

(2) $1 \to \mathbb{Z}/2\mathbb{Z} \to \Delta_g \to \Gamma_0^{2g+2} \to 1$.

Remark 9.13. Any such central extension is determined by a characteristic class. The characteristic class in the case of Theorem 9.12 naturally arises from a $Spin^c(3)$-structure implicit in the underlying topology of the diffeomorphism group [11].

10. On configurations in $S^1 \times S^1$

A second special case arises with configuration spaces for surfaces of genus 1, the subject of this section. Since the upper half-plane is closely connected to the structure of this configuration space, as well as to the applications below, such as the structure of certain 'Brunnian braid groups' as given in the Appendix, Section 23, some introductory information on it is listed next.

Let $\mathbb{H}$ denote the upper half-plane, the complex numbers with strictly positive pure imaginary part. The group $SL(2,\mathbb{Z})$ acts on $\mathbb{H}$ by fractional linear transformations where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is in $SL(2,\mathbb{Z})$, and

$$M(z) = \frac{az+b}{cz+d}$$

for $z \in \mathbb{H}$.

The orbit space $\mathbb{H}/SL(2,\mathbb{Z})$, important in classical number theory and the theory of automorphic forms [71], has the feature that the projection map

$$q : \mathbb{H} \to \mathbb{H}/SL(2,\mathbb{Z})$$

has singular points and is not a covering projection. In this case, the action of $SL(2,\mathbb{Z})$ is not free.

The action by the kernel $\Gamma(2,r)$ of the mod-r reduction map $\rho_r : SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/r\mathbb{Z})$ is free in case $r \geq 2$. Furthermore, the group $\Gamma(2,r)$ is isomorphic to a finitely generated free group in case $r \geq 2$ [48, 71]. For example, if p is an odd prime, $\Gamma(2,p)$ is a free group on $1 + p(p^2-1)/12$ letters while $\Gamma(2,2)$ is a free group on 2 letters [38]. Furthermore, the group $\Gamma(2,4)$ is isomorphic to the 4-stranded Brunnian braid group $\mathrm{Brun}_4(S^2)$, as described in Section 23.

Observe that there is a map

$$\Phi : \mathrm{Top}^+(S^1 \times S^1) \to SL(2,\mathbb{Z})$$

defined by sending an element $f \in \mathrm{Top}^+(S^1 \times S^1)$ to the isomorphism

$$f_* : H_1(S^1 \times S^1) \to H_1(S^1 \times S^1),$$

regarded as an element in $GL(2,\mathbb{Z})$ of determinant $+1$.

There is an analogous map

$$\Phi_g : \text{Top}^+(S_g) \to Sp(2g,\mathbb{Z})$$

defined by the equation

$$\Phi_g(f) = f_* : H_1(S_g) \to H_1(S_g)$$

where S_g is a closed orientable surface of genus g. In this case, the value of $\Phi_g(f)$ is an element in $GL(2g,\mathbb{Z})$ which preserves the cup-product structure for the cohomology of S_g and is thus an element in $Sp(2g,\mathbb{Z})$.

For convenience, let T^2 denote the torus $S^1 \times S^1$. Recall that the group $SL(2,\mathbb{Z})$ also acts on $T^2 = S^1 \times S^1$ with action defined by the equation

$$M(u,v) = (u^a v^b, u^c v^d)$$

for $(u,v) \in T^2$. Notice that this action preserves the point $(1,1)$ and thus restricts to an action on $T^2 - \{(1,1)\}$, a space which is denoted $\widehat{T^2}$ here. Furthermore,

$$M((u,v)\cdot(u',v')) = M((u,v))\cdot M((u',v'))$$

since S^1 is abelian. Furthermore, observe that this action, on the level of the first homology group $H_1(T^2) = \mathbb{Z}\oplus\mathbb{Z}$, is precisely the 'tautological' action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}\oplus\mathbb{Z}$.

Notice that there is an induced homomorphism

$$E : SL(2,\mathbb{Z}) \to \text{Top}^+(T^2)$$

defined by the equation $E(M)(u,v) = M(u,v)$. A lemma using this information and convenient for the proof of Theorem 10.6 below, is stated next.

Lemma 10.1. *The function*

$$E : SL(2,\mathbb{Z}) \to \text{Top}^+(T^2)$$

is a continuous homomorphism. Furthermore, this map splits the natural map

$$\Phi : \text{Top}^+(T^2) \to SL(2,\mathbb{Z})$$

with $\Phi \circ E$ given by the identity self-map of $SL(2,\mathbb{Z})$.

Variations obtained by 'twisting together' configuration spaces on the one-hand give $K(\pi,1)$'s for $\pi = \Gamma_1^k$. On the other-hand, the resulting $K(\Gamma_1^k,1)$-spaces have real cohomology groups which are given in terms of classical modular forms. This section is an exposition of the connection between these analogues of configuration spaces with cohomology given in terms of modular forms. The first theorem in this direction is developed next [12].

It is now convenient to focus on the space $\mathrm{Diff}^+(S^1 \times S^1)$ in order to appeal directly to results of Earle and Eells [27], keeping in mind that the natural inclusion

$$\mathrm{Diff}^+(S^1 \times S^1) \subset \mathrm{Top}^+(S^1 \times S^1)$$

is a homotopy equivalence by Theorem 8.7. Notice that $S^1 \times S^1$ acts by rotations on each coordinate in T^2, giving elements in $\mathrm{Diff}^+(S^1 \times S^1)$ which are isotopic to the identity. These rotations are in the kernel of

$$\Phi : \mathrm{Diff}^+(S^1 \times S^1) \to SL(2, \mathbb{Z}),$$

with this kernel denoted $\widetilde{S^1 \times S^1}$. Earle and Eells prove that the natural inclusion

$$S^1 \times S^1 \to \widetilde{S^1 \times S^1}$$

is a homotopy equivalence [27]. Thus there is a fibration

$$B\Phi : B\mathrm{Diff}^+(S^1 \times S^1) \to BSL(2, \mathbb{Z})$$

with fibre $B(\widetilde{S^1 \times S^1})$ which is homotopy equivalent to $(\mathbb{CP}^\infty)^2$. This information will be used to prove the next result.

Theorem 10.2. *Assume that $k \geq 2$. The spaces*

$$E\mathrm{Diff}^+(S^1 \times S^1) \times_{\mathrm{Diff}^+(S^1 \times S^1)} \mathrm{Conf}(T^2, k)/\Sigma_k,$$

and

$$E\mathrm{Top}^+(S^1 \times S^1) \times_{\mathrm{Top}^+(S^1 \times S^1)} \mathrm{Conf}(T^2, k)/\Sigma_k$$

are both $K(\Gamma_1^k, 1)$.

A proof is given by the following sequence of lemmas.

Lemma 10.3. *The spaces $E\mathrm{Top}^+(T^2) \times_{S^1 \times S^1} T^2$, and $E\mathrm{Top}^+(T^2) \times_{\widetilde{S^1 \times S^1}} T^2$ are contractible. Thus $E\mathrm{Diff}^+(T^2) \times_{S^1 \times S^1} T^2$, and $E\mathrm{Diff}^+(T^2) \times_{\widetilde{S^1 \times S^1}} T^2$ are contractible.*

Proof. Observe $ET^2 \times_{T^2} T^2$ is contractible and so $E\mathrm{Top}^+(T^2) \times_{S^1 \times S^1} T^2$ is also.

Next, consider the natural inclusion $\iota : S^1 \times S^1 \to \widetilde{S^1 \times S^1}$ to obtain a map of orbit spaces

$$E\mathrm{Top}^+(T^2) \times_{S^1 \times S^1} T^2 \to E\mathrm{Top}^+(T^2) \times_{\widetilde{S^1 \times S^1}} T^2$$

which is a homotopy equivalence. Since $E\mathrm{Top}^+(T^2) \times_{S^1 \times S^1} T^2$ is contractible, so is

$$E\mathrm{Top}^+(T^2) \times_{\widetilde{S^1 \times S^1}} T^2.$$

The lemma follows. $\square$

Lemma 10.4. *The space*

$$E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} T^2$$

is a $K(\pi, 1)$ *where* π *is isomorphic to* $SL(2, \mathbb{Z})$*.*

Proof. Observe that there is a fibration

$$E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} T^2 \to BSL(2, \mathbb{Z})$$

with fibre

$$E\mathrm{Top}^+(T^2) \times_{\widetilde{S^1 \times S^1}} T^2.$$

Since $E\mathrm{Top}^+(T^2) \times_{\widetilde{S^1 \times S^1}} T^2$ is contractible by 10.3, the lemma follows. □

Lemma 10.5. *If* $k \geq 2$ *then the space*

$$E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k)$$

is a $K(\pi, 1)$*.*

Proof. Consider the natural first coordinate projection map

$$E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k) \to E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} T^2$$

with fibre $\mathrm{Conf}(\widehat{T^2}, k-1)$.

The base of this fibration, $E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} T^2$ is a $K(SL(2, \mathbb{Z}), 1)$ by Lemma 10.4, and the fibre $\mathrm{Conf}(\widehat{T^2}, k-1)$ is also a $K(\pi, 1)$ by Theorem 8.3. It follows that

$$E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k)$$

is a $K(\pi, 1)$ for all $k \geq 1$. □

Proof of Theorem 10.2. By Lemma 10.5, the space

$$E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k)$$

is a $K(\pi, 1)$ for all $k \geq 2$. Thus the space $E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k)/\Sigma_k$ is also a $K(\pi, 1)$ as the action of Σ_k on $E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k)$ is free.

Furthermore, by Theorem 3.2 $E\mathrm{Top}(T^2) \times_{\mathrm{Top}(T^2)} \mathrm{Conf}(T^2, k)/\Sigma_k$ has fundamental group given by $\pi_0(\mathrm{Top}(T^2, k))$. In addition, $E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k)/\Sigma_k$ has fundamental group given by $\pi_0(\mathrm{Top}^+(T^2, k))$ by Lemma 8.8. □

The next theorem gives information about the pointed mapping class group with k marked points, $\Gamma_g^{k,*}$. Let $\widehat{1}$ denote the element $(1,1) \in T^2$ with $\widehat{T^2} = T^2 - \{\widehat{1}\}$. The above remarks imply that the natural action of $SL(2,\mathbb{Z})$ on T^2 restricts to an action on $\widehat{T^2}$. The next theorem gives information about $ESL(2,\mathbb{Z}) \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(\widehat{T^2}, k)/\Sigma_k$ which has had computational utility [12].

Theorem 10.6. *Assume that $k \geq 1$. The space*

$$ESL(2,\mathbb{Z}) \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(\widehat{T^2}, k)/\Sigma_k$$

is a $K(\Gamma_1^{k,*}, 1)$.

Proof. By Lemma 10.5, the space

$$E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k)$$

is a $K(\pi,1)$ for all $k \geq 2$. Thus the space $E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k+1)/(1 \times \Sigma_k)$ is also a $K(\pi,1)$ as the action of $1 \times \Sigma_k$ on $E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k+1)$ is free. Furthermore, the fundamental group of $E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k+1)/(1 \times \Sigma_k)$ is $\Gamma_1^{k,*}$ by Theorem 3.2.

To finish the proof of the theorem, it suffices to exhibit a homotopy equivalence

$$\Gamma(k) : ESL(2,\mathbb{Z}) \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(\widehat{T^2}, k)/\Sigma_k \to E\mathrm{Top}^+(T^2) \times_{\mathrm{Top}^+(T^2)} \mathrm{Conf}(T^2, k+1)/(1 \times \Sigma_k).$$

To define the map $\Gamma(k)$, first consider the action of $SL(2,\mathbb{Z})$ on T^2 obtained from the homomorphism

$$E : SL(2,\mathbb{Z}) \to \mathrm{Top}^+(T^2)$$

of Lemma 10.1. From this, we see that $\mathrm{Top}^+(T^2)$ is a semi-direct product of $SL(2,\mathbb{Z})$ and $\widetilde{S^1 \times S^1}$.

Now define a map

$$\gamma_k : \mathrm{Conf}(\widehat{T^2}, k) \to \mathrm{Conf}(T^2, k+1)$$

by the formula

$$\gamma_k((z_1, \ldots, z_k)) = (z_1, \ldots, z_k, 1).$$

Next consider the natural quotient map

$$\rho : ESL(2,\mathbb{Z}) \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(T^2, k+1) \to E\mathrm{Top}^+(T_2) \times_{\mathrm{Top}^+(T_2)} \mathrm{Conf}(T^2, k+1)$$

pre-composed with the map $1 \times \gamma_k$; we let

$$\mu : ESL(2,\mathbb{Z}) \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(\widehat{T^2}, k) \to E\mathrm{Top}^+(T_2) \times_{\mathrm{Top}^+(T_2)} \mathrm{Conf}(T^2, k+1)$$

be given by

$$\mu = \rho \circ (1 \times \gamma_k).$$

Since the map μ is equivariant with respect to the action of Σ_k on the source and $1 \times \Sigma_k$ on the target, that μ is a homotopy equivalence then implies that the induced quotient map $\Gamma(k)$ is also a homotopy equivalence and the theorem follows.

That μ is an equivalence follows from a comparison of fibrations as given next. First consider the commutative diagram

$$\begin{array}{ccccc}
E \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(\widehat{T^2}, k) & \xrightarrow{\pi} & BSL(2,\mathbb{Z}) & \xrightarrow{1} & BSL(2,\mathbb{Z}) \\
{\scriptstyle 1\times\gamma_k}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow \\
E \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(T^2, k+1) & \xrightarrow{\pi} & BSL(2,\mathbb{Z}) & \xrightarrow{1} & BSL(2,\mathbb{Z}) \\
{\scriptstyle \rho}\downarrow & & {\scriptstyle BE}\downarrow & & {\scriptstyle 1}\downarrow \\
E \times_{\mathrm{Top}^+(T_2)} \mathrm{Conf}(T^2, k+1) & \xrightarrow{\pi} & B\mathrm{Top}^+(T_2) & \xrightarrow{B\Phi} & BSL(2,\mathbb{Z})
\end{array}$$

for which $\pi : EG \times_G X \to BG$ denotes the natural projection map.

Thus there is a morphism of fibrations

$$\begin{array}{ccccc}
\mathrm{Conf}(\widehat{T^2}, k) & \xrightarrow{1} & E \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(\widehat{T^2}, k) & \xrightarrow{\pi} & BSL(2,\mathbb{Z}) \\
{\scriptstyle 1\times\gamma_k}\downarrow & & {\scriptstyle \rho\circ(1\times\gamma_k)}\downarrow & & \downarrow{\scriptstyle 1} \\
E \times_{\widetilde{S^1\times S^1}} \mathrm{Conf}(T^2, k+1) & \xrightarrow{1} & E \times_{\mathrm{Top}^+(T_2)} \mathrm{Conf}(T^2, k+1) & \xrightarrow{B\Phi\circ\pi} & BSL(2,\mathbb{Z}).
\end{array}$$

for which the induced map

$$1 \times \gamma_k : \mathrm{Conf}(\widehat{T^2}, k) \to E \times_{\widetilde{S^1\times S^1}} \mathrm{Conf}(T^2, k+1)$$

is an equivalence by observing that

(1) the projection

$$E \times_{\widetilde{S^1\times S^1}} \mathrm{Conf}(T^2, k+1) \to E \times_{\widetilde{S^1\times S^1}} T^2$$

is a bundle projection with fibre $\mathrm{Conf}(\widehat{T^2}, k)$,

(2) the space

$$E \times_{\widetilde{S^1\times S^1}} T^2$$

is contractible (by Lemma 10.3), and

(3) the induced self-map of $\mathrm{Conf}(\widehat{T^2}, k)$ is the identity, by inspection.

The theorem follows. □

Remark 10.7. (1) The bundle projection $B\mathrm{Diff}^+(S^1 \times S^1) \to BSL(2,\mathbb{Z})$ with fibre $B(S^1 \times S^1) = (\mathbb{CP}^\infty)^2$ was first exploited in a beautiful paper by Furusawa, Tezuka, and Yagita who showed that the real cohomology of $B\mathrm{Diff}^+(S^1 \times S^1)$ was given in terms of classical modular forms [40]. They also determined the torsion in the cohomology of $B\mathrm{Diff}^+(S^1 \times S^1)$

(2) The naturally associated orbifold $\mathbb{H} \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(S^1 \times S^1, q)$ has cohomology which is that of $ESL(2,\mathbb{Z}) \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(S^1 \times S^1, q)$, as long as the primes 2 and 3 are units.

(3) The real cohomology of $ESL(2,\mathbb{Z}) \times_{SL(2,\mathbb{Z})} \mathrm{Conf}(\widehat{T^2}, k)/\Sigma_k$, and thus the cohomology of $\Gamma_1^{k,*}$, was worked out in [12] where the answer is given in terms of ranks of certain modular forms. In addition, fixing the dimension of the cohomology group while letting k increase gives a 'stable answer' which is equal to the ranks of certain Jacobi forms computed by Eichler and Zagier [30], by a direct comparison. This particular interpretation also gave the cohomology groups (more easily) with coefficients in the sign representation.

11. On configurations in a surface of genus greater than 1

The purpose of this section is to focus on the configuration space of surfaces S_g of genus g greater than 1. One distinguishing feature in this case is the result of Earle and Eells which proves that

$$\mathrm{BDiff}^+(S_g)$$

is a $K(\pi,1)$, namely, each path-component of $\mathrm{Diff}^+(S_g)$ is contractible [27]. Again, the spaces

Theorem 11.1. *Assume that S_g is an orientable surface of genus $g \geq 2$. Then*

1. $E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k)/\Sigma_k$ *is a* $K(\Gamma_g^k, 1)$, *and*
2. $E\mathrm{Top}(S_g)^+ \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k+1)/\{1 \ \times \Sigma_k\}$ *is a* $K(\Gamma_g^{k,*}, 1)$.

Proof. First consider $E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k)$, the total space of a bundle over $B\mathrm{Top}^+(S_g)$ with fibre $\mathrm{Conf}(S_g, k)$. Since $B\mathrm{Top}^+(S_g)$, and $\mathrm{Conf}(S_g, k)$ are $K(\pi,1)$'s, so is $E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k)$ as well as $E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k)/H$ where H is any subgroup of the symmetric group on k letters. Thus both

$$E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k)/\Sigma_k,$$

and

$$E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k+1)/\{1 \ \times \Sigma_k\}$$

are $K(\pi,1)$'s.

To finish, it suffices to identify the fundamental groups of these spaces. Notice that by Theorem 3.2, (i) the fundamental group of $E\mathrm{Top}^+(S_g) \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k)/\Sigma_k$ is Γ_g^k, and (ii) the fundamental group of $E\mathrm{Top}(S_g)^+ \times_{\mathrm{Top}^+(S_g)} \mathrm{Conf}(S_g, k+1)/\{1 \ \times \Sigma_k\}$ is $\Gamma_g^{k,*}$. □

Remark 11.2. Many of these results appeared in a slightly different form in [42, 69, 70], with the exception of the $K(\pi, 1)$ properties.

12. Loop spaces of configuration spaces

The subject of this section, as well Sections 14 through 15, is basic properties of loop spaces of configuration spaces. A subsequent section, Section 21, lists overlapping features describing connections of these structures to low dimensional topology, homotopy theory and number theory.

First recall the definitions of free loop spaces, and pointed loop spaces.

Definition 12.1. Let X denote a topological space.

(1) Define the free loop space of X to be
$$L(X) = \{f : S^1 \to X \mid f \text{ is continuous}\},$$
topologized with the compact-open topology.

(2) If X has a base-point $*$, define the pointed loop space of X to be
$$\Omega(X) = \{f : S^1 \to X \mid f \text{ is continuous and } f(1) = *\}$$
for $1 \in S^1 \subset \mathbb{R}^2$ with $\Omega(X)$ topologized as a subspace of $L(X)$.

(3) The topology on both $L(X)$ and $\Omega(X)$ is the compact-open topology. A more convenient, as well as more general, choice of topology is given by the associated compactly-generated topology. However, this generalization will not be emphasized here.

One way to view $\Omega\mathrm{Conf}(M, k)$ is through the graph of a function
$$f : [0,1] \to \mathrm{Conf}(M, k)$$
given by
$$\mathrm{graph}(f) : [0,1] \to [0,1] \times \mathrm{Conf}(M, k)$$
with
$$\mathrm{graph}(f)(t) = (t, f(t)).$$

FIGURE 1. Picture of a braid in P_{N+1}.

Notice that in case $M = \mathbb{R}^2$, then graph(f) is an embedding whose image is exactly a braid which starts at $(0, f(0))$ and quits at $(1, f(1))$. These graphs represent the precise physical meaning of a braid, see Figure 2.

Notice that by definition, the group of path-components satisfies the properties

$$\pi_0(\Omega(\mathrm{Conf}(\mathbb{R}^2, k)) = P_k$$

and

$$\pi_0(\Omega(\mathrm{Conf}(\mathbb{R}^2, k)/\Sigma_k)) = B_k.$$

Furthermore, the spaces $\mathrm{Conf}(\mathbb{R}^2, k)$, and $\mathrm{Conf}(\mathbb{R}^2, k)/\Sigma_k$ are $K(\pi, 1)$'s by Theorem 8.3. The next classical result follows at once [32, 37].

Theorem 12.2. *The unordered configuration space* $\mathrm{Conf}(\mathbb{R}^2, k)/\Sigma_k$ *is a* $K(B_k, 1)$ *and the ordered configuration space* $\mathrm{Conf}(\mathbb{R}^2, k)$ *is a* $K(P_k, 1)$.

One result proven here gives the structure of the homology of the pointed loop space of $\mathrm{Conf}(\mathbb{R}^m, k)$, a result which reflects elementary properties of linking invariants for pairs of linked spheres. This setting of linking is developed in two ways below. One way is by looking at the way in which these constructions extend to invariants of classical links. The second way is by looking at how the algebras associated to these linking invariants correspond to certain spectral sequences in homotopy theory, as elucidated in the section on simplicial groups, Section 16. This section will start with motivating examples and then continue with the proofs of some basic theorems. A summary of how and where these results fit in is given in the section 'Other connections', Section 21.

Recall the following facts. The projection maps

$$p_k : \mathrm{Conf}(\mathbb{R}^m, k) \to \mathrm{Conf}(\mathbb{R}^m, k-1)$$

admit cross-sections σ defined by the formula

$$\sigma_k(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, w)$$

where

$$w = M(\vec{e}_1)$$

where e_1 is the unit vector $(1, 0, \ldots, 0)$ and $M = 1 + \max_{1 \le i \le k-1} ||x_i||$. Furthermore, the fibre of p_k is $\mathbb{R}^n - Q_{k-1}$ which is homotopy equivalent to $\vee_{k-1} S^{m-1}$.

Next recall the following classical lemma with proof given in [66].

Lemma 12.3. *Let*

$$p : E \to B$$

be a fibration with fibre F and $\iota : F \to E$ the inclusion of the fibre in the total space, for which E, B, and F are path-connected spaces. If p admits a cross-section (up to homotopy), then $\Omega(E)$ is homotopy equivalent to

$$\Omega(B) \times \Omega(F).$$

One consequence of the existence of the cross-sections σ_k above, and Lemma 12.3, is stated next.

Proposition 12.4. *Assume that M is a manifold without boundary, of dimension $m \ge 2$ such that the natural first coordinate projection*

$$p_1 : \mathrm{Conf}(M, k) \to M$$

admits a section. Then there is a homotopy equivalence

$$\prod_{0 \le i \le k-1} \Omega(M - Q_i) \to \Omega\mathrm{Conf}(M, k).$$

If $m \ge 3$, then there is a homotopy equivalence

$$\prod_{1 \le i \le k-1} \Omega(\vee_i S^{m-1}) \to \Omega\mathrm{Conf}(\mathbb{R}^m, k).$$

Proof. Since the projection p_1 admits a cross-section by hypotheses, it follows from Lemma 12.3 that there is a homotopy equivalence

$$\Omega(M) \times \Omega\mathrm{Conf}(M - Q_1, k-1) \to \Omega\mathrm{Conf}(M, k).$$

□

The homology of the loop space of certain configuration spaces is worked out next. One reason for including this computation is that it appears in several different natural mathematical contexts. A second reason is that the homology of $\Omega\mathrm{Conf}(\mathbb{R}^m, k))$ arises in terms of Vassiliev invariants of pure braids as developed by Toshitake Kohno [50]. It then turns out that these structures are intimately tied to the homotopy groups of spheres, as described below, and then to certain natural structures involving derivations of free Lie algebras, as described in several sections below. Since the Lie algebras encountered here are free as modules over the integers, the definitions given next will be restricted to free modules over the integers.

Certain graded Lie algebras are basic here. The first one is the free Lie algebra generated by a graded free abelian group V. First recall that any graded, associative algebra A inherits the structure of a Lie algebra with the bracket

$$[-,-] : A \otimes A \to A$$

defined by the formula

$$[a, b] = a \cdot b + (-1)^{|a||b|} b \cdot a$$

for elements a and b of degree $|a|$ and $|b|$ respectively.

Definition 12.5. Let V denote a graded free abelian group with $T[V]$ the tensor algebra generated by V. Then

$$L[V]$$

is the smallest sub-Lie algebra of $T[V]$ generated by V.

A related Lie algebra arises which is universal for the (graded) 'infinitesimal braid relations', also known as the 'horizontal $4T$-relations', or the 'Yang-Baxter Lie algebra relations', as in [49, 52, 13, 31]. That is the largest Lie algebra over a fixed commutative ring R for which the 'infinitesimal braid relations' are satisfied.

Definition 12.6. Fix a strictly positive integer q. Define

$$\mathcal{L}_k(q)$$

to be the free (graded) Lie algebra over the integers $\mathbb{Z}$ generated by elements $B_{i,j}$ of degree q, $k \geq i > j \geq 1$, modulo the graded infinitesimal braid relations:

- **(i):** $[B_{i,j}, B_{s,t}] = 0$ if $\{i,j\} \cap \{s,t\} = \emptyset$,
- **(ii):** $[B_{i,j}, B_{i,t} + (-1)^q B_{t,j}] = 0$ if $1 \leq j < t < i \leq k$, and
- **(iii):** $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ if $1 \leq j < t < i \leq k$.

T. Kohno [49] gives a slightly different description of these relations as follows: Introduce new generators $B_{j,i}$ of degree q, $k \geq i > j \geq 1$ with the relations $B_{i,j} = (-1)^q B_{j,i}$. Then Kohno's description of the above relations simplifies to (i), and (ii) above with distinct i, j, and t.

These relations appear as special cases in the Vassiliev invariants of braids [52, 13]. They also arise in the study of the KZ (Knishnik-Zamolodchikov) equations as integrability conditions for certain flat bundles [10], as well as in work of Kohno [49, 52], and Drinfel'd [25, 26] on the Kohno-Drinfel'd monodromy theorem [10]. The next theorem, an algebraic reflection, was proven in [13, 31] where the notation $\mathrm{Prim}H_*(\Omega\mathrm{Conf}(\mathbb{R}^m,k);\mathbb{Z})$ denotes the module of primitive elements (in a torsion free Hopf algebra).

Theorem 12.7. *If $m \geq 3$, the homology of $\Omega\mathrm{Conf}(\mathbb{R}^m,k)$ is torsion free and there is an isomorphism of Lie algebras on the level of the module of primitives:*

$$\mathfrak{L}_k(m-2) \to \ \mathrm{Prim}H_*(\Omega\mathrm{Conf}(\mathbb{R}^m,k);\mathbb{Z}).$$

Furthermore, the universal enveloping algebra of $\mathfrak{L}_k(m-2)$,

$$U[\mathfrak{L}_k(m-2)]$$

is isomorphic to

$$H_*(\Omega\mathrm{Conf}(\mathbb{R}^m,k);\mathbb{Z})$$

as a Hopf algebra.

There is more topology behind this theorem. The loop space $\Omega\mathrm{Conf}(\mathbb{R}^m,k)$ is homotopy equivalent to a product of loop spaces

$$\prod_{1\leq i\leq k-1} \Omega(\vee_i S^{m-1}),$$

thus it is natural to construct representative cycles. It is these cycles, in dimension $m-2$, which represent the elements $B_{i,j}$. In addition, this geometric decomposition has the following algebraic consequence. There are embeddings of Lie algebras

$$g_j : \mathfrak{L}_j(m-2) \to H_*(\Omega\mathrm{Conf}(\mathbb{R}^m,k))$$

such that the natural additive extension

$$\bigoplus_{1\leq j\leq k-1} \mathfrak{L}_j(m-2) \to \mathrm{Prim}H_*(\Omega\mathrm{Conf}(\mathbb{R}^m,k))$$

is an isomorphism, but does not preserve the structure as Lie algebras. The failure to preserve the Lie algebra structure is important in applications.

13. Planetary motion in configuration spaces

The purpose of this section is to give pairs of naively linked spheres in $\mathbb{R}^m$ which correspond to 'planetary motion' and which reflect properties of the loop space of a configuration space. To illustrate, start with $\mathrm{Conf}(\mathbb{R}^m, 3)$, and regard S^{m-1} as the standard locus of points in $\mathbb{R}^m$ of norm equal to one. There is a map

$$\gamma : S^{m-1} \times S^{m-1} \to \mathrm{Conf}(\mathbb{R}^m, 3)$$

defined by

$$\gamma(v, w) = (0, v, v + w/4).$$

Observe that one may regard 0 as the coordinates of a sun S_0 with v the coordinates of a planet P_v in orbit about the sun S_0, and with $v + w/4$ the coordinates of a moon in orbit around the planet P_v.

These maps, as well as analogous maps obtained by permuting the coordinates

$$(0, v, v + w/4),$$

induce relations in the homology of the loop space of the configuration space by considering

$$\Omega(\gamma) : \Omega(S^{m-1} \times S^{m-1}) \to \Omega(\mathrm{Conf}(\mathbb{R}^m, 3)).$$

These relations give precisely the 'horizontal 4T relations' or 'infinitesimal braid relations' as stated in Definition 12.6.

14. Homological calculations for $\mathbb{R}^m$

The purpose of this section is to give the computation stated in Theorem 12.7. Recall that $\mathrm{Conf}(\mathbb{R}^m, k)$ is $(m-2)$-connected and that the algebra

$$H^*(\mathrm{Conf}(\mathbb{R}^m, k); \mathbb{Z})$$

is generated by classes $A_{i,j}$, $k \geq i > j \geq 1$, of degree $m-1$ [15, 20]. Thus the homology suspension induces an isomorphism

$$\sigma_* : H_{m-2}(\Omega\mathrm{Conf}(\mathbb{R}^m, k); \mathbb{Z}) \to H_{m-1}(\mathrm{Conf}(\mathbb{R}^m, k); \mathbb{Z})$$

for $m > 2$.

Definition 14.1. Define the homology class $B_{i,j}$ to be the unique class specified by

$$\sigma_*(B_{i,j}) = A_{i,j*},$$

the dual basis element dual to $A_{i,j}$ with $k \geq i > j \geq 1$.

Alternatively, the classes $B_{i,j}$ are represented by maps of spheres, as described in the next section. Furthermore, commutation relations for the $B_{i,j}$ are obtained by (1) exhibiting maps

$$\gamma : S^{m-1} \times S^{m-1} \to \mathrm{Conf}(\mathbb{R}^m, 3),$$

(2) looping the map γ, and (3) using the commutativity of the two fundamental cycles in

$$H_*((\Omega S^{m-1})^2; \mathbb{Z}).$$

The maps $\gamma : S^{m-1} \times S^{m-1} \to \mathrm{Conf}(\mathbb{R}^m, 3)$ correspond to naive planetary motion as given in the previous section. These relations are analyzed in the next section.

15. On spheres embedded in the configuration space

The purpose of this section is to give pairs of spheres in $\mathbb{R}^m$ which will reflect homological properties of the loop space of a configuration space. Fix integers s, t, and ℓ such that $k \geq s > t \geq 1$, $k \geq \ell \geq 1$, with $\ell \notin \{s, t\}$.

Definition 15.1. Define a map

$$\gamma(s,t,\ell) : S^{m-1} \times S^{m-1} \to \mathrm{Conf}(\mathbb{R}^m, k)$$

by the formula

$$\gamma(s,t,\ell)(u,v) = (x_1, \ldots, x_k)$$

with $\|u\| = \|v\| = 1$ such that

(1) $z_i = (4i, 0, 0, \ldots, 0)$ for $k \geq i \geq 1$,
(2) $x_i = z_i$ if $i \neq \{s, t\}$, and
(3) $x_s = z_\ell + 2v$, $x_t = z_\ell + u$.

Notice that $(x_1, \ldots, x_k)$ is indeed in $\mathrm{Conf}(\mathbb{R}^m, k)$.

Next, recall that the class $A_{i,j}$ is defined by the equation

$$A_{i,j} = \pi^*_{i,j}(\iota)$$

where $\pi_{i,j} : \mathrm{Conf}(\mathbb{R}^m, k) \to \mathrm{Conf}(\mathbb{R}^m, 2)$ denotes projection on the (i, j) coordinates and ι is a fixed fundamental cycle for $H^{m-1}(S^{m-1})$ [15, 20]. Furthermore,

$$A_{i,j} = (-1)^m A_{j,i}$$

by inspection. The notation of Definition 15.1 is used in the proof of the next lemma.

Lemma 15.2. (1) *If* $\{i, j\} \cap \{s, t, \ell\}$ *has cardinality* 0 *or* 1*, then*

$$\gamma(s,t,\ell)^*(A_{i,j}) = 0.$$

(2) *If* $\{i,j\} \cap \{s,t,\ell\}$ *has cardinality* 2*, then*
  (a) *for* $\ell < t < s$,

$$\gamma(s,t,\ell)^*(A_{i,j}) = \begin{cases} \iota \otimes 1 & \text{if } j = \ell \text{ and } i = t, \\ 1 \otimes \iota & \text{if } j = \ell \text{ and } i = s, \\ 1 \otimes \iota & \text{if } j = t \text{ and } i = s\,, \end{cases}$$

  (b) *for* $t < \ell < s$,

$$\gamma(s,t,\ell)^*(A_{i,j}) = \begin{cases} (-1)^m \iota \otimes 1 & \text{if } j = t \text{ and } i = \ell\,, \\ 1 \otimes \iota & \text{if } j = t \text{ and } i = s, \\ 1 \otimes \iota & \text{if } j = \ell \text{ and } i = s\,, \end{cases}$$

  (c) *for* $t < s < \ell$,

$$\gamma(s,t,\ell)^*(A_{i,j}) = \begin{cases} (-1)^m \iota \otimes 1 & \text{if } j = t \text{ and } i = \ell\,, \\ 1 \otimes \iota & \text{if } j = t \text{ and } i = s\,, \\ (-1)^m 1 \otimes \iota & \text{if } j = s \text{ and } i = \ell\,. \end{cases}$$

Proof. Notice that if $\{i,j\} \cap \{s,t,\ell\} = \emptyset$, then $\pi_{i,j} \circ \gamma(s,t,\ell)$ is constant. If $\{i,j\} \cap \{s,t,\ell\}$ has cardinality 1, then all but one of the x_r coordinates are constant. Furthermore,

$$\pi_{i,j} \circ \gamma(s,t,\ell)(u,v) = (x_i, x_j)$$

where either

(1) $x_i = z_i$ and $x_j = z_\ell + 2v$ with $i \neq \ell$,
(2) $x_i = z_i$ and $x_j = z_\ell + u$ with $i \neq \ell$, or
(3) $x_i = z_\ell + 2v$ or $z_\ell + u$ with $j \neq \ell$.

In either of these three cases $\pi_{i,j} \circ \gamma(s,t,\ell)$ is null-homotopic. Thus, part (1) follows.

Part (2) is obtained by considering three cases. One case is listed as the others are similar. Thus assume that $t < s < \ell$.

Case 1: $j = t$ and $i = \ell$.

In this case $\pi_{i,j} \circ \gamma(s,t,\ell)(z,w) = (y_\ell + z, y_\ell)$. This last map is evidently homotopic to the map which sends (z,w) to $(y_\ell, y_\ell - z)$ and thus $A_{i,j}$ pulls back to $(-1)^m \iota \otimes 1$.

Case 2: $j = t$ and $i = s$.

In this case $\pi_{i,j} \circ \gamma(s,t,\ell)(z,w) = (y_\ell + z, y_\ell + 2w)$. By shrinking z to 0, this last map is homotopic to the map which sends (z,w) to $(y_\ell, y_\ell + 2w)$. Hence $A_{i,j}$ pulls back to $1 \otimes \iota$.

Case 3: $j = s$ and $i = \ell$.

In this case $\pi_{i,j} \circ \gamma(s,t,\ell)(z,\omega) = (y_\ell + 2w, y_\ell)$ which is homotopic to the map that sends (z,w) to $(y_\ell, y_\ell - w)$. Hence $A_{i,j}$ pulls back to $(-1)^m 1 \otimes \iota$. □

Let γ denote a fixed choice of fundamental cycle for $H_{m-1}(S^{m-1})$. Consider the natural basis for $H_{m-1}(\mathrm{Conf}(\mathbb{R}^m,k))$ obtained by taking the linear duals $A_{i,j*}$ to the elements $A_{i,j}$ in Lemma 15.2.

Lemma 15.3. (1) *If $\ell < t < s$, then*

$$\gamma(s,t,\ell)_*(i \otimes 1) = A_{t,\ell*}\,,\ \text{and}$$
$$\gamma(s,t,\ell)_*(1 \otimes i) = A_{s,\ell*} + A_{s,t*}\,.$$

(2) *If $t < \ell < s$, then*

$$\gamma(s,t,\ell)_*(i \otimes 1) = (-1)^m A_{\ell,t*}\,,\ \text{and}$$
$$\gamma(s,t,\ell)_*(1 \otimes i) = A_{s,\ell*} + A_{s,\ell*}\,.$$

(3) *If $t < \ell < s$, then*

$$\gamma(s,t,\ell)_*(i \otimes 1) = (-1)^m A_{\ell,t*},\ \text{and}$$
$$\gamma(s,t,\ell)_*(1 \otimes i) = A_{s,t*} + (-1)^m A_{\ell,s*}\,.$$

Next consider the two 'axial' inclusions $S^{m-1} \to S^{m-1} \times S^{m-1}$. Passage to adjoints gives two classes x_i in $H_{m-2}(\Omega(S^{m-1})^2;\mathbb{Z})$ such that $H_*(\Omega(S^{m-1})^2;\mathbb{Z})$ is isomorphic to the tensor product of tensor algebras $T[x_1] \otimes T[x_2]$, as an algebra, where

$$[x_1,x_2] = x_1x_2 - (-1)^m x_2x_1 = 0.$$

Thus $\Omega\gamma(s,t,\ell)_* = [x_1,x_2] = 0$ by naturality. The infinitesimal braid relations arise by applying this formula to Lemma 15.3.

Corollary 15.4. *If $m \geq 3$, then the following relations hold in*

$$H_*(\Omega\mathrm{Conf}(\mathbb{R}^m,k);\mathbb{Z}):$$

(1) $[B_{i,j}, B_{i,t} + (-1)^m B_{t,j}] = 0$ *if* $1 \leq j < t < i \leq k$.
(2) $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ *if* $1 \leq j < t < i \leq k$.

Proof. By Lemma 15.3, and the definition that $B_{i,j}$ is the unique element such that $\sigma_* B_{i,j} = A_{i,j*}$, the following holds:

(i) If $\ell < t < s$, then

$$\Omega\gamma(s,t,\ell)_*(x_1 \otimes 1) = B_{t,\ell}, \text{ and}$$
$$\Omega\gamma(s,t,\ell)_*(1 \otimes x_2) = B_{s,\ell} + B_{s,t} \ .$$

(ii) If $t < \ell < s$, then

$$\Omega\gamma(s,t,\ell)_*(x_1 \otimes 1) = (-1)^m B_{\ell,t}, \text{ and}$$
$$\Omega\gamma(s,t,\ell)_*(1 \otimes x_2) = B_{s,t} + B_{s,\ell} \ .$$

(iii) If $t < s < \ell$, then

$$\Omega\gamma(s,t,\ell)_*(x_1 \otimes 1) = (-1)^m B_{\ell,t}, \text{ and}$$
$$\Omega\gamma(s,t,\ell)_*(1 \otimes x_2) = B_{s,t} + (-1)^m B_{\ell,s} \ .$$

Thus part (i) gives $[B_{t,\ell}, B_{s,\ell} + B_{s,t}] = 0$ for $\ell < t < s$. This is a restatement of equation (2). In addition, part (iii) gives

$$[B_{\ell,t}, (-1)^m B_{s,t} + B_{\ell,s}] = 0 \text{ if } t < s < \ell \ .$$

This is a restatement of equation (1). The corollary follows. □

Proposition 15.5. *If $m \geq 3$ and $\{i,j\} \cap \{s,t\} = \emptyset$, then*

$$[B_{i,j}, B_{s,t}] = 0$$

in $H_*(\Omega\mathrm{Conf}(\mathbb{R}^m, k);\mathbb{Z})$.

Proof. If $\{i,j\} \cap \{s,t\} = \emptyset$, then define

$$\theta : S^{m-1} \times S^{m-1} \to \mathrm{Conf}(\mathbb{R}^m, k)$$

by the formula

$$\theta(u,v) = (x_1, \ldots, x_k)$$

where the x_i are defined by the formula (1) $x_v = z_v$ for $v \notin \{i,j,s,t\}$, (2) $x_j = z_j$, $x_i = z_j + u$, (3) $x_s = z_s$, and (4) $x_t = z_s + v$. Then

$$\theta_*(\iota \otimes 1) = A_{i,j_*} \text{ and}$$
$$\theta_*(1 \otimes \iota) = A_{s,t*} \ .$$

Furthermore, $(\Omega\theta)_*(x_1 \otimes 1) = B_{i,j}$ and $(\Omega\theta)_*(1 \otimes x_2) = B_{s,t}$. Since $[x_1, x_2] = 0$, it follows that $[B_{i,j}, B_{s,t}] = 0$ by naturality. □

One way in which these relations arise is through the structure of the projection maps $p_i : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1)$, and 'doubling maps' $\sigma_i : \mathrm{Conf}(\mathbb{R}^m, k) \to \mathrm{Conf}(\mathbb{R}^m, k+1)$. These maps correspond to the structure of a simplicial object as developed next. The role of the Lie algebras which satisfy the 'horizontal 4T relations' will be described below in this framework.

16. Simplicial objects, and Δ-objects

This purpose of this section, as well as Sections 17 through 21, is to give descriptions of naive properties of configurations spaces and their relationship to simplicial groups. We will consider specific concrete cases which on the one hand give classical structures for describing the homotopy groups of the 2-sphere. The natural connection to the homology of the pointed loop space of the configuration space, given in Theorem 12.7, is also described. One of the main features here is the interplay between the structure of the braid groups, the homology of the loop space of certain configuration spaces, and the Bousfield-Kan spectral sequence associated to the homotopy groups of a simplicial group.

The goal of subsequent sections is to describe the connections between the fundamental groups of configuration spaces, homotopy groups of spheres, Vassiliev invariants and T. Kohno's Lie algebra arising from the infinitesimal braid relations.

Before embarking in this direction, an overview will be given to clarify the connections here. First, consider the projection maps out of configuration spaces

$$p_i : \mathrm{Conf}(M, k) \to \mathrm{Conf}(M, k-1)$$

which are defined in this section by deleting the i-th coordinate. These projection maps satisfy the compatibility condition

$$p_i \circ p_j = p_{j-1} \circ p_i$$

in case $i < j$. With mild conditions concerning base-points for the space M, the analogous formulas are satisfied on the level of fundamental groups with

$$p_{i*} \circ p_{j*} = p_{j-1*} \circ p_{i*}$$

in case $i < j$.

This compatibility property is precisely the condition for a collection of groups to form a Δ-group, as first developed in [65] and defined below. In addition, in the case $M = \mathbb{R}^m$, the projection maps, together with certain additional maps, give the collection of fundamental groups of the configuration spaces the structure of a simplicial group, also defined below.

The basic combinatorial invariant framework is that of a Δ-set and simplicial set, which model the combinatorics of a simplicial complex. Basic properties of simplicial sets appear in the excellent references [61, 22, 4, 57, 83].

Definition 16.1. A Δ-*set* is a collection of sets

$$K_\bullet = \{K_0, K_1, \ldots\}$$

with functions, *face operations,*

$$d_i : K_t \to K_{t-1} \text{ for } \; 0 \leq i \leq t$$

which satisfy the identities

$$d_i d_j = d_{j-1} d_i \text{ if } i < j.$$

A Δ-*group* is a Δ-set for which all $d_i : K_t \to K_{t-1}$ are group homomorphisms.

A natural example of a Δ-group arises from the pure braid groups $P_n(S) = \pi_1(\mathrm{Conf}(S, n))$ for a path-connected surface S, as follows, see [5].

Example 16.2. There are $(n + 1)$ homomorphisms

$$d_i \colon P_{n+1}(S) \to P_n(S),$$

with $0 \leq i \leq n$, where d_i is obtained by deleting the $(i + 1)$-st strand of a braid in $P_{n+1}(S)$. The homomorphisms d_i are induced on the level of fundamental groups of configuration spaces by the projection maps

$$p_{i+1} : \mathrm{Conf}(S, n + 1) \to \mathrm{Conf}(S, n)$$

given be deleting the $(i + 1)$-st coordinate. These satisfy the identities $p_i \circ p_j = p_{j-1} \circ p_i$ for $i < j$ and induce the structure of Δ-group on the collection $\pi_1(\mathrm{Conf}(S, n))$, $n \geq 1$, for a path-connected surface S, as recorded in the next Definition, see [5].

Definition 16.3. Let S be a connected surface. Define

$$\Delta_\bullet(S)$$

by

$$\Delta_n(S) = P_{n+1}(S),$$

the $(n + 1)$-st pure braid group for the surface S. By the previous example (together with a check of base-points), $\Delta_\bullet(S)$ is a Δ-group.

In case $S = \mathbb{CP}^1 = S^2$, the associated Δ-group gives basic information about the homotopy groups of the 2-sphere [5]. In case S_g is a closed oriented surface, the Δ-group $\Delta_\bullet(S_g)$ does not admit the structure of a simplicial group as given in the next definition. (See Example 4.4.)

Definition 16.4. A *simplicial set* is

(1) a Δ-set
$$K_\bullet = \{K_0, K_1, \ldots\}$$
together with

(2) functions, *degeneracy operations,*
$$s_j : K_t \to K_{t+1} \text{ for } \ 0 \le j \le t$$
which satisfy the *simplicial identities*

$$d_i d_j = d_{j-1} d_i \text{ if } i < j, \qquad s_i s_j = s_{j+1} s_i \text{ if } i \le j, \text{ and}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ \text{identity} & \text{if } i = j \text{ or } i = j+1, \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

A *simplicial-group*
$$G_\bullet = \{G_0, G_1, \ldots\}$$
is a simplicial-set for which all of the G_i are groups with faces and degeneracies given by group homomorphisms.

Example 16.5. Two examples of simplicial sets are given next.

(1) The simplicial 1-simplex $\Delta[1]$ has two 0-simplices $< 0 >$ and $< 1 >$. The n-simplices of $\Delta[1]$ are sequences $< 0^i, 1^{n+1-i} >$ for $0 \le i \le n+1$. The non-degenerate simplices are $< 0 >$, $< 1 >$, and $< 0, 1 >$.

(2) The simplicial circle S^1 is a quotient of the simplicial 1-simplex $\Delta[1]$ obtained by identifying $< 0 >$ and $< 1 >$. There are exactly two equivalence classes of non-degenerate simplices given by $< 0 >$, and $< 0, 1 >$. Furthermore, the simplicial circle S^1 is given in degree k by
   (a) a single point $< 0 >$ in case $k = 0$, and
   (b) $n+1$ points $< 0^i, 1^{n+1-i} >$ for $0 \le i < n+1$ in case $k = n$ for which $< 0^{n+1} >$ and $< 1^{n+1} >$ are identified.

In what follows below, it is useful to label these simplices by

$$y_{n+1-i} =< 0^i, 1^{n+1-i} >$$

for $0 < i \leq n+1$ with $y_0 = s_0^n(< 0 >)$.

Classical, elegant constructions for the standard simplicial n-simplex $\Delta[n]$ as well as the n-sphere are given in [4, 22, 57, 83].

Homotopy groups are defined for simplicial sets which satisfy an additional condition known as the (Kan) extension condition.

Example 16.6. A simplicial group $G_\bullet = \{G_0, G_1, \ldots\}$ always satisfies the extension condition, as shown in [61].

An example of a simplicial group obtained naturally from Artin's pure braid groups is described next.

Example 16.7. Consider Δ-groups with $\Delta_n(S) = P_{n+1}(S)$ as given in Example 16.2 for surfaces S. Specialize to the surface

$$S = \mathbb{R}^2.$$

In this case, there are also $n+1$ homomorphisms

$$s_i\colon P_{n+1} \to P_{n+2},$$

with $0 \leq i \leq n$, where s_i is obtained by 'doubling' the $(i+1)$-st strand. The homomorphisms s_i are induced on the level of fundamental groups by the maps for configuration spaces

$$\mathbb{S}_i : \mathrm{Conf}(\mathbb{R}^2, n+1) \to \mathrm{Conf}(\mathbb{R}^2, n+2)$$

defined by the formula

$$\mathbb{S}_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i+1}, \lambda(x_{i+1}), x_{i+2}, \ldots, x_{n+1})$$

where $\lambda(x_{i+1}) = x_{i+1} + (\epsilon, 0)$ for $(\epsilon, 0)$ a point in $\mathbb{R}^2$ with

$$\epsilon = (1/2) \cdot \min_{t \neq i+1} ||x_{i+1} - x_t||.$$

The homomorphisms d_i and s_j satisfy the simplicial identities [21, 5].

Thus the pure braid groups, in the case $S = \mathbb{R}^2$, provide an example of a simplicial group, denoted

$$\mathrm{AP}_\bullet,$$

with

$$\mathrm{AP}_n = P_{n+1}$$

for $n = 0, 1, 2, 3, \ldots$.

Consider a pointed topological space $(X, *)$. The pointed loop space of X, $\Omega(X)$, has a natural multiplication coming from 'loop sum' which is not associative, but homotopy associative. Milnor proved that the loop space of a connected simplicial complex is homotopy equivalent to a topological group [60]. James [47] proved that the loop space of the suspension of a connected CW-complex is naturally homotopy equivalent to a free monoid as explained in [43], page 282. Milnor [58] realized that the James construction could be translated directly into the the language of simplicial groups as described next.

Definition 16.8. Let $K_\bullet$ denote a pointed simplicial set (with base-point $* \in K_0$ and $s_0^n(*) \in K_n$). Define *Milnor's free simplicial group* $F[K]_\bullet$ by

$$F[K]_n = F[K_n]/s_0^n(*) = 1$$

for which

$$F[K]$$

denotes the free group generated by the set K.

Then $F[K]_\bullet$ is a simplicial group with face and degeneracy operations given by the natural multiplicative extension of those for $K_\bullet$. In addition, the face and degeneracy operations applied to a generator give either another generator or the identity element.

Example 16.9. An example of $F[K]_\bullet$ is given by $K_\bullet = S^1_\bullet$, the simplicial circle. Notice that $F[S^1]_n = F[y_1, \dots, y_n]$, the free group on n generators, by Example 16.5.

Milnor defined the geometric realization of a simplicial set $K_\bullet = \{K_0, K_1, \dots\}$ for which the underlying topology of $K_\bullet$ is discrete [59]. Recall the inclusion of the i-th face $\delta_i\colon \Delta[n-1] \to \Delta[n]$ together with the projection maps to the j-th face $\sigma_j\colon \Delta[n+1] \to \Delta[n]$ [4, 22, 57].

Definition 16.10. The *geometric realization of* $K_\bullet$ is

$$|K_\bullet| = (\coprod K_n \times \Delta[n])/\sim$$

where $\sim$ denotes the equivalence relation generated by requiring

(1) if $x \in K_{n+1}$ and $\alpha \in \Delta[n]$, then $(d_i(x), \alpha) \sim (x, \delta_i(\alpha))$, and
(2) if $y \in K_n$ and $\beta \in \Delta[n+1]$, then $(x, \sigma_j(\beta)) \sim (s_j(x), \beta)$.

Theorem 16.11. *If $K_\bullet$ is a reduced simplicial set (that is K_0 is equal to a single point $\{*\}$), then the geometric realization $|F[K]_\bullet|$ is homotopy equivalent to $\Omega\Sigma|K_\bullet|$. Thus the homotopy groups of $F[K]_\bullet$ (as given in* [61] *and Definition 16.6) are isomorphic to the homotopy groups of the space $\Omega\Sigma|K_\bullet|$.*

Example 16.12. Consider the special case of $K_\bullet = S^1_\bullet$. Then the geometric realization $|F[S^1]_\bullet|$ is homotopy equivalent to ΩS^2, and there are isomorphisms

$$\pi_n(F[S^1]_\bullet) \to \pi_n \Omega S^2 \cong \pi_{n+1} S^2.$$

A partial synthesis of this information is given in Sections 19, 20, and 21.

17. Pure braid groups, and Vassiliev invariants

The section addresses a naive construction with the braid groups arising as a 'cabling' construction. This construction is interpreted in later sections in terms of the structure of braid groups, Vassiliev invariants of pure braids as developed by Toshitake Kohno [49, 51], associated Lie algebras and the homotopy groups of the 2-sphere [21, 5, 81].

Recall from Definition 8.1, B_k denotes Artin's k-stranded braid group while P_k denotes the pure k-stranded braid group. Furthermore, the group B_k is the fundamental group of the orbit space

$$\mathrm{Conf}(\mathbb{R}^2, k)/\Sigma_k,$$

and the pure braid group $P_k(S)$ is the fundamental group

$$\pi_1(\mathrm{Conf}(S, k)).$$

The pure braid groups P_k and $P_k(S^2)$ are closely related to the loop space of the 2-sphere as elucidated below in Section 16. Similar properties are satisfied for any sphere, as described in Section 21. We will now start to address this point.

Figure 2. The braid x_i in P_{N+1}.

Consider the free group on N letters $F_N = F[y_1, \dots, y_N]$ and elements x_i in P_{N+1} for $1 \leq i \leq N$, with x_i given by the naive 'cabling' pictured in Figure 2 above. The braid x_1 with $N = 1 = i$ in Figure 2.1 is Artin's generator $A_{1,2}$ of P_2. The braids x_i yield homomorphisms from F_N to P_{N+1},

$$\Theta_N \colon F[y_1, \dots, y_N] \to P_{N+1}$$

defined on generators y_i in F_N by the formula

$$\Theta_N(y_i) = x_i.$$

The maps Θ_N are the subject of [21] where it is shown that $\Theta_N \colon F_N \to P_{N+1}$ is faithful for every N. Three natural questions arise: (1) Why would one want to know whether Θ_n is faithful, (2) are there sensible applications and (3) why is Θ_n faithful? The answers to these three questions provide the main content of this expository article.

18. On Θ_n

This section addresses one reason why the map Θ_n is faithful [21]. The method of proof is to appeal to the structure of the Lie algebras obtained from the descending central series for both the source and the target of Θ_n. The structure of these Lie algebras is reviewed below.

Recall the descending central series of a discrete group π, given by

$$\pi = \Gamma^1(\pi) \geq \Gamma^2(\pi) \geq \cdots$$

where $\Gamma^i(\pi)$ is the subgroup of π generated by all commutators

$$[[\cdots[[x_1, x_2]x_3]\cdots]x_t]$$

for $t \geq i$ with $x_i \in \pi$. The group $\Gamma^i(\pi)$ is a normal subgroup of π with the successive sub-quotients

$$\mathrm{gr}_i(\pi) = \Gamma^i(\pi)/\Gamma^{i+1}(\pi),$$

which are abelian groups, having additional structure as follows [55].

Consider the direct sum of all of the $\mathrm{gr}_i(\pi) = \Gamma^i(\pi)/\Gamma^{i+1}(\pi)$ denoted

$$\mathrm{gr}_*(\pi) = \oplus_{i \geq 1}\Gamma^i(\pi)/\Gamma^{i+1}(\pi).$$

The commutator function

$$[-,-] : \pi \times \pi \to \pi,$$

given by

$$[x, y] = xyx^{-1}y^{-1},$$

passes to quotients to give a bilinear map

$$[-,-] : \mathrm{gr}_s(\pi) \otimes_{\mathbb{Z}} \mathrm{gr}_t(\pi) \to \mathrm{gr}_{s+t}(\pi)$$

which satisfies both the antisymmetry law and Jacobi identity for a Lie algebra.

Remark 18.1. The abelian group $\mathrm{gr}_*(\pi)$ is both a graded abelian group and a Lie algebra, but not a graded Lie algebra as the sign conventions do not work properly in this context. This situation can be remedied by doubling all degrees of elements in $\mathrm{gr}_*(\pi)$.

The associated graded Lie algebra obtained from the descending central series for the target yields Vassiliev invariants of pure braids, by work of Kohno [49, 51]. This Lie algebra has been used by both Kohno and Drinfel'd [25] in their work on the KZ equations. The Lie algebra obtained from the descending central series of the free group F_N is a free Lie algebra, by a classical result due to P. Hall [41, 67].

The proof described next yields more information than just the fact that Θ_N is faithful. The method of proof gives a natural connection of Vassiliev invariants of braids to a classical spectral sequence abutting to the homotopy groups of the 2-sphere. Sections 19, 20, and 21 below provide an elucidation of this interconnection.

A discrete group π is said to be residually nilpotent provided

$$\bigcap_{i\geq 1} \Gamma^i(\pi) = \{\text{identity}\}$$

where $\Gamma^i(\pi)$ denotes the i-th stage of the descending central series for π. Examples of residually nilpotent groups are free groups, and P_n.

Lemma 18.2. (1) *Assume that π is a residually nilpotent group. Let*

$$\alpha\colon \pi \to G$$

be a homomorphism of discrete groups such that the morphism of associated graded Lie algebras

$$\mathrm{gr}_*(\alpha) : \mathrm{gr}_*(\pi) \to \mathrm{gr}_*(G)$$

is a monomorphism. Then α is a monomorphism.

(2) *If π is a free group, and $\mathrm{gr}_*(\alpha)$ is a monomorphism, then α is a monomorphism.*

Thus one step in the proof of Theorem 18.3 below is to describe the map

$$\Theta_n\colon F[y_1, y_2, \ldots, y_n] \to P_{n+1}$$

on the level of associated graded Lie algebras

$$\mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1, y_2, \ldots, y_n]) \to \mathrm{gr}_*(P_{n+1}).$$

Recall Artin's generators $A_{i,j}$ for P_{n+1} together with the projections of the $A_{i,j}$ to $\mathrm{gr}_*(P_{n+1})$, labeled $B_{i,j}$ [21]. The next theorem was proven in [21] by a direct computation.

Theorem 18.3. *The induced morphism of Lie algebras*

$$\mathrm{gr}_*(\Theta_n) \colon \mathrm{gr}_*(F[y_1, y_2, \ldots, y_n]) \to \mathrm{gr}_*(P_{n+1})$$

satisfies the formula

$$\mathrm{gr}_*(\Theta_n)(y_q) = \Sigma_{1 \le i \le n-q+1 < j \le n+1} B_{i,j}.$$

Examples of this theorem are listed next.

Example 18.4. (1) If $q = 1$, then

$$\mathrm{gr}_*(\Theta_n)(y_1) = B_{1,n+1} + B_{2,n+1} + \cdots + B_{n,n+1}.$$

Thus if $q = 1$, and $n = 3$,

$$\mathrm{gr}_*(\Theta_3)(y_1) = B_{1,4} + B_{2,4} + B_{3,4}.$$

(2) If $q = 2$, then

$$\mathrm{gr}_*(\Theta_n)(y_2) = (B_{1,n+1} + B_{2,n+1} + \cdots + B_{n-1,n+1}) + (B_{1,n} + B_{2,n} + \cdots + B_{n-1,n}).$$

Thus if $q = 2$, and $n = 3$,

$$\mathrm{gr}_*(\Theta_3)(y_2) = (B_{1,4} + B_{2,4}) + (B_{1,3} + B_{2,3}).$$

(3) In general,

$$\mathrm{gr}_*(\Theta_n)(y_q) = V_{n-q+2} + V_{n-q+3} + \cdots + V_{n+1}$$

where

$$V_r = B_{1,r} + B_{2,r} + \cdots + B_{r-1,r}.$$

Thus if $q = 3$, and $n = 3$,

$$\mathrm{gr}_*(\Theta_3)(y_3) = B_{1,2} + B_{1,3} + B_{1,4}.$$

To determine the map of Lie algebras with a more global view, the structure of the Lie algebra $\mathrm{gr}_*(P_n)$ is useful, and is given as follows. Let $L[S]$ denote the free Lie algebra over $\mathbb{Z}$ generated by a set S. The next theorem was given in work of Kohno [49, 51] and Falk-Randell [34].

Theorem 18.5. *The Lie algebra* $\mathrm{gr}_*(P_n)$ *is the quotient of the free Lie algebra generated by* $B_{i,j}$ *for* $1 \leq i < j \leq n$ *given by*

$$\mathrm{gr}_*(P_n) = L[B_{i,j} | \ 1 \leq i < j \leq n]/I$$

where I *denotes the* 2*-sided (Lie) ideal generated by the infinitesimal braid relations listed next:*

(1) $[B_{i,j}, B_{s,t}] = 0$, *if* $\{i, j\} \cap \{s, t\} = \emptyset$.
(2) $[B_{i,j}, B_{i,s} + B_{s,j}] = 0$.
(3) $[B_{i,j}, B_{i,t} + B_{j,t}] = 0$.

Remark 18.6. The Lie algebra in Theorem 18.5, apart from a natural degree shift, is precisely the one arising in Theorem 12.7 and gives the homology of the loop space of the configuration space. We will see below, in Section 20, that this same Lie algebra gives the E^1-term of the Bousfield-Kan spectral sequence abutting to certain homotopy groups.

A computation with these maps gives the following result of [21]. Further connections regarding this result are elucidated in Sections 19 and 21.

Theorem 18.7. *The maps* $\Theta_n\colon F[y_1, y_2, \dots, y_n] \to P_{n+1}$ *on the level of associated graded Lie algebras*

$$\mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1, y_2, \dots, y_n]) \to \mathrm{gr}_*(P_{n+1})$$

are monomorphisms. Thus the maps Θ_n *are monomorphisms.*

Remark 18.8. Two remarks concerning Θ_n are given next.

(1) That Θ_n is a monomorphism identifies $F[y_1, y_2, \dots, y_n]$ as a free subgroup of rank n in P_{n+1}. However, there are other, natural free groups of rank n in P_{n+1}. These arise from the fibrations of Fadell and Neuwirth given by the projection maps

$$p_{i+1} : \mathrm{Conf}(\mathbb{R}^2, n+1) \to \mathrm{Conf}(\mathbb{R}^2, n)$$

which delete the $(i+1)$st coordinate and have fibre of the homotopy type of an n-fold wedge of circles $\vee_n S^1$ [32].

Recall that $d_i : P_{n+1} \to P_n$ denotes the map induced by p_{i+1} on the level of fundamental groups. The kernel of d_i is a free group of rank n.

The image of Θ_n has a contrasting feature: Any composite of the natural projection maps, $d_I : P_{n+1} \to P_2$, precomposed with Θ_n,

$$F[y_1, y_2, \dots, y_n] \xrightarrow{\Theta_n} P_{n+1} \xrightarrow{d_I} P_2,$$

is a surjection.

(2) The combinatorial behavior of the map $\mathrm{gr}_*(\Theta_n)$ is intricate even though the definition is elementary as well as natural. For example, various powers of 2 arise in the computation of the map

$$\mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1, y_2, \ldots, y_n]) \to \mathrm{gr}_*(P_{n+1})$$

for $n > 2$. One example is listed next.

Example 18.9. $\Theta_3([[[y_1, y_2]y_3]y_2]) = -[[[\gamma_1, \gamma_2]\gamma_3]\gamma_2] + 2[[[\gamma_1, \gamma_3]\gamma_2]\gamma_2] + \delta$ where δ is independent of the other terms with $\gamma_1 = B_{1,4} + B_{2,4} + B_{3,4}$, $\gamma_2 = B_{3,4}$ and $\gamma_3 = B_{2,4} + B_{3,4}$. At first glance, these elements may appear to be 'random'. However, this formula represents a systematic behavior which arises naturally from kernels of certain morphisms of Lie algebras.

The crucial feature which makes the computations effective is the 'infinitesimal braid relations'. In addition, the behavior of the map $\mathrm{gr}_*(\Theta_n)$ is more regular after restricting to certain sub-Lie algebras arising in the third stage of the descending central series [21]. Finally, the maps Θ_n also induce monomorphisms of restricted Lie algebras on passage to the Lie algebras obtained from the mod-p descending central series [21].

19. Pure braid groups of surfaces as simplicial groups and Δ-groups

The homomorphism $\Theta_n\colon F[y_1, y_2, \ldots, y_n] \to P_{n+1}$ which arises from the cabling operation described in Figure 2 satisfies the following properties.

(1) The homomorphisms $\Theta_n\colon F[y_1, y_2, \ldots, y_n] \to P_{n+1}$ give a morphism of simplicial groups

$$\Theta\colon F[S^1]_\bullet \to \mathrm{AP}_\bullet$$

for which the homomorphism Θ_n is the restriction of Θ to $F[S^1]_n$.

(2) By Theorem 18.7, the homomorphisms $\Theta_n\colon F[y_1, y_2, \ldots, y_n] \to P_{n+1}$ are monomorphisms and so the morphism $\Theta\colon F[S^1] \to \mathrm{AP}_\bullet$ is a monomorphism of simplicial groups.

(3) There is exactly one morphism of simplicial groups Θ with the property that $\Theta_1(y_1) = A_{1,2}$.

Thus, the picture given in Figure 2 is a description for generators of $F[S^1]_n$ in the simplicial group $F[S^1]_\bullet$. These features are summarized next.

Theorem 19.1. *The homomorphisms*

$$\Theta_n\colon F[y_1, y_2, \ldots, y_n] \to P_{n+1}$$

('pictured' in Figure 2) give the unique morphism of simplicial groups

$$\Theta\colon F[S^1]_\bullet \to \mathrm{AP}_\bullet$$

with $\Theta_1(y_1) = A_{1,2}$. The map Θ is an embedding. Hence the n-th homotopy group of $F[S^1]$, isomorphic to $\pi_{n+1}(S^2)$, is a natural sub-quotient of AP_n. Furthermore, the smallest sub-simplicial group of $\mathrm{AP}_\bullet$ which contains the element $\Theta_1(y_1) = A_{1,2}$ is isomorphic to $F[S^1]_\bullet$.

On the other-hand, the homotopy sets for the Δ-group $\Delta_\bullet(S^2)$ also give the homotopy groups of the 2-sphere, via a different occurrence of $F[S^1]_\bullet$. The homeomorphism of spaces

$$\mathrm{Conf}(S^2, k) \to PGL(2,\mathbb{C}) \times \mathrm{Conf}(S^2 - Q_3, k-3),$$

for $k \geq 3$ and where Q_3 denotes a set of there distinct points in S^2, is basic for the next theorem [5].

Theorem 19.2. *If $S = S^2$ and $n \geq 4$, then there are isomorphisms*

$$\pi_n(\Delta_\bullet(S^2)) \to \pi_n(S^2).$$

The descriptions of homotopy groups implied by these theorems admit interpretations in terms of classical, well-studied features of the braid groups as given in the next section. An extension to all spheres is given in [21], as pointed out in Section 21.

20. Brunnian braids, 'almost Brunnian' braids, and homotopy groups

The homotopy groups of a simplicial group, or the homotopy sets of a Δ-group, admit a combinatorial description, as discussed in Lemma 20.3 below. These homotopy sets are the set of left cosets Z_n/B_n where Z_n is the group of n-cycles and B_n is the group of n-boundaries for the Δ-group.

Recall Example 16.2 in which the Δ-group $\Delta_\bullet(S)$ is specified by $\Delta_n(S) = P_{n+1}(S)$, the $(n+1)$-stranded pure braid group for a connected surface S. The main point of this section is that the n-cycles Z_n are given by the 'Brunnian braids' while the n-boundaries B_n are given by the 'almost Brunnian braids', subgroups considered next which are also important in other applications [56].

Definition 20.1. Consider the n-stranded pure braid group for any (connected) surface S, the fundamental group of $\mathrm{Conf}(S, n)$. The group of Brunnian braids $\mathrm{Brun}_n(S)$ is the subgroup of $P_n(S)$ given by those braids which become trivial after deleting any single strand. That is,

$$\mathrm{Brun}_n(S) = \bigcap_{0 \leq i \leq n-1} \ker(d_i\colon P_n(S) \to P_{n-1}(S))$$

for which

$$d_i = (p_{i+1})_* : P_n(S) \to P_{n-1}(S).$$

The 'almost Brunnian' $(n+1)$-stranded braid group is

$$\mathrm{QBrun}_{n+1}(S) = \bigcap_{1\leq i\leq n} \ker(d_i\colon P_{n+1}(S) \to P_n(S)).$$

The subgroup $\mathrm{QBrun}_{n+1}(S)$ of $P_{n+1}(S)$ consists of those braids which are trivial after deleting any one of the strands $2, 3, \ldots, n+1$, but not necessarily the first.

Example 20.2. Consider the simplicial group $\mathrm{AP}_\bullet$ with

$$\mathrm{AP}_n = P_{n+1}$$

for $n = 0, 1, 2, 3, \ldots$ as given in Example 16.7.

In this case, notice that that the map $d_0\colon \mathrm{QBrun}_{k+2} \to \mathrm{Brun}_{k+1}$ is a split epimorphism. Thus the homotopy groups of the simplicial group $\mathrm{AP}_\bullet$ are all trivial.

An inspection of definitions gives the next lemma.

Lemma 20.3. *Let S denote a connected surface with associated Δ-group $\Delta_\bullet(S)$ (as given in Example 16.2). Then the following hold.*

1. *The group $\mathrm{Brun}_{n+1}(S)$ is equal to the group of n-cycles $Z_n(S)$.*
2. *The group of boundaries $B_n(S)$ is $d_0(\mathrm{QBrun}_{n+2}(S))$.*
3. *There is an isomorphism*

$$\pi_k(\mathrm{AP}_\bullet) \to \mathrm{Brun}_{k+1}/d_0(\mathrm{QBrun}_{k+2}).$$

   Furthermore, $\pi_k(\mathrm{AP}_\bullet)$ is the trivial group.

4. *There is an isomorphism of left cosets which is natural for pointed embeddings of connected surfaces S*

$$\pi_k(\Delta_*(S)) \to \mathrm{Brun}_{k+1}(S)/d_0(\mathrm{QBrun}_{k+2}(S)).$$

Properties of the Δ-group for the 2-sphere $S = \mathbb{CP}^1 = S^2$ is the main subject of [5] where the next result is proven.

Theorem 20.4. *If $S = S^2$ and $k \geq 4$, then*

$$\pi_k(\Delta_\bullet(S^2)) = \mathrm{Brun}_{k+1}(S^2)/d_0(\mathrm{QBrun}_{k+2}(S^2))$$

is a group which is isomorphic to the classical homotopy group $\pi_k(S^2)$.

Furthermore, if $k \geq 4$, there is an exact sequence of groups

$$1 \to \mathrm{Brun}_{k+2}(S^2) \to \mathrm{Brun}_{k+1}(\mathbb{R}^2) \to \mathrm{Brun}_{k+1}(S^2) \to \pi_k(S^2) \to 1.$$

Remark 20.5. Recently, the authors have proven (unpublished) that the Brunnian braid group $\mathrm{Brun}_4(S^2)$ is isomorphic to the principal congruence subgroup of level 4 in $PSL(2, \mathbb{Z})$ [5]. (This fact is checked in the appendix here.)

This identification may admit an extension by considering the Brunnian braid groups $\mathrm{Brun}_{2g}(S^2)$ as natural subgroups of mapping class groups for genus g surfaces. The subgroups $\mathrm{Brun}_{2g}(S^2)$ may embed naturally in $Sp(2g, \mathbb{Z})$ via classical surface topology using branched covers of the 2-sphere (work in progress). It seems reasonable to conjecture that this is correct.

The next lemma follows by a direct check of the long exact homotopy sequence obtained from the Fadell-Neuwirth fibrations for configuration spaces [32, 31].

Lemma 20.6. *If S is a surface not homeomorphic to either S^2 or $\mathbb{RP}^2$, and $k \geq 3$, then* $\mathrm{Brun}_k(S)$ *and* $\mathrm{QBrun}_k(S)$ *are free groups. If S is any surface, and $k \geq 4$, then* $\mathrm{Brun}_k(S)$ *and* $\mathrm{QBrun}_k(S)$ *are free groups.*

Example 20.7. One classical example of a Brunnian braid group is $\mathrm{Brun}_4(S^2)$ which is isomorphic to the principle congruence subgroup of level 4 in $PSL(2, \mathbb{Z})$, as given in Section 21 below.

Lemma 20.8. *If $k \geq 3$, then $\Theta_k(F_k) \cap \mathrm{Brun}_{k+1}$ as well as $\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2})$ are countably infinitely generated free groups.*

The standard Hall collection process or natural variations can be used to give inductive recipes rather than closed forms for generators. T. Stanford has given a related elegant exposition of the Hall collection process [74]. The analogous process was applied in joint work of Ran Levi and the author to give group theoretic models for iterated loop spaces (available on Levi's website).

The connection to the homotopy groups of S^2, as well as to the Lie algebra attached to the descending central series of the pure braid groups, is discussed next.

Theorem 20.9. *The group $\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2})$ is a normal subgroup of $\Theta_k(F_k) \cap \mathrm{Brun}_{k+1}$. There are isomorphisms*

$$\Theta_k(F_k) \cap \mathrm{Brun}_{k+1} / \Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2}) \to \pi_{k+1} S^2.$$

The method of proving that the maps $\Theta_n \colon F[y_1, y_2, \ldots, y_n] \to P_{n+1}$ are monomorphisms via Lie algebras admits an interpretation in terms of classical homotopy theory. The method is to filter both simplicial groups $F[S^1]_\bullet$ and $\mathrm{AP}_\bullet$ via the descending central series, and then to analyze the natural map on the level of associated graded Lie algebras.

On the other-hand, the Lie algebra arising from filtering any simplicial group by its descending central series gives the E^0-term of the Bousfield-Kan spectral sequence for the simplicial group in question [4]. Similarly, filtering via the mod-p descending central series gives the classical unstable Adams spectral sequence [4, 22, 81].

Thus the method of proof of Theorem 18.7 is precisely an analysis of the behavior of the natural map $\Theta : F[S^1]_\bullet \to \mathrm{AP}_\bullet$ on the level of the E^0-term of the Bousfield-Kan spectral sequence. This method exhibits a close connection between Vassiliev invariants of pure braids and these natural spectral sequences. The next result is restatement of Theorem 18.7 proven in [21].

Corollary 20.10. *The maps* $\Theta_n \colon F[y_1, y_2, \dots, y_n] \to P_{n+1}$ *on the level of associated graded Lie algebras*

$$\mathrm{gr}_*(\Theta_n) \colon \mathrm{gr}_*(F[y_1, y_2, \dots, y_n]) \to \mathrm{gr}_*(P_{n+1})$$

are monomorphisms. Thus the maps Θ_n *induce embeddings on the level of the* E^0*-term of the Bousfield-Kan spectral sequences for* $E^0(\Theta) \colon E^0(F[S^1]_\bullet) \to E^0(\mathrm{AP}_\bullet)$*.*

21. Other connections

Connections to other spheres: The work above has been extended to all spheres, as well as other connected CW-complexes [21]. One way in which other spheres arise is via the induced embedding of free products of simplicial groups

$$\Theta \amalg \Theta : F[S^1]_\bullet \amalg F[S^1]_\bullet \to \mathrm{AP}_\bullet \amalg \mathrm{AP}_\bullet.$$

The geometric realization of $F[S^1]_\bullet \amalg F[S^1]_\bullet$ is homotopy equivalent to $\Omega(S^2 \vee S^2)$ by Milnor's theorem stated above as 16.11. Furthermore, $\Omega(S^2 \vee S^2)$ is homotopy equivalent to a weak infinite product of spaces $\Omega(S^n)$ for all $n > 1$.

Connection to certain Galois groups: Consider automorphism groups $\mathrm{Aut}(H)$ where H is one of F_n, the pro-finite completion $\widehat{F_n}$ or the pro-ℓ completion $\widehat{(F_n)}_\ell$. Certain Galois groups G are identified as natural subgroups of these automorphism groups in [2, 23, 25, 26, 45, 46, 68]. One example is Drinfel'd's Grothendieck-Teichmüller Galois group $G = \widehat{GT}$, a subgroup of $\mathrm{Aut}(\widehat{F_2})$.

Let $\mathrm{Der}(L^R[V_n])$ denote the Lie algebra of derivations of the free Lie algebra $L^R[V_n]$ where V_n denotes a free module of rank n over R a commutative ring with identity. Two natural morphisms of Lie algebras, which take values in $\mathrm{Der}(L^R[V_n])$, occur in this context, as follows.

Properties of the infinitesimal braid relations, as stated in Theorem 18.5 above, give a second natural map

$$\mathrm{Ad} : \mathrm{gr}_*(P_{n+1}) \to \mathrm{Der}(L^{\mathbb{Z}}[V_n])$$

whose kernel is precisely the center of $\mathrm{gr}_*(P_{n+1})$ [17]. Combining this last fact with Theorem 18.3 gives properties of the composite morphism of Lie algebras

$$\mathrm{gr}_*(F_n) \xrightarrow{\mathrm{gr}_*(\Theta_n)} \mathrm{gr}_*(P_{n+1}) \xrightarrow{\mathrm{Ad}} \mathrm{Der}(L^{\mathbb{Z}}[V_n]).$$

Proposition 21.1. *If $n \geq 2$, the induced morphism of Lie algebras*

$$\mathrm{Ad} \circ \mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1, y_2, \ldots, y_n]) \to \mathrm{Der}(L^{\mathbb{Z}}[V_n])$$

is a monomorphism.

In addition, certain Galois groups G above are filtered with induced morphisms of Lie algebras

$$\mathrm{gr}_*(G) \to \mathrm{Der}(L^{\widehat{\mathbb{Z}}}[V_n])$$

where $\widehat{\mathbb{Z}}$ denotes the pro-finite completion of the integers. One example is $G = \widehat{GT}$ with

$$\mathrm{gr}_*(\widehat{GT}) \to \mathrm{Der}(L^{\widehat{\mathbb{Z}}}[V_2]),$$

as given in [23, 45, 46, 68].

This raises the question of (i) whether the images of

$$\mathrm{Ad} \circ \mathrm{gr}_*(\Theta_2),$$

and

$$\mathrm{gr}_*(\widehat{GT})$$

in $\mathrm{Der}(L^{\widehat{\mathbb{Z}}}[V_2])$ have a non-trivial intersection, or (ii) whether this intersection is meaningful.

22. Questions

The point of this section is to consider whether the connections between the braid groups and homotopy groups above are useful. Some natural, as well as speculative, problems are listed next.

The combinatorial problem of distinguishing elements in the pure braid groups has been well-studied. For example, the Lie algebra associated to the descending central series of the pure braid group P_n has been connected with Vassiliev theory and has been shown to be a complete set of invariants which distinguish all elements in P_n [51]. Furthermore, these Lie algebras have been applied to other questions arising from the classical KZ-equations [49, 25] as well as the structure of certain Galois groups [45, 23, 25, 26].

Question 1: One description of homotopy groups is given in Theorem 20.9. This relation is coarser than that given by Vassiliev invariants. Give methods of understanding this coarser relation.

Question 2: Consider Brunnian braids Brun_k. Fix a braid γ with image in the kth symmetric group Σ_k given by a k-cycle. For any braid α in Brun_k, the braid closure of $\alpha \circ \gamma$ is a knot. Describe features of these knots or those obtained from the analogous constructions for $\Theta_k(F_{k-1}) \cap \mathrm{Brun}_k$. Where do these fit in Budney's description of the space of long knots [8]?

Question 3: Give combinatorial properties of the natural map $\mathrm{Brun}_{k+1}(\mathbb{R}^2) \to \mathrm{Brun}_{k+1}(S^2)$ which provide information about the cokernel. Two concrete problems are stated next.

(1) Give group theoretic reasons why the order of the 2-torsion in $\pi_*(S^2)$ is bounded above by 4 and why the p-torsion for an odd prime p is bounded above by p.

(2) If $k+1 \geq 5$, the image of $\mathrm{Brun}_{k+1}(\mathbb{R}^2) \to \mathrm{Brun}_{k+1}(S^2)$ is a normal subgroup of finite index.

This fact follows from Serre's classical theorem that $\pi_k(S^2)$ is finite for $k > 3$ and Theorems 19.2 and 20.4, proven in [5].

Do natural features of the braid groups imply this result?

Question 4: Let F_n denote the image of $\Theta_n(F_n)$. Observe that the groups $\mathrm{QBrun}_{n+2} \cap F_{n+1}$, and $\mathrm{Brun}_{n+1} \cap F_n$ are free. Furthermore, there is a short exact sequence of groups

$$\text{(Extension 1)}: \ 1 \to F_n \cap d_0(\mathrm{QBrun}_{n+2}) \to F_n \cap \mathrm{Brun}_{n+1} \to \pi_{n+1}S^2 \to 1$$

as well as isomorphisms

$$F_n \cap \mathrm{Brun}_{n+1}/(F_n \cap d_0(\mathrm{QBrun}_{n+2})) \to \pi_{n+1}S^2,$$

by Theorem by 20.9.

Consider the Serre 5-term exact sequence for the group extension given by Extension 1 directly above to obtain information about the induced surjection

$$H_1(F_n \cap \mathrm{Brun}_{n+1}) \to \pi_{n+1}(S^2).$$

This 5-term exact sequence specializes to

$$\begin{array}{rcccl} H_2(\pi_{n+1}(S^2)) & \to & H_1(F_n \cap d_0(\mathrm{QBrun}_{n+2}))_{\pi_{n+1}(S^2)} & & \\ & \to & H_1(F_n \cap \mathrm{Brun}_{n+1}) & \to & \pi_{n+1}(S^2) \end{array}$$

where A_π denotes the group of co-invariants of a π-module A. Thus $\pi_{n+1}(S^2)$ is a quotient of the free abelian group $H_1(F_n \cap \mathrm{Brun}_{n+1})$ with relations given by the image of the co-invariants $H_1(F_n) \cap d_0(\mathrm{QBrun}_{n+2})_{\pi_{n+1}(S^2)}$.

Give combinatorial descriptions of the induced map on the level of the first homology groups

$$H_1(F_n \cap d_0(\mathrm{QBrun}_{n+2})) \to H_1(F_n \cap \mathrm{Brun}_{n+1}).$$

A similar problem arises with the epimorphism $\mathrm{Brun}_{n+1}(S^2) \to \pi_n S^2$ with kernel in the image of $\mathrm{Brun}_{k+1}(\mathbb{R}^2)$ for $n+1 \geq 5$.

Question 5: Let $L[V]$ denote the free Lie algebra over the integers generated by the free abelian group V. Let $\mathrm{Der}(L[V])$ denote the Lie algebra of derivations of $L[V]$ and consider the classical adjoint representation

$$\mathrm{Ad}\colon L[V] \to \mathrm{Der}(L[V]).$$

Recall that the map $\Theta_k\colon F_k \to P_{k+1}$ induces a monomorphism of Lie algebras

$$\mathrm{gr}_*(\Theta_k)\colon \mathrm{gr}_*(F_k) \to \mathrm{gr}_*(P_{k+1})$$

where $\mathrm{gr}_*(F_k)$ is isomorphic to the free Lie algebra $L[V_k]$ with V_k a free abelian group of rank k. In addition, properties of the 'infinitesimal braid relations' give a representation

$$\rho_k\colon \mathrm{gr}_*(P_{k+1}) \to \mathrm{Der}(L[V_k])$$

appearing in work on certain Galois groups [45] (with the integers $\mathbb{Z}$ replaced by the pro-finite completion of $\mathbb{Z}$).

Identify F_k with $\Theta_k(F_k)$ in what follows below. Give methods to describe combinatorial properties of the composite

$$\mathrm{gr}_*(F_k \cap \mathrm{Brun}_{k+1}) \xrightarrow{\mathrm{gr}_*(i_k)} \mathrm{gr}_*(F_k) \xrightarrow{\mathrm{gr}_*(\Theta_k)} \mathrm{gr}_*(P_{k+1}) \xrightarrow{\rho_k} \mathrm{Der}(L[V_k])$$

where $i_k : F_k \cap \mathrm{Brun}_{k+1} \to F_k$ is the natural inclusion. Let Φ_{k+1} denote this composite. When restricted to $\mathrm{gr}_*(F_k) = L[V_k]$, this map is a monomorphism. Give methods to describe the sub-quotient

$$\Phi_{k+1}(\mathrm{gr}_*(F_k \cap \mathrm{Brun}_{k+1}))/\Phi_{k+1}(\mathrm{gr}_*(F_k \cap d_0(\mathrm{QBrun}_{k+2}))).$$

Question 6: Assume that the pure braid groups $P_n(S)$ are replaced by either their pro-finite completions $\widehat{P_n(S)}$ or their pro-ℓ completions. Describe the associated changes for the homotopy groups arising in Theorems 19.2, 20.4, or 19.1. For example, is the torsion in these homotopy groups left unchanged by replacing $P_n(S)$ by $\widehat{P_n(S)}$?

23. Appendix: Brunnian braids and principal congruence subgroups

The Brunnian braid group has features in common with the Borromean rings dating back at least to Carlo Borromeo in 1560. Related structures for Brunnian links are given in [56].

The purpose of this section is to record an observation concerning Brunnian braid groups for the 2-sphere and to pose a related question as well as to describe a starting point to appear in joint work with Berrick, Wong and Wu.

Recall that there are classical maps

$$r : B_4 \to B_3$$

and

$$\Theta : B_3 \to SL(2, \mathbb{Z}).$$

The map $r : B_4 \to B_3$ is defined by the formula

$$r(\sigma_i) = \begin{cases} \sigma_i & \text{if } i = 1 \text{ or } i = 2\,, \\ \sigma_1 & \text{if } i = 3. \end{cases}$$

The map

$$\Theta : B_3 \to SL(2, \mathbb{Z})$$

is defined by the formula

$$\Theta(\sigma_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and

$$\Theta(\sigma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The map Θ arises from a map of B_3 to the mapping class group for genus 1 surfaces, $SL(2, \mathbb{Z})$, obtained via two Dehn twists along two circles which intersect in a single point [3].

Recall that $\Gamma(2, r)$ denotes the kernel of the mod-r reduction map

$$\rho_r : SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/r\mathbb{Z})$$

(in Section 10). Similarly, let $P\Gamma(2, r)$ denote the kernel of the mod-r reduction map

$$\rho_r : PSL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z}/r\mathbb{Z}).$$

The following is the main result of this section in which D_8 denotes the dihedral group of order 8.

Theorem 23.1. *The classical maps*

$$r : B_4 \to B_3$$

and

$$\Theta : B_3 \to SL(2, \mathbb{Z})$$

induce maps which give

(1) *a homomorphism* $B_4(S^2) \to PSL(2,\mathbb{Z})$ *together with a short exact sequence of groups*
$$1 \to D_8 \to B_4(S^2) \to PSL(2,\mathbb{Z}) \to 1,$$
(2) *a homomorphism* $P_4(S^2) \to P\Gamma(2,2)$ *together with a split short exact sequence of groups*
$$1 \to \mathbb{Z}/2\mathbb{Z} \to P_4(S^2) \to P\Gamma(2,2) \to 1$$
with an isomorphism
$$P_4(S^2) \to \Gamma(2,2),$$
and
(3) *a homomorphism* $\mathrm{Brun}_4(S^2) \to P\Gamma(2,4) = \Gamma(2,4)$ *which is an isomorphism.*

The previous theorem may admit an extension to other Brunnian braid groups
$$\mathrm{Brun}_{2g+2}(S^2).$$
Preparation for this possible extension is given by two digressions, one concerning the 'hyperelliptic mapping class group' given by the central extension Δ_{2g+2}, described in Example 9.10. The second digression concerns principle congruence subgroups.

Recall the 'hyperelliptic mapping class group' Δ_{2g+2}, a subgroup of Γ_g the mapping class group for genus g surfaces. By 9.10 or [11], there is a central extension
$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Delta_{2g+2} \to B_{2g+2}(S^2) \to 1.$$
This extension, as well as information about its characteristic class, is given in [11]. A construction for the classifying space $B\Delta_{2g+2}$ arises from complex 2-plane bundles as described in Example 9.10 or [11], given by
$$EU(2) \times_{U(2)} \mathrm{Conf}(S^2, 2g+2) \times_{\Sigma_{2g+2}} S^1$$
for g even.

Definition 23.2. Given $n \geq 3$, define

(1) $\mathbb{X}_n = EU(2) \times_{U(2)} \mathrm{Conf}(S^2, n) \times S^1$, and
(2) $\mathbb{Y}_n = EU(2) \times_{U(2)} \mathrm{Conf}(S^2, n) \times_{\Sigma_n} S^1$ with
(3) $q_n : \mathbb{X}_n \to \mathbb{Y}_n$ the standard Σ_n-cover.

Then there are n natural projection maps
$$p_i : \mathbb{X}_n \to \mathbb{X}_{n-1}$$
which are also bundle projections with fibre $S^2 - Q_{n-1}$ for which Q_{n-1} denotes a set of $(n-1)$ distinct points in S^2 by Theorem 3.3.

Define

$$\mathrm{Brun}_n(\mathbb{X})$$

as the intersection of the kernels of the induced maps

$$p_{i*} : \pi_1(\mathbb{X}_n) \to \pi_1(\mathbb{X}_{n-1})$$

for $1 \leq i \leq n$.

Further, define $P\Delta_{2g+2}$ as the kernel of the natural map $\Delta_{2g+2} \to \Sigma_{2g+2}$.

Then consider the classical maps

$$\Delta_{2g+2} \to \Gamma_g \to Sp(2g, \mathbb{Z}).$$

Notice that mod-2 reduction gives a map

$$\rho_2 \colon Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2\mathbb{Z})$$

and that the composite map

$$\Delta_{2g+2} \to Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2\mathbb{Z})$$

factors through

$$\Sigma_{2g+2} \subset Sp(2g, \mathbb{Z}/2\mathbb{Z}).$$

The second digression concerns classical principle congruence subgroups of $Sp(2g, \mathbb{Z})$ as follows where one example is the kernel of the map $\rho_2 \colon Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2\mathbb{Z})$.

Definition 23.3. Let

$$\Gamma^{Sp}(2g, r)$$

denote the principle congruence subgroup of level r, the kernel of the mod-r reduction map

$$\rho_r : Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/r\mathbb{Z}).$$

Regarding $SL(2, \mathbb{Z}) = Sp(2, \mathbb{Z})$, the kernel of $\rho_r \colon SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/r\mathbb{Z})$, denoted $\Gamma(2, r)$ above, is equal to $\Gamma^{Sp}(2, r)$.

A sketch of a possible natural extension of Theorem 23.1 is given next. This sketch consists of 5 steps.

(1) Consider the classical map

$$\Delta_{2g+2} \to \Gamma_g \to Sp(2g, \mathbb{Z})$$

and the 'mod-2 reduction map' $Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/2\mathbb{Z})$ together with the composite map $\Delta_{2g+2} \to Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/2\mathbb{Z})$. Restricting this composite to $P\Delta_{2g+2}$ gives a factorization of the composite

$$P\Delta_{2g+2} \to Sp(2g,\mathbb{Z})$$

through $\Gamma^{Sp}(2g,2)$ as described above.

(2) Inspection of n elements $\{x_1,\ldots,x_n\}$ in $\Gamma^{Sp}(2g,2)$ gives that their commutator $[[\cdots[[x_1,x_2],x_3]\cdots,]x_n]$ is in the principal congruence subgroup of level 2^n in $Sp(2g,\mathbb{Z})$, $\Gamma^{Sp}(2g,2^n)$. This fact is a special case of the statement that the commutator

$$[x,y]$$

of an element x in the principal congruence subgroup of level p, and an element y in the principal congruence subgroup of level q, is in the principal congruence subgroup level pq [24].

(3) A direct computation gives an isomorphism

$$\mathrm{Brun}_{2g+2}(\mathbb{X}) \to \mathrm{Brun}_{2g+2}(S^2) \times \mathbb{Z}/2\mathbb{Z}.$$

(4) Observe that $\mathrm{Brun}_{2g+2}(\mathbb{X})$ is generated by commutators of length at least $(g+1)$ as an iterated application of the fibrations $p_i : \mathbb{X}_n \to \mathbb{X}_{n-1}$ together with the analogous argument in [21] in the case of pure braid groups. Thus the image of $\mathrm{Brun}_{2g+2}(\mathbb{X})$ in $Sp(2g,\mathbb{Z})$ lies in the principle congruence subgroup of level 2^{g+2}.

(5) It is natural to conjecture that $\mathrm{Brun}(S^2, 2g+2)$ is isomorphic to a subgroup of $\Gamma^{Sp}(2g,2^{g+1})$ in $Sp(2g,\mathbb{Z})$, thus extending Theorem 23.1.

If this conjecture is, in fact, correct, is there some natural additional geometry associated to the homotopy groups of ΩS^2, arising from this connection to $\Gamma^{Sp}(2g,2^{g+1})$, which informs on the associated homotopy groups?

A proof of Theorem 23.1 is given via a sequence of lemmas.

Lemma 23.4. *The image of the natural map*

$$\pi_1(\mathrm{Conf}(S^2,4)) \to \pi_1(\mathrm{Conf}(S^2,3)^4)$$

is exactly

$$\oplus_3\mathbb{Z}/2\mathbb{Z}.$$

Thus the 4*-stranded Brunnian braid group*

$$\mathrm{Brun}_4(S^2),$$

which is the kernel of $\pi_1(\mathrm{Conf}(S^2,4)) \to \pi_1(\mathrm{Conf}(S^2,3)^4)$*, is equal to the kernel of*

$$P_4(S^2) \to \oplus_3 \mathbb{Z}/2\mathbb{Z}.$$

Proof. Recall that there are homeomorphisms

$$\theta_q : PSL(2,\mathbb{C}) \times \mathrm{Conf}(S^2 - \{0,1,\infty\}, q) \to \mathrm{Conf}(S^2, q+3)$$

given by

$$\theta_q(\rho, (z_1, z_2, \ldots, z_q)) = (\rho(0), \rho(1), \rho(\infty), \rho(z_1), \rho(z_2), \ldots, \rho(z_q)).$$

In addition, the group $\mathrm{Brun}_n(S^2)$ is the kernel of the map

$$\pi_1(\mathrm{Conf}(S^2,n)) \to \pi_1(\mathrm{Conf}(S^2,n-1)^n)$$

induced by the n different choices of projection maps

$$p_i : \mathrm{Conf}(S^2,n) \to \mathrm{Conf}(S^2,n-1)$$

where the projection p_i deletes the i-th coordinate. Thus the map

$$\pi_1(\mathrm{Conf}(S^2,4)) \to \pi_1(\mathrm{Conf}(S^2,3)^4)$$

is given by

$$\mathbb{Z}/2\mathbb{Z} \times F_2 \to \oplus_4 \mathbb{Z}/2\mathbb{Z}.$$

The next step is to identify the image.

It suffices to check the behavior of the map

$$\mathbb{R}^2 - \{0,1\} \to \mathrm{Conf}(S^2,3)^4$$

where $\mathbb{R}^2 - \{0,1\}$ is identified as the subspace of $\mathrm{Conf}(S^2,4))$ given by

$$\{(\infty, 0, 1, z) \mid z \in \mathbb{R}^2 - \{0,1\}\}$$

because the free group of rank 2, F_2, is given by the image of this map on the level of fundamental groups.

Notice that the image $\pi_1(\mathrm{Conf}(S^2,4)) \to \pi_1(\mathrm{Conf}(S^2,3)^4)$ is exactly $\oplus_3\mathbb{Z}/2\mathbb{Z}$ as follows.

(1) The induced map $p_{1*} : \pi_1(\mathrm{Conf}(S^2,4)) \to \pi_1(\mathrm{Conf}(S^2,3))$ is an isomorphism by comparing θ_1 and the projection maps. That is, $p_1((\infty,0,1,z)) = (0,1,z)$ represents a generator γ_1 of $\pi_1(\mathrm{Conf}(S^2,3))$ in $\pi_1(\mathrm{Conf}(S^2,3)^4)$.

(2) The induced map $p_{2*} : \pi_1(\mathrm{Conf}(S^2,4)) \to \pi_1(\mathrm{Conf}(S^2,3))$ is an epimorphism by comparing θ_2 and the projection maps. That is, $p_1((\infty,0,1,z)) = (\infty,1,z)$ which carries a generator x of $F[x,y]$ in $\pi_1(\mathrm{Conf}(S^2,4))$ to an independent generator γ_2 in $\pi_1(\mathrm{Conf}(S^2,3)^4)$.

(3) The induced map $p_{3*} : \pi_1(\mathrm{Conf}(S^2, 4)) \to \pi_1(\mathrm{Conf}(S^2, 3))$ is an isomorphism by comparing θ_3 and the projection maps. That is, $p_1((\infty, 0, 1, z)) = (\infty, 0, z)$ which carries a generator y of $F[x, y]$ to an independent generator γ_2 in $\pi_1(\mathrm{Conf}(S^2, 3)^4)$.

(4) The induced map

$$p_{4*} : \pi_1(\mathrm{Conf}(S^2, 4)) \to \pi_1(\mathrm{Conf}(S^2, 3))$$

is trivial as $p_4((\infty, 0, 1, z)) = (\infty, 0, 1)$ which is constant.

The lemma follows.

□

Lemma 23.5. *The kernels of the natural maps*

$$SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/4\mathbb{Z})$$

and

$$PSL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z}/4\mathbb{Z})$$

are equal.

Proof. Observe that there is a commutative diagram

$$\begin{array}{ccccc}
\{1\} & \longrightarrow & \Gamma(2, 2^2) & \longrightarrow & P\Gamma(2, 2^2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & SL(2, \mathbb{Z}) & \longrightarrow & PSL(2, \mathbb{Z}) \\
\downarrow 1 & & \downarrow & & \downarrow \\
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & SL(2, \mathbb{Z}/4\mathbb{Z}) & \longrightarrow & PSL(2, \mathbb{Z}/4\mathbb{Z})
\end{array}$$

□

Lemma 23.6. *The map*

$$\Theta : B_4 \to SL(2, \mathbb{Z})$$

induces a map

$$\Phi : B_4(S^2) \to PSL(2, \mathbb{Z})$$

together with a commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & P_4(S^2) & \longrightarrow & P\Gamma(2,2) \\
\downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & B_4(S^2) & \longrightarrow & PSL(2,\mathbb{Z}) \\
\downarrow 1 & & \downarrow & & \downarrow \\
\oplus_2\mathbb{Z}/2\mathbb{Z} & \longrightarrow & \Sigma_4 & \longrightarrow & PSL(2,\mathbb{Z}/2\mathbb{Z})
\end{array}$$

where K denotes the kernel of the map

$$\Phi : B_4(S^2) \to PSL(2,\mathbb{Z}).$$

Proof. Observe that the kernel of

$$B_4 \to B_4(S^2)$$

is generated by the two elements $(\sigma_1\sigma_2\sigma_3)^4$ and $(\sigma_1\sigma_2)^3$. Furthermore

$$\Theta((\sigma_1\sigma_2)^3) = -Id$$

where

$$-Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In addition,

$$\Theta((\sigma_1\sigma_2\sigma_3)^4) = \Theta((\sigma_1\sigma_2\sigma_1)^4) = \Theta((\sigma_1\sigma_2)^6) = Id.$$

Thus there is an induced epimorphism

$$\Phi : B_4(S^2) \to PSL(2,\mathbb{Z})$$

together with the commutative diagram stated in the lemma.

Notice that

(1) the kernel of $\Sigma_4 \to PSL(2,\mathbb{Z}/2\mathbb{Z})$ is $\oplus_2\mathbb{Z}/2\mathbb{Z}$, generated by the images of the two elements $A = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2$ and $B = \sigma_1\sigma_3^{-1}$.
(2) the kernel of $P_4(S^2) \to P\Gamma(2,2)$ is generated by the image of the element $C = (\sigma_1\sigma_2)^3$.
(3) The elements A, B, and C generate a subgroup of $B_4(S^2)$ isomorphic to D_8 by a direct check.

The lemma follows.

□

One proof of Theorem 23.1 is as follows. To check the first assertion, the kernel of the natural epimorphism

$$B_4(S^2) \to PSL(2, \mathbb{Z}),$$

denoted K in Lemma 23.6, is isomorphic to D_8. Thus, the first assertion follows.

By the proof of Lemma 23.6, there is a central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to P_4(S^2) \to P\Gamma(2, 2) \to 1$$

(as the $\mathbb{Z}/2\mathbb{Z}$ is generated by $(\sigma_1\sigma_2)^3$). Since $P\Gamma(2, 2)$ is free on two generators, the extension is split. Notice that this is overkill as Lemma 23.4 has been re-proven. By inspection, there is a commutative diagram

$$\begin{array}{ccccc}
\{1\} & \longrightarrow & \mathrm{Brun}_4(S^2) & \longrightarrow & P\Gamma(2, 2^2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & P_4(S^2) & \longrightarrow & P\Gamma(2, 2) \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & \oplus_3\mathbb{Z}/2\mathbb{Z} & \longrightarrow & \oplus_2\mathbb{Z}/2\mathbb{Z}
\end{array}$$

where the rows and columns are all group extensions (using the fact that $P\Gamma(2, 2)$ is generated by the two matrices

$$x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

This suffices.

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