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Higher Bruhat Orders and Cyclic Hyperplane Arrangements

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Higher Bruhat Orders and Cyclic Hyperplane Arrangements

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Abstract.
We study the higher Bruhat orders $B(n,k)$ of Manin & Schechtman [MaS] and

- characterize them in terms of inversion sets,
- identify them with the posets $\mathcal{U}(C^{n+1,r}, n+1)$ of uniform extensions of the alternating oriented matroids $C^{n,r}$ for $r := n-k$ (that is, with the extensions of a cyclic hyperplane arrangement by a new oriented pseudoplane),
- show that $B(n,k)$ is a lattice for $k = 1$ and for $r \leq 3$, but not in general,
- show that $B(n,k)$ is ordered by inclusion of inversion sets for $k = 1$ and for $r \leq 4$.

However, $B(8,3)$ is not ordered by inclusion. This implies that the partial order $B_{\subseteq}(n,k)$ defined by inclusion of inversion sets differs from $B(n,k)$ in general. We show that the proper part of $B_{\subseteq}(n,k)$ is homotopy equivalent to S^{r-2}. Consequently,

- $B(n,k) \simeq S^{r-2}$ for $k = 1$ and for $r \leq 4$.

In contrast to this, we find that the uniform extension poset of an affine hyperplane arrangement is in general not graded and not a lattice even for $r = 3$, and that the proper part is not always homotopy equivalent to $S^{r(\mathcal{M})-2}$.

1. Introduction

The *higher Bruhat orders* $B(n,k)$ were introduced by Manin & Schechtman [MaS, §2] [MaS1]. In this paper we clarify the geometric interpretation of the higher Bruhat orders (as suggested by Kapranov & Voevodsky [KaV, Sect. 4]). We use the geometric picture to analyze the main structural properties of $B(n,k)$, including new proofs for the results of Manin & Schechtman.

We start with a review of the weak ordering of the symmetric group, see also [YaO] [Bj1] [BLSWZ, Sect. 2.3]. For this denote the set of integers $\{1,\ldots,n\}$ by $[n]$, and the set of k-subsets of $[n]$ by $\binom{[n]}{k}$. We write $U \subset\!\!\cdot\, U'$ if U, U' are finite sets with $U \subset U'$ and $|U'| = |U|+1$. For any collection $\mathcal{U}$ of finite sets, we define the partial order by *single step inclusion* on $\mathcal{U}$ by the condition that $U \leq U'$ if and only if there exist sets $U_i \in \mathcal{S}$ with $U = U_0 \subset\!\!\cdot\, U_1 \subset\!\!\cdot\, \ldots \subset\!\!\cdot\, U_t = U'$, where $t = |U'| - |U|$ is implied.

Definition 1.1.

(i) Let $A(n,1)$ denote the set of permutations of the n-element set $[n]$.

(ii) For every permutation $\rho = (\rho_1, \dots, \rho_n)$, the inversion set $\mathrm{inv}(\rho) := \{ij : i<j, \rho_i>\rho_j\}$ is a set of pairs, that is, a subset of $\binom{[n]}{2}$.

(iii) Define $B(n,1) := \{\mathrm{inv}(\rho) : \rho \in A(n,1)\}$. Every permutation is determined by its inversion set, thus $A(n,1)$ is in bijection to the collection $B(n,1)$ of inversion sets.

(iv) The *weak Bruhat order* is the set $B(n,1)$, partially ordered by single step inclusion.

Some main structural properties of the weak ordering are the following:

(1) $B(n,1)$ is a graded poset of length $\binom{n}{2}$, whose rank function is $r(B) = |B|$,

(2) $U \subseteq \binom{[n]}{2}$ is an inversion set, $U \in B(n,1)$, if and only for every triple $i < j < l$ the intersection $U \cap \{ij, il, jl\}$ is neither $\{il\}$ nor $\{ij, jl\}$, [YaO, Prop. 2.2]

(3) $B(n,1)$ is a lattice, [YaO, Thm. 2.1]

(4) $U \leq U'$ holds if and only if $U \subseteq U'$, [YaO, Prop. 2.1]

(5) the proper part of $B(n,1)$ has the homotopy type of the $(r-2)$-sphere. [Bj1]

Furthermore, $B(n,1)$ has various geometric interpretations. For example, it is the "poset of regions" of the Coxeter arrangement A_{n-1}, which is the arrangement of all hyperplanes spanned by n vectors in general position in $\mathbb{R}^{n-1}$. This also suggests far-reaching generalizations of the weak order, to the posets of regions of arbitrary arrangements. The analogues of (1), (4) and (5) are still true in this context [Ed1] [EdW]. If the arrangement is simplicial, then the poset of regions is a lattice, but not in general [BEZ]. If the poset is a lattice, then an analogue of (2) holds, see [BEZ].

We will now generalize the construction of the weak orders $B(n,1)$ to give a definition of the higher Bruhat orders of Manin & Schechtman. The equivalence of our version of $B(n,k)$ with the original definition is non-trivial; it will be demonstrated in Corollaries 2.3 and 4.2. Define a *k-packet* as the set $P(I) := \{J \in \binom{[n]}{k} : J \subset I\}$ of all k-subsets of a $(k+1)$-set $I = \{i_1 < i_2 < \ldots < i_{k+1}\} \in \binom{[n]}{k+1}$. In the lexicographic order the elements of $P(I)$ are $I \backslash i_{k+1} < I \backslash i_k < \ldots < I \backslash i_1$.

Definition 1.2.

(i) A permutation ρ of $\binom{[n]}{k}$ is *admissible* if every k-packet $P(I)$ occurs in it either in lexicographic order or in reversed lexicographic order. Let $A(n,k)$ denote the set of all admissible permutations of $\binom{[n]}{k}$.

(ii) For each $\rho \in A(n,k)$ the *inversion set* $\mathrm{inv}(\rho) \subseteq \binom{[n]}{k+1}$ is the set of packets that appear in reversed lexicographic order in ρ.

(iii) The set $B(n,k)$ is defined as the collection of all inversion sets $B(n,k) := \{\mathrm{inv}(\rho) : \rho \in A(n,k)\}$.

(iv) The *higher Bruhat order* $B(n,k)$ is the partial order on $B(n,k)$ given by single step inclusion.

In this paper, we will treat the questions for higher Bruhat orders that correspond to the five structural features of the weak order listed above. In the course of our work, we will also show that our definition is equivalent to the original one given by Manin & Schechtman [MaS]. Specifically we prove the following results, where $r := n - k$.

(1) $B(n,k)$ is a graded poset of length $\binom{n}{k}$, whose rank function is $r(B) = |B|$, [MaS, §2 Thm. 3b]. (Theorem 4.1(G))

(2) $U \subseteq \binom{[n]}{k+1}$ is an inversion set, $U \in B(n,k)$, if and only for every $K \in \binom{[n]}{k+2}$ and for $\{i < j < l\} \subseteq K$, the intersection $U \cap \{K\backslash l, K\backslash j, K\backslash i\}$ is neither $\{K\backslash j\}$ nor $\{K\backslash l, K\backslash i\}$. (Theorem 4.1(B))

(3) $B(n,1)$ is a lattice for $k=1$ and for $r \leq 3$, but not in general. (Theorem 4.4)

(4) $U \leq U'$ holds if and only if $U \subseteq U'$, provided that $k = 1$ or $r \leq 4$, but not in general. (Theorem 4.5)

This last fact shows that the (simpler) partial order $B_{\subseteq}(n,k)$ on the set $B(n,k)$ defined by *inclusion* does <u>not</u> in general coincide with the partial order by *single step inclusion* defined by Manin & Schechtman. However, the combinatorics of $B(n,k)$ is intimately related to that of $B_{\subseteq}(n,k)$, so all main results on $B(n,k)$ have counterparts for $B_{\subseteq}(n,k)$, see Theorem 5.1. The partial order of $B_{\subseteq}(n,k)$ is easier to study, however. We prove the following result, which applies to $B(n,k)$ whenever $B_{\subseteq}(n,k) = B(n,k)$:

(5) the proper part of $B_{\subseteq}(n,k)$ has the homotopy type of $S^{(r-2)}$. (Theorem 5.2)

The key to our development is the interpretation of $B(n,k)$ and of $B_{\subseteq}(n,k)$ as "posets of oriented matroid extensions" of a cyclic configuration of n vectors in $\mathbb{R}^r$ by a new element. Choosing a particular vector representation, $B(n,k)$ includes elements that correspond to the regions of the "adjoint" arrangement of hyperplanes spanned by the vectors, plus in general many more extensions that correspond to other extensions, realizable or not. We refer to [BLSWZ, Sect. 5.3] for the fact that the regions of the adjoint arrangement correspond to only a part of the realizable single element extensions of the corresponding oriented matroid. In this paper, we will treat oriented matroids as arrangements of pseudo-hyperplanes. So we get the interpretation of $B(n,k)$ as the poset of extensions of the cyclic hyperplane arrangement $\mathbf{X}_c^{n,n-k-1}$ by a new pseudo-hyperplane.

This paper is organized as follows. In Section 2 we collect elementary facts about admissible orderings and show that their inversion sets are "consistent", while in Section 3 we discuss affine hyperplane arrangements and show that for "cyclic" arrangements the extensions by a new pseudohyperplane correspond to consistent sets. In Theorem 4.1, this is used for a geometric characterization of the sets $A(n,k)$ and the higher Bruhat orders $B(n,k)$. From this, we get in Section 4 structural information about the posets $B(n,k)$, whose homotopy types are determined in Section 5. The geometric interpretation of $B(n,k)$ also suggests a generalization: one can consider the poset of all extensions of any affine arrangement in general position by a new pseudo-hyperplane. This poset, however, does not retain any of the above structural features, see Section 6. Enumerative results are collected in Section 7.

2. Admissible Orders and Consistent Sets

We will now review the original construction of the set $B(n,k)$ by Manin & Schechtman.

Definition 2.1. [MaS, Def. 2.2]

(i) A permutation of $\binom{[n]}{k}$ is *admissible* if its restriction to each k-packet $I \in \binom{[n]}{k+1}$ is either the lexicographic order or the reversed lexicographic order. $A(n,k)$ is defined as the set of all admissible orders on $\binom{[n]}{k}$.

(ii) Two permutations $\rho, \rho' \in A(n,k)$ are *elementarily equivalent* ($\rho \sim \rho'$) if they differ by an interchange of two neighbors not contained in a common packet. Let $B(n,k)$ be the quotient by the induced equivalence relation and $A(n,k) \to B(n,k)$, $\rho \mapsto [\rho]$ the quotient map.

(iii) For each $\rho \in A(n,k)$ the *inversion set* $\mathrm{inv}(\rho)$ is the set of packets that appear in reversed lexicographic order in ρ. Here $\rho \sim \rho'$ implies $\mathrm{inv}(\rho) = \mathrm{inv}(\rho')$, so the inversion set $\mathrm{inv}[\rho] := \mathrm{inv}(\rho)$ is well-defined for $[\rho] \in B(n,k)$.

We will view permutations of $\binom{[n]}{k}$ as *linear orders* on the set $\binom{[n]}{k}$. For $[\rho] \in B(n,k)$ let $Q[\rho]$ be the intersection of the linear orders in $[\rho]$, that is, the *partial order* on $\binom{[n]}{k}$ defined by $I' < I$ if and only if $I' <_\tau I$ for all $\tau \in [\rho]$. Similarly, let $Q'[\rho]$ be the intersection of all admissible orders τ with $\mathrm{inv}(\tau) = \mathrm{inv}(\rho)$.

Lemma 2.2. *The following four sets coincide:*
A_1: *$[\rho]$, the set of linear orders of $\binom{[n]}{k}$ equivalent to ρ,*
A_2: *the linear extensions of $Q[\rho]$,*
A_3: *the admissible orders of $\binom{[n]}{k}$ with inversion set $\mathrm{inv}(\rho)$,*
A_4: *the linear extensions of $Q'[\rho]$.*

Proof. Every τ that is equivalent to ρ is admissible with inversion set $\mathrm{inv}(\rho)$, and thus it is also a linear extension of $Q'[\rho]$. Now we use that any two linear extensions of a poset Q' can be connected by a sequence of transpositions of adjacent elements that are incomparable in Q'. Furthermore, if $I, J \in \binom{[n]}{k}$ are incomparable in $Q'[\rho]$, then they are not contained in a common k-packet. Thus every linear extension of $Q'[\rho]$ is in $[\rho]$.

With this we have shown that $A_1 = A_3 = A_4$. But the equality of the first and the third set also implies $Q[\rho] = Q'[\rho]$, that is, $A_1 = A_3$ implies $A_2 = A_4$. □

Corollary 2.3. [MaS, §2 Thm. 3d] *Every $[\rho] \in B(n,k)$ is uniquely determined by its inversion set $\mathrm{inv}(\rho)$.*

In particular we get from this corollary that our Definition 1.2(iii) of the set $B(n,k)$ is equivalent to that of Definition 2.1(ii) due to Manin & Schechtman. Our goal is now to characterize inversion sets. For this we consider any $(k+1)$-packet $P(I)$, with $I = \{i_1 < i_2 < \ldots < i_{k+2}\}$, in its lexicographic order. Thus a *beginning segment* is of the form $\{I \backslash i_{k+1}, I \backslash i_k, \ldots, I \backslash i_j\}$ for some j. An *ending segment* of $P(K)$ is of the form

$\{I\backslash i_j, I\backslash i_{j-1}, \ldots, I\backslash i_1\}$ for some j. The subsets $\emptyset$ and $P(I)$ are considered both as beginning and as ending segments of $P(I)$. We get the following lemma, which identifies the characterizing property of inversion sets. Its converse will be proved in Theorem 4.1(B).

Lemma 2.4. *Every inversion set $U \in B(n,k)$ satisfies the following equivalent conditions:*
(1) *U and its complement are both convex: if $\{j_1<j_2<j_3\} \subseteq K$ for some $K \in \binom{[n]}{k+2}$, then the intersection of U with $\{K\backslash j_3, K\backslash j_2, K\backslash j_1\}$ is neither $\{K\backslash j_3, K\backslash j_1\}$ nor $\{K\backslash j_2\}$,*
(2) *U is consistent, that is, the intersection of U with any $(k+1)$-packet is a beginning or an ending segment of it.*

Proof. The condition (2) that $U \cap P(I)$ is either a beginning or a final segment of $P(I)$ means the following: if we consider the $k+1$-packet in its lexicographic order $I\backslash i_{k+2} < i_{k+1} < \ldots < i_1$, then there is at most one switch between elements of U and between non-elements of U. This yields (2) $\Longleftrightarrow$ (1).

Now assume that $\rho \in A(n,k)$ is an admissible order on $\binom{[n]}{k}$, and let $\{j_1<j_2<j_3\} \subseteq K$ for some $K \in \binom{[n]}{k+2}$. Now if

$$K\backslash j_3 \in \mathrm{inv}(\rho), \qquad K\backslash j_2 \notin \mathrm{inv}(\rho), \qquad K\backslash j_1 \in \mathrm{inv}(\rho).$$

then this implies

$$K\backslash\{j_3,j_1\} >_\rho K\backslash\{j_3,j_2\}, \qquad K\backslash\{j_2,j_3\} <_\rho K\backslash\{j_2,j_1\}, \qquad K\backslash\{j_1,j_2\} >_\rho K\backslash\{j_1,j_3\},$$

which yields a contradiction. An analogous contradiction arises if we find $\{j_1<j_2<j_3\}$ with $K\backslash j_3 \notin \mathrm{inv}(\rho)$, $K\backslash j_2 \in \mathrm{inv}(\rho)$, $K\backslash j_1 \notin \mathrm{inv}(\rho)$. Thus $\mathrm{inv}(\rho)$ satisfies (2). □

Let $U \subseteq \binom{[n]}{k+1}$ be a consistent set. Then the *complement* $\overline{U} := \binom{[n]}{k+1}$ of U is consistent as well. Define the *boundary* of U by $\partial U := \{I=\{i_1<\ldots<i_{k+1}\} \in \binom{[n]}{k+2} : I\backslash i_1 \notin U,\ I\backslash i_{k+2} \in U\}$. Similarly, let $U*\{n+1\} := \{K\cup\{n+1\} : K \in U\}$, and define the *extension* $\widehat{U} \subseteq \binom{[n+1]}{k+2}$ of U as $\widehat{U} := U*\{n+1\} \cup \partial U$.

The following two lemmas contain the key to an inductive treatment of consistent sets. Their geometric significance will become clear in Section 3. In fact, both the statements and the arguments in the proofs can be identified in Figure 1.

Lemma 2.5. *Let U' be a consistent subset of $\binom{[n]}{l}$, and let U'' be a consistent subset of $\binom{[n]}{l+1}$. Then $U := U'' \cup U'*\{n+1\} \subseteq \binom{[n+1]}{l+1}$ is consistent if and only if $\partial U' \subseteq U''$ and $\partial\overline{U'} \subseteq \overline{U''}$.*

Proof. Let $K \in \binom{[n+1]}{l+2}$. If $n+1 \notin K$, then $P(K) \cap U = P(K) \cap U''$ is a beginning or ending subset of $P(K)$, because U'' is consistent.

If $K = \{i_1 < \ldots < i_{l+1} < n+1\}$, we let $I := K\backslash n+1$ and get

$$P(K) = \{I < K\backslash i_{l+1} < \ldots < K\backslash i_1\}, \qquad P(I) = \{I\backslash i_{l+1} < \ldots < I\backslash i_1\}.$$

Now if $U' \cap P(I)$ is a beginning, but not an ending subset of $P(I)$, so that $I \in \partial U'$, then $U \cap P(K)$ cannot be an ending subset of $P(K)$, so consistency if U implies $I \in U''$. From this we get the requirement that $\partial U' \subseteq U''$.

Similarly, if $U' \cap P(I)$ is an ending, but not a beginning subset of $P(I)$, so that $I \in \partial \overline{U'}$, then $U \cap P(K)$ cannot be a beginning subset of $P(K)$, so consistency of U implies $I \notin U''$. From this we get the requirement that $\partial \overline{U'} \subseteq \overline{U''}$.

If $U' \cap P(I)$ is both a beginning and an ending segment of $P(I)$, then either $P(I) \subseteq U'$, so that $U \cap P(K)$ automatically is an ending segment of $P(K)$, or $P(I) \cap U' = \emptyset$, so that $U \cap P(K)$ automatically is a beginning segment of $P(K)$. From this we get that the two conditions of the lemma are also sufficient for consistency of U. □

Lemma 2.6. *Let $U \subseteq \binom{[n]}{l}$ be a consistent set. Then*
(i) *the boundary ∂U of U is a consistent subset of $\binom{[n]}{l+1}$,*
(ii) *the extension $\widehat{U}$ of U is a consistent subset of $\binom{[n+1]}{l+1}$,*
(iii) *U is also a consistent subset of $\binom{[n+1]}{l}$.*

Proof. (i) Choose $K = \{i_1 < \ldots < i_{l+2}\} \in \binom{[n]}{l+2}$. We have to show that $\partial U \cap P(K)$ is a beginning or ending segment of $P(K)$.

As the <u>first case</u> assume that $K \backslash \{i_1, i_{l+2}\} \in U$. This implies that $K \backslash i_{l+2} \notin \partial U$. If $K \backslash \{i_1, i_2\} \in U$, then this implies $P(K \backslash i_1) \subseteq U$ and thus $\partial U \cap P(K) = \emptyset$. Otherwise we get the existence of values s, t, with $3 \leq s \leq l+2$, $1 \leq t \leq l+1$ with
$P(K \backslash i_1) \cap U = \{K \backslash \{i_1, i_j\} : s \leq j \leq l+2\}$,
$P(K \backslash i_{l+2}) \cap U = \{K \backslash \{i_{l+2}, i_j\} : 1 \leq j \leq t\}$.
From this we can compute
$\partial U \cap P(K) = \{\{K \backslash i_j : 1 \leq j \leq \min(s-1, t)\}$,
which is a beginning segment of $P(K)$.

As the <u>second case</u> now assume that $K \backslash \{i_1, i_{l+2}\} \notin U$. This implies that $K \backslash i_1 \notin \partial U$. If $K \backslash \{i_{l+1}, i_{l+2}\} \notin U$, then this implies $P(K \backslash i_{l+2}) \subseteq \overline{U}$ and thus $\partial U \cap P(K) = \emptyset$. Otherwise we get the existence of values s, t, with $2 \leq s \leq l+2$, $1 \leq t \leq l$ with
$P(K \backslash i_1) \cap U = \{K \backslash \{i_1, i_j\} : 2 \leq j < s\}$,
$P(K \backslash i_{l+2}) \cap U = \{K \backslash \{i_{l+2}, i_j\} : t < j \leq l+1\}$.
From this we can compute
$\partial U \cap P(K) = \{\{K \backslash i_j : \max(s, t+1) \leq j \leq l+2\}$,
which is a beginning segment of $P(K)$.

(ii) Here $\partial U'$ is consistent by part (i), hence we can apply Lemma 2.5 for $U'' = \partial U'$: we also have $\partial \overline{U'} \subseteq \overline{U''}$, because $\partial U'$ and $\partial \overline{U'}$ are disjoint by definition. (iii) This is also a special case of Lemma 2.5: for $U' = \emptyset$ we get also have $\partial U' = \partial \overline{U'} = \emptyset$, so $U = U'$ is also a consistent subset of $\binom{[n+1]}{k+1}$. □

3. Cyclic Arrangements

Consider any arrangement $\mathbf{X} = \{H_1, \ldots, H_n\}$ of n affine hyperplanes in general position in $\mathbb{R}^d$. Then every vertex is determined as the intersection of d hyperplanes. Associating the vertex with the set of $n-d$ hyperplanes that do <u>not</u> pass through it, we get a bijection $V = V(\mathbf{X}) \longleftrightarrow \binom{[n]}{n-d}$ between the vertices of $\mathbf{X}$ and the $(n-d)$-subsets of $[n]$. Similarly, every 1-dimensional line in $\mathbf{X}$ is the intersection of $d-1$ of the hyperplanes. Associating every line with the $n-d-1$ hyperplanes that do <u>not</u> contain it, we get a bijection between the lines of $\mathbf{X}$ and the $(n-d+1)$-sets in $[n]$. Furthermore, under these bijections the vertices on a line of $\mathbf{X}$ correspond to the $(n-d)$-sets in the corresponding $(n-d+1)$-set, i.e., the vertices on a line correspond to an $(n-d)$-packet.

The key observation is now that if $\mathbf{X}$ is the cyclic arrangement of n hyperplanes in $\mathbb{R}^{n-k}$, then the vertices on a line correspond to a k-packet *in lexicographic order.*

Definition 3.1. The *cyclic arrangement* $\mathbf{X}_c^{n,d}$ is the arrangement $\{H_1, \ldots, H_n\}$ in $\mathbb{R}^d$ given by

$$H_i = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 + t_i x_2 + \ldots + t_i^{d-1} x_d + t_i^d = 0\}$$

for $1 \le i \le d$, with arbitrary real parameters $t_1 < t_2 < t_3 < \ldots < t_n$.

For every choice of the parameters t_i this arrangement represents the alternating oriented matroid $\mathrm{C}^{n,d+1}$. Here the hyperplane at infinity corresponds to the extension $\mathrm{C}^{n+1,d+1}$ of $\mathrm{C}^{n,d+1}$ by a new element $g := n+1$. Thus the combinatorial type of this affine arrangement does not depend on the choice of the parameters t_i.

Lemma 3.2. *The vertices of $\mathbf{X}_c^{n,d}$ correspond to $\binom{[n]}{n-d}$ in such a way that the vertices on an affine line correspond to the $(n-d)$-packets in lexicographic order (or its reverse).*

Proof. There are many ways to derive this basic fact, either by elementary linear algebra (the vertices V_I corresponding to $I \in \binom{[n]}{k}$ can be explicitly determined in terms of Vandermonde determinants), or using simple oriented matroid tools to compute the contractions (any contraction of $\mathrm{C}^{n,d}$ is a reorientation of a cyclic oriented matroid, with the induced linear order of the ground set), or by exploiting orthogonality resp. oriented matroid duality. □

Now consider any extension of the cyclic arrangement $\mathbf{X}_c^{n,d}$ by a new oriented hyperplane H_f in general position. For this, two extensions by hyperplanes $H_f, H_{f'}$ are *equivalent* if on their negative sides they have the same set $V_f = V_{f'}$ of affine vertices of the arrangement. From Lemma 3.2 we see

$$V_f \;:=\; \{K \in \binom{[n]}{k} : K \text{ corresponds to a vertex on the negative side of } H_f\}$$

is a consistent set $V_f \subseteq \binom{[n]}{k}$. The same is true for any extension of $\mathbf{X}_c^{n,r}$ by a new oriented *pseudo-hyperplane* (topologically deformed hyperplane) H_f in general position. The proper

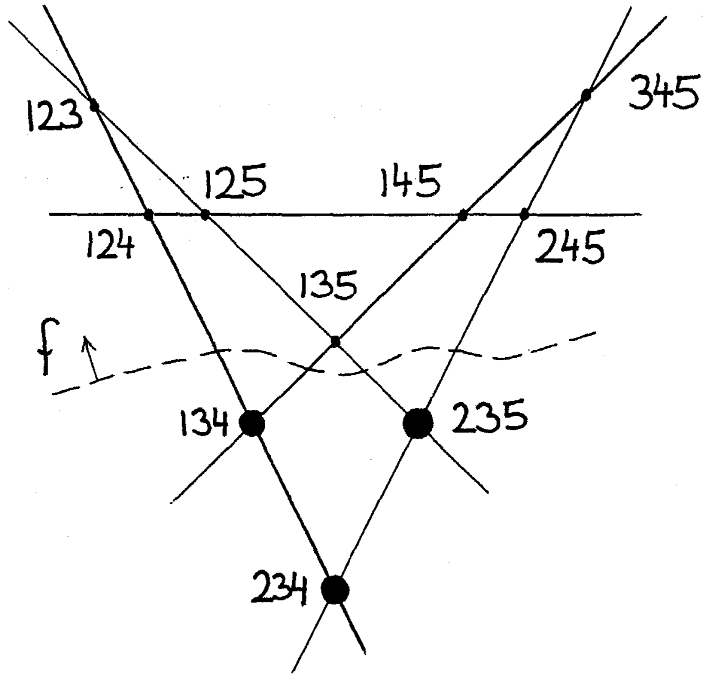

Figure 1: The cyclic arrangement $\mathbf{X}_c^{5,2}$ with a pseudoline extension f and the corresponding vertex set.

framework to study such extensions of an arrangement $\mathbf{X}$ by a pseudohyperplane is the theory of *oriented matroids.*

We will only sketch the connection, and refer to [BLSWZ] for the details. Let $\mathbf{X} = \{H_1, \ldots, H_n\}$ be an affine arrangement in $\mathbb{R}^d$. The affine space $\mathbb{R}^d$ can be identified with a hemisphere of S^d, where the hyperplanes H_i correspond to intersections of $(d-1)$-subspheres of S^d with the hemisphere. Assuming that a positive side has been chosen for every hyperplane, the hyperplane arrangement (resp. the corresponding sphere arrangement) represents an oriented matroid $\mathcal{M}_0$ of rank $d+1$ on the ground set $[n]$, so the hyperplane H_i of the arrangement corresponds to the element $i \in [n]$ of the oriented matroid. By representing $\mathcal{M}_0$ by an affine arrangement we have distinguished the hyperplane at infinity, which corresponds to the extension of $\mathcal{M}_0$ by a new element $g = n+1$. In this sense we say that $\mathbf{X}$ represents the *affine oriented matroid* $(\mathcal{M}, g)$, that is the oriented matroid $\mathcal{M}_0 = \mathcal{M} \backslash g$ together with a distinguished extension of $\mathcal{M}_0$ by g. In particular, the cyclic arrangement $\mathbf{X}_c^{n,d}$ represents the affine *alternating oriented matroid* $(\mathrm{C}^{n+1,d+1}, n+1)$, whose structure is well understood [BLSWZ, Sect. 8.1].

The fact that the extensions of an affine arrangement by a new pseudohyperplane correspond to oriented matroid extensions is due to the "topological representation theorem", see [BLSWZ, Chap. 5]. Here two extensions are considered equivalent if and only if they have the same set of vertices of **X** on their negative side, since this is equivalent to the condition that they determine the same oriented matroid extension. Denoting by V the vertices of **X** (which correspond to half of the vertices/cocircuits of the sphere representation of $\mathcal{M}_0$), we know that every extension $\mathcal{M}_0 \cup f$ of $\mathcal{M}_0$ is determined by its *localization*, a function $\sigma_f : V \longrightarrow \{+,-\}$ that indicates for every affine vertex whether it is on the positive or on the negative side of the extension pseudo-hyperplane. See [BLSWZ, Sect. 7.1] for details and proofs. A key technical result is Las Vergnas' characterization of single element extensions, which in our picture can be stated as follows.

Lemma 3.3. *Let V be the set of vertices of an affine hyperplane arrangement* **X**. *A subset $V_f \subseteq V$ is the vertex set of an extension of* **X** *by a a new pseudohyperplane in general position if and only if it contains a beginning or an ending segment of the set of vertices on every (arbitrarily directed) line.*

Definition 3.4. The *uniform extension poset* of **X** is the set of all extensions of **X** by a new pseudohyperplane H_f in general position, partially ordered by single-step inclusion of their vertex sets.

The uniform extension poset of **X** only depends on the affine matroid $(\mathcal{M}, g)$ represented by **X**, and will thus be denoted by $\mathcal{U}(\mathcal{M}, g)$. It is the set of all uniform single element extensions of $\mathcal{M}_0 = \mathcal{M}\backslash g$, whose partial order depends on the extension $\mathcal{M}$ of $\mathcal{M}_0$.

Corollary 3.5. *The uniform extension poset $\mathcal{U}(\mathrm{C}^{n+1,r}, n+1)$ of $\mathrm{C}^{n,r}$ is naturally isomorphic to the set of all consistent subsets of $\binom{[n]}{k+1}$, ordered by single-step inclusion.*

Proof. This follows directly from the Lemmas 3.2 and 3.3. □

To understand the geometry of $\mathcal{U}(\mathrm{C}^{n+1,r}, n+1)$ we use the partial orders of oriented matroid programs, following the lines of [StZ, Sect. 3]. Let $(\mathcal{M}, g)$ be the affine matroid of an arrangement and $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$ an extension of $\mathcal{M}$ (!) corresponding to a new pseudo-hyperplane H_f. The tripel $(\widetilde{\mathcal{M}}, g, f)$ is an *oriented matroid program*, where H_f is interpreted as (a level plane of) a linear objective function on the affine arrangement $(\mathcal{M}, g)$, see [BLSWZ, Sect. 10.1]. The *graph* G_f has the affine vertices of $(\mathcal{M}, g)$ as its nodes, and the edges between them are the bounded edges of $(\mathcal{M}, g)$, directed according to increasing f, that is, according to the direction in which their line cuts the level-plane H_f. Assuming that $\widetilde{\mathcal{M}}$ is uniform, there are no horizontal (undirected) edges. In general, the program can be *non-euclidean* [EdM], so that there are directed cycles in the graph G_f. The following non-trivial result is the technical key to our development.

Proposition 3.6. *If $(\mathcal{M}, g) = (\mathrm{C}^{n+1,r+1}, n+1)$, then the graph G_f is acyclic for any program $(\mathcal{M} \cup f, g, f)$.*

Proof. See [StZ, Prop. 4.7/Thm. 4.12]. □

4. Structure of Higher Bruhat Orders

With the preparations of Sections 2 and 3, we can now prove the following main theorem.

Theorem 4.1. *Let $1 \leq k \leq n$ and $r := n-k$.*

(B) *There is a natural isomorphism of posets between*

1. *the higher Bruhat order $B(n,k)$,*
2. *the set of all consistent subsets of $\binom{[n]}{k+1}$, ordered by single-step inclusion,*
3. *the set of extensions of the cyclic arrangement $\mathbf{X}_c^{n,r-1}$ by a new pseudo-hyperplane in general position, ordered by single-step inclusion of their vertex sets, and*
4. *the poset $\mathcal{U}(\mathrm{C}^{n+1,r}, n+1)$ of all uniform single element extensions of $\mathrm{C}^{n,r}$.*

(G) *The poset $B(n,k)$ is a graded poset of length $\binom{n}{k+1}$. Its rank function is $r(U) = |U|$. The unique minimal element is $\hat{0} = \emptyset$, the unique maximal element is $\hat{1} = \binom{[n]}{k+1}$.*

(A′) *There is a natural bijection between*

1. *the set $B(n,k)$,*
2. *the posets $Q[\rho]$, for $\rho \in A(n,k)$, and*
3. *the different ways to assign directions to the 1-dimensional lines of $X_c^{n,r}$ without creating directed cycles.*

(A) *There is a natural bijection between*

1. *the set of admissible orderings $A(n,k)$,*
2. *the maximal chains of the poset $B(n,k-1)$, and*
3. *the different ways to sweep the arrangement $\mathbf{X}_c^{n,r}$ by a generic pseudo-hyperplane.*

We note that the geometric statements of (A3) and (B3) have precise geometric meaning in the axiomatic setting of pseudoarrangements provided by oriented matroid theory, see [BLSWZ, Chap. 5].

Theorem 4.1 also contains the main results of Manin & Schechtman: the bijection (A1)↔(A2) is [MaS, §2 Thm. 3c], while the part (G) is [MaS, §2 Thm. 3b]. Furthermore, after applying oriented matroid duality (B1)↔(B4) is a bijection between $B(n,k)$ and the single element liftings of $\mathrm{C}^{n,k}$: such a bijection is stated (without a proof) by Kapranov & Voevodsky [KaV, Thm. 4.9].

Proof. We start with part **(B)**. For this let $U \subseteq \binom{[n]}{k+1}$ be consistent. By Corollary 2.6(ii), U is also consistent as a subset of $\binom{[n+1]}{k+1}$, and thus by Lemma 3.3 it defines an extension of $\mathbf{X}_c^{n+1,r}$ by a new pseudo-hyperplane H_f (cf. Figure 2). We now treat H_{n+1} as the hyperplane at infinity, with $g := n+1$. With this we get an oriented matroid program $(\mathrm{C}^{n+1,r+1} \cup f, n+1, f)$. Its affine vertices are the vertices of $\mathbf{X}_c^{n+1,r}$ that do not lie on H_{n+1}, so they correspond to the $(k+1)$-subsets of $[n+1]$ that contain $n+1$. The lines that are not contained in H_{n+1} correspond to the $(k+1)$-packets $P(K)$ with $n+1 \in K$. Setting $K = \{i_1 < i_2 < \ldots < i_{k+2}\}$ and $J := K \backslash n+1$, we get from Lemma 3.2 that the vertices on such a line are given by

$$J = K\backslash n+1 \quad — \quad K\backslash i_{k+1} \quad — \quad \ldots \quad — \quad K\backslash i_2 \quad — \quad K\backslash i_1.$$

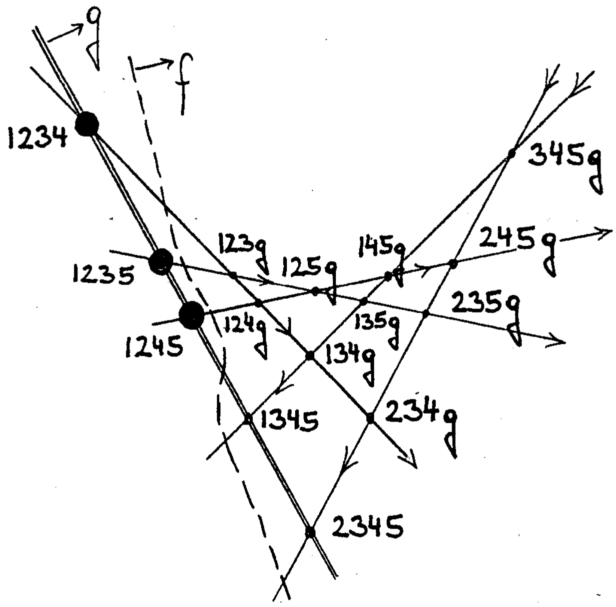

Figure 2: The cyclic arrangement $\mathrm{X}_c^{6,2}$, with $g := n{+}1 = 6$. The consistent set $U = \{1234, 1235, 1245\}$ induces the extension by H_f, directions of the lines of $\mathrm{X}_c^{5,2}$ and thus a partial order on $\binom{[5]}{3}$.

Thus the graph G_f of the program has the vertex set $V = \{I \in \binom{[n+1]}{k+1} : n{+}1 \in I\} = \binom{[n]}{k} * \{n{+}1\}$, with directed edges (cf. Figure 2)

$$\begin{cases} K\backslash i_{k+1} \longrightarrow \dots \longrightarrow K\backslash i_2 \longrightarrow K\backslash i_1 & \text{if } J \in U, \\ K\backslash i_{k+1} \longleftarrow \dots \longleftarrow K\backslash i_2 \longleftarrow K\backslash i_1 & \text{if } J \notin U. \end{cases}$$

By Proposition 3.6, the graph is acyclic. Thus G_f defines a partial order "$\leq$" on $\binom{[n]}{k}$ by

$$I \leq I' \quad :\Longleftrightarrow \quad G_f \text{ contains a directed path from } I' \cup \{n{+}1\} \text{ to } I \cup \{n{+}1\},$$

and by construction we have

$$\begin{cases} I\backslash i_{k+1} > \dots > I\backslash i_2 > I\backslash i_1 & \text{if } J \in U, \\ I\backslash i_{k+1} < \dots < I\backslash i_2 < I\backslash i_1 & \text{if } J \notin U. \end{cases}$$

Hence every linear extension ρ of the partial order "$\leq$" is an admissible on $\binom{[n]}{k}$ with $\mathrm{inv}(\rho) = U$. With Lemma 2.4, this proves part (B).

For part (**A′**), this also shows that every $U \in B(n,k)$ directs the lines of $X_c^{n,r}$ in an acyclic way, and this determines a partial order $Q[\rho]$ on $\binom{[n]}{k}$. Finally U can be reconstructed from $Q[\rho]$ as $U = \mathrm{inv}(\rho)$ for every linear extension ρ of $Q[\rho]$.

For part (**G**), we have to verify that indeed $\emptyset \leq U \leq \binom{[n]}{k+1}$ for every consistent set $U \subseteq \binom{[n]}{k+1}$. The rest is then clear from the definition of "$\leq$" by single-step inclusion. Given

U, we note that $\widehat{U} = \partial U \cup U*\{n+1\} \subseteq \binom{[n+1]}{k+2}$ is consistent by Lemma 2.6(ii), and thus by (B) defines and extension H_f of $\mathbf{X}_c^{n+1,r-1}$. This defines a graph G_f which is acyclic by Proposition 3.6 and thus defines a partial order $\preceq$ on $\binom{[n]}{k+1}$ as in part (A'). Any linear extension ρ of this partial order is admissible, $\rho \in A(n, k+1)$. By construction, U is an order ideal of $\preceq$, hence the linear extension

$$\rho = (S_1 < S_2 < \ldots < S_{\binom{n}{k+1}}) \in A(n, k+1)$$

can be chosen in such a way that U is a beginning segment of ρ, that is, $U = \{S_1, S_2, \ldots, S_i\}$ for some i. However, every beginning segment $\{S_1, S_2, \ldots, S_m\}$ is consistent. Thus ρ induces a maximal chain $\hat{0} = \emptyset < \{S_1\} < \{S_1, S_2\} < \ldots < \{S_1, S_2, \ldots, S_{\binom{n}{k+1}}\} = \hat{1}$ of length $\binom{n}{k+1}$ in $B(k, n)$ that contains U.

From the same argument we also see **(A)**: every admissible ordering $\rho \in A(n, k)$ induces a maximal chain of length $\binom{n}{k}$ in $B(n, k-1)$. According to part (G) every maximal chain in $B(n, k-1)$ has this form, and the linear orderings on $\binom{[n]}{k}$ induced by maximal chains in $B(n, k-1)$ are clearly admissible. By (B), every maximal chain in $B(n, k-1)$ corresponds to a sequence of pseudohyperplane extensions of $\mathbf{X}_c^{n,r}$ that describes a topological sweep, and conversely. □

The higher Bruhat order $B(5, 2)$ is drawn in Figure 3. Here every element is denoted by the corresponding consistent vertex set of a cyclic arrangement $\mathbf{X}_c^{5,2}$.

We now use Theorem 4.1 to verify that our definition of the partial order on $B(n, k)$ coincides with the one used by Manin & Schechtman.

Corollary 4.2. *$B \subset\!\!\cdot\; B'$ holds for sets $B, B' \in B(n, k)$ if and only if there are admissible orders $\rho, \rho' \in A(n, k)$ with* $\mathrm{inv}(\rho) = B$, $\mathrm{inv}(\rho') = B'$ *and ρ' is obtained from ρ by reversing a single k-packet $P(I)$ whose elements appear in ρ in lexicographic order, with no other elements in between.* (That is, $\mathbf{r} \leq \mathbf{r}' \iff = p_I(\mathbf{r}')$ in the notation of [MaS].)

Proof. The "if" part is clear. For the converse, let $U \subset\!\!\cdot\; U'$, $U' \backslash U = \{I\}$ with $I = \{i_1 < \ldots < i_{k+1}\}$. Consider the line orientations of $\mathbf{X}_c^{n,r}$ corresponding to U and to U' according to Theorem 4.1(A'). They only differ by the reversal of one line, and both are acyclic. For the associated partial orders Q, Q' on $\binom{[n]}{k}$ this means that Q contains the packet $P(I)$, ordered lexicographically, as an interval $[I \backslash i_{k+1}, I \backslash i_1]$, while Q' contains the packet $P(I)$, ordered lexicographically, as an interval $[I \backslash i_1, I \backslash i_{k+1}]$, and Q and Q' differ only in the reversal of this interval. Thus linear extensions ρ of Q and ρ' of Q' can be constructed to satisfy the conditions of the lemma. □

Now we use Theorem 4.1(B) to derive structural properties of the posets $B(n, k)$.

Proposition 4.3. *$B(n, k)$ is isomorphic to a lower interval of $B(n+1, k)$.*

Proof. This is immediate from Lemma 2.6(iii), which shows that there is an order preserving inclusion $B(n, k) \hookrightarrow B(n+1, k)$. □

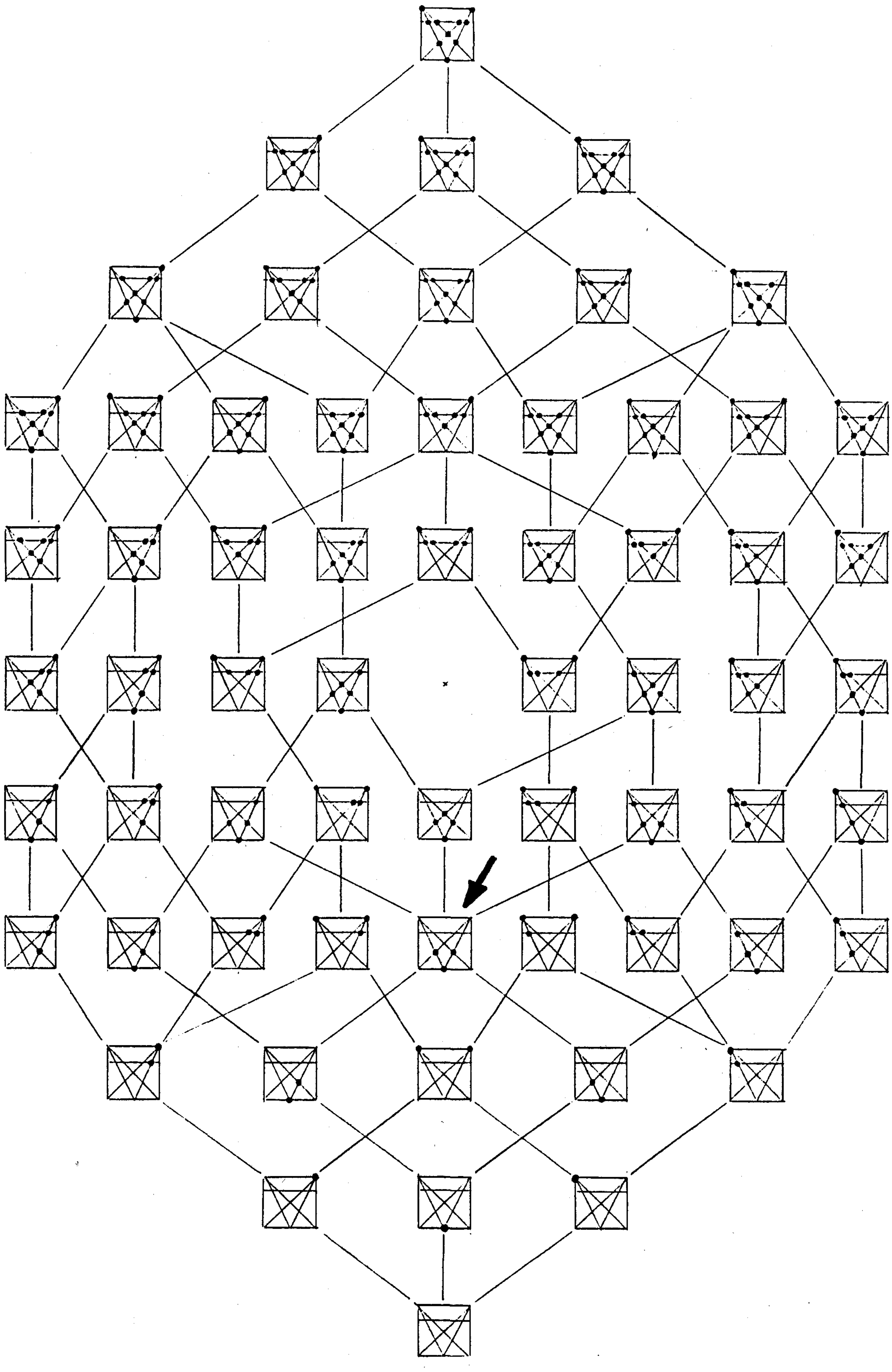

Figure 3: The higher Bruhat order $B(5,2)$.

In contrast to this, it is not clear whether there is an embedding of $B(n,k)$ into $B(n+1,k+1)$ as a subposet. The case $r = 2$ shows that $B(n,k)$ is not an interval of $B(n+1,k+1)$ in general. The map $U \longrightarrow \widehat{U}$ suggested by Lemma 2.6(ii) is an injection, but not order-preserving in general.

The proofs of the following two theorems are linked. We prove, in effect, that

1) $B(n,k+1)$ is a lattice and ordered by inclusion $\Longrightarrow$ $B(n,k)$ is ordered by inclusion,
2) $B(n,k)$ is ordered by inclusion, and $n-k \leq 3$ $\Longrightarrow$ $B(n,k)$ is a lattice.

This can be avoided if one gives an independent proof that $B(n,k)$ is a lattice for $n-k = 3$, which is possible for example by relying on geometric intuition from Figure 1.

Theorem 4.4. *The poset $B(n,k)$ is a lattice for $k = 1$ and for $n-k \leq 3$.*
However, $B(6,2)$ is not a lattice.

Proof. For $k = 1$ the poset $B(n,1)$ is the weak Bruhat order of S_n, which is known to be a lattice, see [YaO, Thm. 2.1] [BEZ].

For $r \leq 2$, the result is trivial. For $r = 3$, we use that $B(n,k)$ is ordered by inclusion by Theorem 4.5. If it is not a lattice, then by [Zie, Crit. 2] there exist six consistent sets $S \subset S \cup \{K_i\} \subset T \backslash \{L_j\} \subset T$ for $i,j \in \{1,2\}$ so that neither $S \cup \{K_1, K_2\}$ nor $T \backslash \{L_1, T_2\}$ are consistent. From this we get that $K_1 \cup K_2 =: K = [n] \backslash h$, where (without loss of generality) K_1 is the smallest and K_2 is the largest set in $P(K)$. Similarly, we get $L_1 \cup L_2 =: L = [n] \backslash h'$, where L_1 is the smallest and L_2 is the largest set in $P(L)$.

By symmetry, we may assume $h < h'$. Now if $h = 1$, then we get $L_2 \in P(K)$, but $T \backslash \{L_2\}$ is consistent, so we get that K_1 or K_2 is not contained in $T \backslash \{L_2\}$, a contradiction. If $h' = n$, then we get $K_1 \in P(L)$, but $S \cup \{K_1\}$ is consistent, so we get that L_1 or L_2 is contained in $S \cup \{K_1\}$, a contradiction. Thus we have $1 < h < h' < n$.

From $K_2 = [n] \backslash \{1,h\} \in T \backslash \{L_2\}$ and $L_2 = [n] \backslash \{1,h'\} \notin T \backslash \{L_2\}$ with $K_2 >_{lex} L_2$ we get $[n] \backslash \{1,n\} \notin T \backslash \{L_2\}$. From $K_1 = [n] \backslash \{n,h\} \in S \cup \{K_1\}$ and $L_2 = [n] \backslash \{n,h'\} \notin S \cup \{K_1\}$ with $K_1 >_{lex} L_1$ we get $[n] \backslash \{1,n\} \in S \cup \{K_1\}$. But this contradicts $S \cup \{K_1\} \subset T \backslash \{L_2\}$.

Now consider $B(6,2)$ and let

$$S = \{123, 124, 356, 456\}, \qquad K_1 = 134, \quad K_6 = 346.$$

Then S, $S \uplus \{K\}$ and $S \uplus \{L\}$ are consistent, while $S \uplus \{K_1, K_6\}$ is not consistent on $P(1346)$. The minimal consistent sets that contain $S \uplus \{K_1, K_6\}$ are

$$S_i := \{123, 124, 356, 456,\ 134, 346,\ 156, 126,\ 25i\} \qquad \text{for } i = 1, 6.$$

They satisfy $S \cup \{K_i\} \leq S_j$ for $i,j \in \{1,6\}$, so $(S \uplus \{K_1\}) \vee (S \uplus \{K_6\})$ does not exist. □

Theorem 4.5. *$B(n,k)$ is ordered by inclusion for $k = 1$ and for $n-k \leq 4$.*
However, $B(8,3)$ is not ordered by inclusion.

Proof. For $k = 1$ this is well known [YaO, Prop. 2.1].

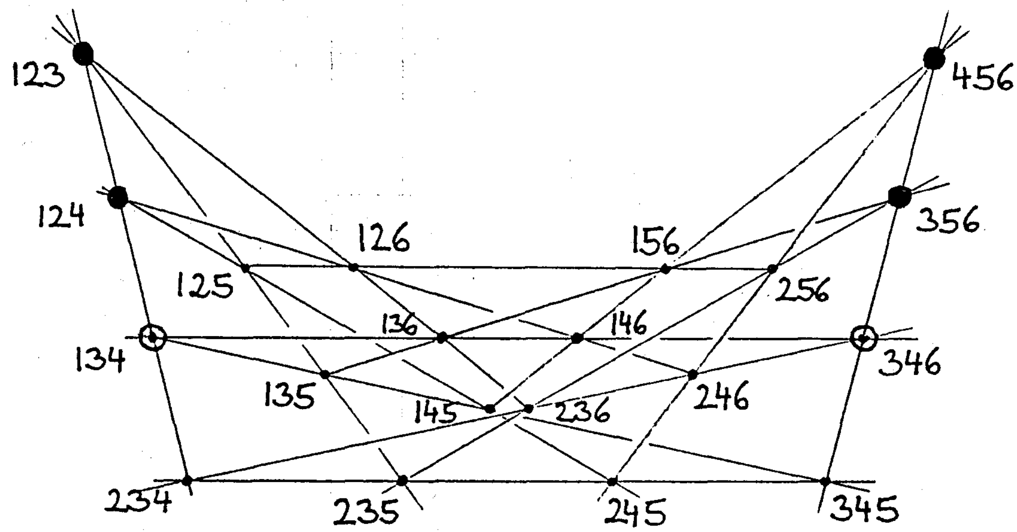

Figure 4: The vertices and lines of $B(6,2)$. The vertices marked by a black dot are in S. The vertices with a white circle are K_1 and K_6.

Let $U_1 \subseteq U_2 \subseteq \binom{[n]}{k+1}$ be consistent. Then by, Theorem 4.1(B), $U_1 \leq U_2$ holds if and only if there exists an admissible linear order ρ of $\binom{[n]}{k+1}$ of which U_1 and U_2 are both beginning segments, that is, so that $U_1 \subseteq U_2$ are ideals of the poset $Q[\rho]$. The consistent set $U \subseteq \binom{[n]}{k+2}$ that corresponds to $Q[\rho]$ by Theorem 4.1(A′) has to satisfy that $U \cup U_1 * \{n+1\}$ and $U \cup U_2 * \{n+1\}$ are both consistent. By Lemma 2.5, this means that

$$\partial U_1 \cup \partial U_2 \subseteq U \qquad \text{and} \qquad \partial\overline{U_1} \cup \partial\overline{U_2} \subseteq \overline{U}. \qquad (*)$$

The sets $\partial U_1, \partial U_2, \partial\overline{U_1}, \partial\overline{U_2} \in B(n,k+1)$ are consistent, with $\partial U_i \cap \partial\overline{U_j} = \emptyset$ for $i,j \in \{1,2\}$, so $\partial U_i \subseteq \overline{\partial\overline{U_j}}$. By induction on $r := n-k$ we can assume that $B(n,k)$ is ordered by inclusion, so $\partial U_i \leq \overline{\partial\overline{U_j}}$ for $i,j \in \{1,2\}$. For $r \leq 4$ we can assume that $B(n,k+1)$ is a lattice (Theorem 4.4), so a consistent set U that satisfies $(*)$ can be chosen arbitrarily from the interval $[\partial U_1 \vee \partial U_2,\ \overline{\partial\overline{U_1}} \leq \overline{\partial\overline{U_2}}]$ of $B(n,k+1)$. Thus $B(n,k)$ is ordered by inclusion for all $r \leq 4$.

The smallest example we know for which U does not exist occurs in $B(8,4)$. This leads to consider the consistent sets

$$U_1 = \{1234, 5678\}$$

$$U_2 = \binom{[8]}{4} \setminus \left\{ \begin{array}{l} 1235, 1245, 1345, 2345, 1236, 1246, 1346, 2346, 1256, \\ 4678, 4578, 4568, 4567, 3678, 3578, 3568, 3567, 3478 \end{array} \right\}$$

in $B(8,3)$ which satisfy $U_1 \subseteq U_2$. We will now give a direct proof for $U_1 \not\leq U_2$, which does avoid the discussion "on the boundary". For this one first has to check that U_1 and

$\overline{U}_2 := \binom{[8]}{4} \backslash U_2$ are consistent. For U_1 this is obvious. For $\overline{U}_2$ one can use that the two rows of our listing both correspond to consistent sets, which can be checked in a situation of rank 3, since no set in the first line contains 7 or 8, while no set in the second line contains 1 or 2. The union of both lines is consistent since no 4-packet contains sets from both lines.

Now assume that $U_1 \leq U_2$. With Theorem 4.1(B) this would imply that there is a linear order "$\prec$" on $\binom{[8]}{4}$ that orders every 4-packet either in lexicographic ("*lex*") or in reversed lexicographic ("*r-lex*") order, and so that if $K \in U_i$, $K' \in \overline{U_i}$ for some i, then $K \prec K'$. Now we get the following sequence of implications:

$$
\begin{array}{llll}
1234 \in U_1,\ 2347 \notin U_1 & \Longrightarrow P(12347)\ \ lex & \Longrightarrow 1237 \prec 2347 & (1)\\
2347 \in U_2,\ 3478 \notin U_2 & \Longrightarrow P(23478)\ \ lex & \Longrightarrow 2347 \prec 2378 & (2)\\
(1)\&(2) \Longrightarrow\ 1237 \prec 2378 & \Longrightarrow P(12378)\ \ lex & \Longrightarrow 1237 \prec 1278 & (3)\\
2567 \notin U_1,\ 5678 \in U_1 & \Longrightarrow P(25678)\ r\text{-}lex & \Longrightarrow 2567 \succ 2678 & (4)\\
1256 \notin U_2,\ 2567 \in U_2 & \Longrightarrow P(12567)\ r\text{-}lex & \Longrightarrow 1267 \succ 2567 & (5)\\
(4)\&(5) \Longrightarrow\ 1267 \succ 2678 & \Longrightarrow P(12678)\ r\text{-}lex & \Longrightarrow 1267 \succ 1278 & (6)\\
1236 \notin U_2,\ 2367 \in U_2 & \Longrightarrow P(12367)\ r\text{-}lex & \Longrightarrow 1237 \succ 1267 & (7)
\end{array}
$$

where (3)&(6)&(7) yield $1237 \succ 1267 \succ 1278 \succ 1237$, a contradiction. □

The question whether $B(n,2)$ is ordered by inclusion for all n remains open. We close this section with a list of the geometric interpretations of $A(n,k)$ and of $B(n,k)$ that are available for small values of r and of k:

- $r = 1$: $A(n,n-1) = B(n,n-1) = \{\hat{0}, \hat{1}\}$.
- $r = 2$: $A(n,n-2)$ is the set of "topological sweeps" on the cyclic line arrangement $\mathbf{X}_c^{n,2}$. $B(n,n-2)$ is the poset of consistent subsets of the affine line $L_{[n]} = P([n])$. Thus $\overline{B}(n,n-2)$ consists of two chains of $n-1$ elements [MaS, §2 Lemma 7]. $B(n,n-2)$ can also be identified with the weak Bruhat order of the dihedral group $I_2(n)$.
- $r = 3$: $B(n,n-3)$ is the set of extensions of the cyclic line arrangement $\mathbf{X}_c^{n,2}$ by a new pseudoline. All these extensions are in fact realizable [Ric, Thm. 8.3].
- $k = 1$: $A(n,1) = B(n,1)$ is the weak order on $\mathcal{S}_n$.
- $k = 2$: $A(n,2)$ is the set of maximal chains in the weak order on $\mathcal{S}_n$, i.e., simple allowable sequences, or arrangements of n pseudolines in "braid form" [GoP] [BLSWZ, Chapt. 6].

  $B(n,2)$ is the set of arrangements of $n+1$ pseudolines that are labeled 1 to n cyclicly at the line $g := n+1$ at infinity. This includes non-realizable arrangements for $n \geq 8$. (See also [KaV, Sect. 4].) The partial order is by single-step inclusion of the triples of pseudolines which determine a triangle of counter-clockwise orientation.

The higher Bruhat orders model the set of minimal paths through a discriminantal arrangement [MaS, §1]. By Theorem 4.1 shows that we have to choose a cyclic arrangement for this. However, in general the poset $B(n,k)$ contains "non-realizable" elements which might not occur in the path space of the arrangement.

5. Sphericity

There are two very natural orderings of the set $B(n,k)$. Up to now, we have taken the ordering by *single-step inclusion* as the primitive one, since it is equivalent to the ordering defined by Manin & Schechtman, by Corollary 4.2. However, it is similarly natural (both from a combinatorial and a geometric viewpoint), to consider the ordering of $B(n,k)$ by *inclusion* as the appropriate generalization of the weak Bruhat order on $\mathcal{S}_n$. We will denote this poset by $B_{\subseteq}(n,k)$. The following theorem collects its main properties.

Theorem 5.1. *Let $B_{\subseteq}(n,k) := \{\mathrm{inv}(\rho) : \rho \in A(n,k)\}$ be the family of consistent sets, ordered by inclusion.*

(1) *$\hat{0} = \emptyset$ is the minimal and $\hat{1} = \binom{[n]}{k+1}$ is the maximal element of $B_{\subseteq}(n,k)$. The length of $B_{\subseteq}(n,k)$ is $\binom{n}{k+1}$, and every element of $B_{\subseteq}(n,k)$ is contained in a maximal chain of this length.*

(2) *$B_{\subseteq}(n,k)$ is graded for $k=1$ and for $r := n-k \leq 4$, but not in general,*

(2′) *$B_{\subseteq}(n,k) = B(n,k)$ for $k=1$ and for $r \leq 4$, but not in general,*

(3) *$B_{\subseteq}(n,k)$ is a lattice for $k=1$ and for $r \leq 3$, but not in general.*

Proof. (1) is Theorem 4.1(G). Note that (2) and (2′) are equivalent restatements of Theorem 4.5. With this, (3) is equivalent to Theorem 4.4. □

The combinatorics of $B_{\subseteq}(n,k)$ is easier to handle than that of $B(n,k)$. Also, in some respects its combinatorics behaves nicer. We will now demonstrate this by computing the homotopy type of $B_{\subseteq}(n,k)$.

Theorem 5.2. *The proper part of the poset $B_{\subseteq}(n,k)$ is homotopy equivalent to an $(r-2)$-sphere:*

$$B_{\subseteq}(n,k) \simeq S^{r-2}.$$

In the case $k=1$ this is a result of Björner [Bj1], which also follows from a theorem of Edelman & Walker [EdW]. The geometric idea of the proof is "adjoint" to that in the proof of [EdW]: it considers the convex hull $conv(V)$ of the set of vertices of the affine arrangement $\mathbf{X}(\mathrm{C}^{n,r})$, which is a simplex, and shows that the poset $\mathcal{U}(\mathrm{C}^{n+1,r}, n+1)$ is homotopy equivalent to the face lattice of the simplex $conv(V)$. A map between these posets is obtained by mapping every extension to the set of vertices of $conv(V)$ that lie on its negative side. To see that this in fact induces a homotopy equivalence, we have to establish several facts, which are collected in the following lemmas.

Denote by $[i,j]$ the interval $\{i, i+1, \ldots, j\}$ in $[n]$, which is empty if $i > j$, and let $K_i := [i, i+k]$. The following lemma also follows by induction from Lemma 2.6(iii).

Lemma 5.3. *For all $i,j \in [n]$, the set $U(i,j) := \{I \in \binom{[n]}{k+1} : I \subseteq [i,j]\}$ is consistent.*

Proof. Let $K \in \binom{[n]}{k+2}$, and note that $U(i,j) \cap P(K) = \{I \in \binom{[n]}{k+1} : I \subseteq K \cap [i,j]\}$.

If $|K \cap [i,j]| \leq k$, then $U(i,j) \cap P(K) = \emptyset$. If $|K \cap [i,j]| = k+2$, then $U(i,j) \cap P(K) = P(K)$. In both cases $U(i,j) \cap P(K)$ is a beginning segment.

Now assume $|K \cap [i,j]| = k+1$, with $K \backslash [i,j] = \{l\}$. In this case we have $U(i,j) = \{[i,j]\}$. But $[i,j]$ is an interval, thus l is either the smallest or the largest element of K. In the first case $[i,j]$ is the last set in the $(k+1)$-packet $P(K)$, and $U(i,j) = \{[i,j]\}$ is an ending segment. In the other case $[i,j]$ is the first set in $P(K)$, and $U(i,j) = \{[i,j]\}$ is a beginning segment. □

Lemma 5.4. *The minimal elements of $B_{\subseteq}(n,k)\backslash\emptyset$ are the sets $\{K_i\} = \{[i,i+k]\}$, for $1 \leq i \leq r$.*

Proof. Note that $\{K_i\} = U(i,i+k)$ is consistent by Lemma 5.3. By Theorem 5.1(1), the minimal non-empty consistent subsets have exactly one element. Let $K = \{i_1 < \ldots < i_{k+1}\}$. If $i_{k+1} - i_1 = k$, then $K = K_{i_1}$. Otherwise we find $j \notin K$ with $i_1 < j < i_{k+1}$, and see that $\{K\}$ is not a beginning or ending segment of $P(K \cup j)$, so $\{K\}$ is not consistent. □

Lemma 5.5. *If $U \subseteq \binom{[n]}{k+1}$ is consistent and $K_l \in U$ for all $l \in [i,j]$, then $U(i,j+k) \subseteq U$.*

Proof. We proceed by induction on j, the claim being trivial for $j < i$, where $U(i,j+k) = \emptyset$, and for $j = k$, where $U(i,j+k) = \{K_i\}$.

Let $K \in U(i,j+1+k)\backslash U(i,j+k) = \{I \in \binom{[n]}{k+1} : j+1+k \in I \subseteq [i,j+1+k]\}$ be an element of $U(i,j+1+k)$ that is not in U, and assume that it is selected such that the sum of its elements is maximal.

Since $K \neq [j+1,j+1+k] = K_{j+1}$, we can find $l \in [j+1,j+1+k]\backslash K$. Consider the $(k+1)$-packet $P(K \cup l)$: its smallest element $(K \cup l)\backslash j+1+k$ is in $[i,j+k]$ and hence in U by induction. Its largest element $(K \cup l)\backslash k_1$ is in U, since $k_1 := \min K < l$ by construction, hence $(K \cup l)\backslash k_1$ has a smaller sum of elements than K, and hence it is in U by the choice of K. Thus we get a contradiction to consistency on the $(k+1)$-packet $P(K \cup l)$. □

Lemma 5.6. *If $i_1, j_1, \ldots, i_l, j_l$ are such that $|[i_s,j_s] \cap [i_t,j_t]| < k$ for all $s < t$, then $U(i_1,j_1) \cup \ldots \cup U(i_l,j_l)$ is consistent.*

Proof. Suppose not, then this union is inconsistent on some $(k+1)$-packet $P(K)$. However, since each of the sets $U(i_l,j_l)$ is consistent by Lemma 5.5, this requires that there are $I \in U(i_s,j_s) \cap P(K)$ and $J \in U(i_t,j_t) \cap P(K)$ with $s < t$. From this we get $I \cap J \subseteq [i_s,j_s] \cap [i_t,j_t]$, hence $|I \cap J| < k$, and I, J cannot be contained in the same $(k+1)$-packet $P(K)$. □

Finally we need the following "Crosscut Lemma" to establish the homotopy equivalence of posets. It is a very special case of the Crosscut Theorem [Bj2, (10.8)]. It can easily be derived from Quillen's Fiber Theorem [Bj2, (10.5)].

Lemma 5.7. *Let Q be a poset with $\hat{0}$ and $\hat{1}$, and let $A := \min(\overline{Q})$ be the set of $a := |A|$ atoms in Q. Assume that every subset of A has a join in Q, with $\vee A = \hat{1}$ and $\vee B < \hat{1}$ for $B \subset A$. Then Q is homotopy equivalent to the $(a-2)$-sphere, $Q \simeq S^{a-2}$.*

Note that the joins $\vee B$ exist in particular if Q is a lattice. However, we will have to use the greater generality of the above formulation, since by Theorem 5.1(3) the posets $B_{\subseteq}(n,k)$ are not in general lattices.

Proof of Theorem 5.2. We apply Lemma 5.7 to the bounded poset $Q = B_{\subseteq}(n,k)$, whose minimal elements are $\{K_1\}, \ldots, \{K_r\}$, by Lemma 5.4. We write $v(i) := \{K_i\}$, so $A = \{v(1), \ldots, v(r)\}$, and $a = |A| = r$.

Every $B \subseteq A$ can be written uniquely as

$$B = \{v(i) : i \in [i_1, j_1] \cup [i_2, j_2] \cup \ldots \cup [i_l, j_l]\} \qquad \text{with} \quad i_{t+1} - j_t \geq 2.$$

Now if $U \in B_{\subseteq}(n,k) = Q$ satisfies $B \subseteq U$, then $U(i_1, j_1+k) \cup \ldots \cup U(i_l, j_l+k) \subseteq U$ follows from consistency of U (Lemma 5.5). We get that $U(i_1, j_1+k) \cup \ldots \cup U(i_l, j_l+k)$ is consistent from Lemma 5.6: $j_s+k - i_t = k - (i_t - j_s) \leq k - (i_{t+1} - j_s) \leq k - 2$ for $s < t$ implies $|[i_s, j_s+k] \cap [i_t, j_t+k]| = |[i_t, j_s+k]| \leq k - 1$. Thus $U(i_1, j_1+k) \cup \ldots \cup U(i_l, j_l+k)$ is the join of B, and this is $\hat{1} = U(1,n) = U(1, r+k)$ exactly if $B = \{v(1), \ldots, v(r)\} = A$. By Lemma 5.7, $B_{\subseteq}(n,k)$ is homotopy equivalent to the $(a-2)$-sphere. □

The same proof technique cannot be used to determine the homotopy type of $B(n,k)$ in general: in $B(8,3)$ we have atoms $v(1) = \{1234\}$ and $v(5) = \{5678\}$. The sets $U_1 = \{1234, 5678\}$ and U_2 are both upper bounds of $v(1)$ and $v(5)$ in $B(8,3)$: for U_1 this is obvious, while for U_2 it is implied by the proof technique of Theorem 4.5, since $\partial\{5678\}$ and $\partial\overline{\{1234\}} = \emptyset$. However, in Theorem 4.5 we have shown $U_1 \not\leq U_2$, so the atoms $v(1)$ and $v(5)$ do not have a join in $B(n,k)$. However, Theorem 5.2 implies that $B(n,k)$ is spherical in all cases where $B(n,k)$ and $B_{\subseteq}(n,k)$ coincide.

Corollary 5.8. *For $k = 1$ and for $n - k \leq 4$, the proper part of the higher Bruhat order $B(n,k)$ is homotopy equivalent to an $(r-2)$-sphere:*

$$B(n,k) \simeq S^{r-2} \qquad \text{for } k = 1 \text{ and for } n - k \leq 4.$$

It would be interesting to study the combinatorics of intervals both in $B_{\subseteq}(n,k)$ and in $B(n,k)$. In particular, one should try to compute the Möbius function, and to determine whether the intervals are always spherical or contractible, as they are in the case $k = 1$ [Bj1].

6. Uniform Extension Spaces

By Theorem 4.1(B), we can interpret $B(n,k)$ as $\mathcal{U}(\mathrm{C}^{n+1,r}, n+1)$ for $r = n-k$, that is, $B(n,k)$ is the set of all uniform extensions of the cyclic oriented matroid $\mathrm{C}^{n,r}$, ordered away from the single element extension $\mathrm{C}^{n+1,r} = \mathrm{C}^{n,r} \cup n+1$.

In this section, we consider the mild generalization obtained by ordering the same set $B(n,k)$ away from a different single element extension $\mathcal{M} = \mathrm{C}^{n,r} \cup g$. This poset is again isomorphic to $B(n,k)$ in the case $k = 1$, but not in general. The poset we get is the uniform extension poset $\mathcal{U}(\mathcal{M}, g)$ of Definition 3.4, if we order by *single-step inclusion*. Also we want to consider $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$ in this case, the same set ordered by *inclusion* of inversion sets. We will see that the main structural properties (Sections 4 and 5) of $B(n,k)$ and $B_{\subseteq}(n,k)$ do not generalize to $\mathcal{U}(\mathcal{M}, g)$ or to $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$, even in the case of $\mathcal{M}\backslash g \cong \mathrm{C}^{n,r}$ and in the case of rank case $r = 3$, where $B(n,k) = B_{\subseteq}(n,k)$ is very well-behaved.

For the following $(\mathcal{M}, g)$ will denote a uniform affine rank 3 oriented matroid on the ground set $[n] \uplus g$, with $\mathcal{M}\backslash g = \mathrm{C}^{n,3}$. $\mathbf{X}$ denotes a realization of $(\mathcal{M}, g)$ by an affine arrangement of n hyperplanes in $\mathbb{R}^2$. A set of vertices of $\mathbf{X}$ is *consistent* if it is the vertex set (in the sense of Lemma 3.3) of a uniform extension $f \in \mathcal{U}(\mathcal{M}, g)$. Again we set $k := n - r$.

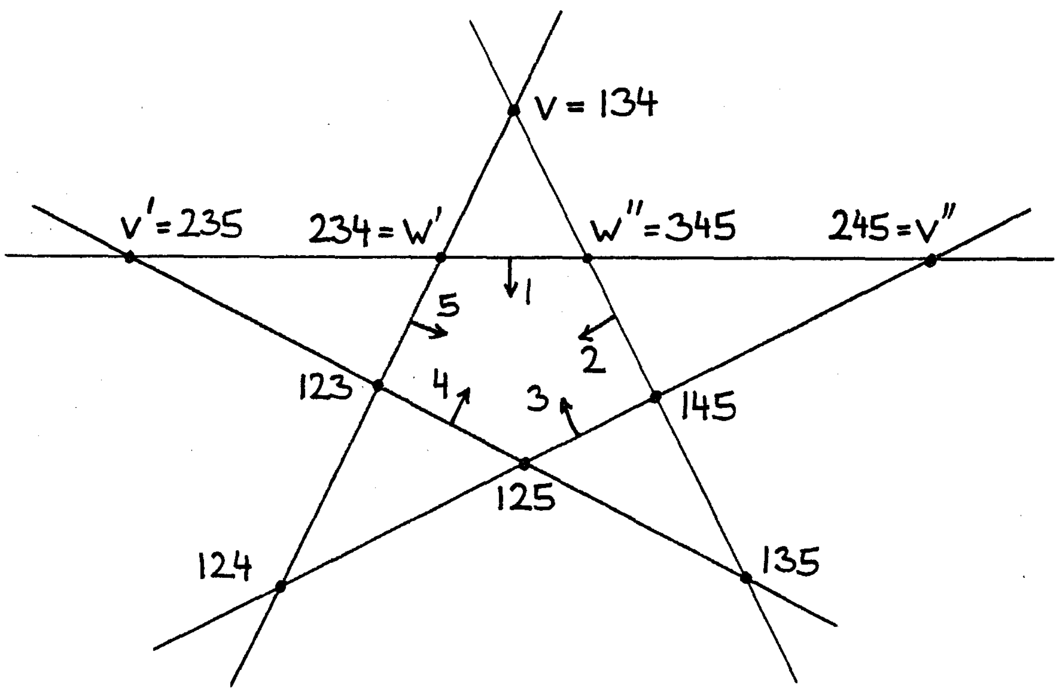

Figure 5: An affine arrangement in $\mathbb{R}^2$ for which $\mathcal{U}(\mathcal{M}, g)$ is not a lattice.

Example 6.1. Let $\mathbf{X}$ be the affine arrangement in $\mathbb{R}^2$ sketched in Figure 5. We have $d = 2$, $r = 3$, $n = 5$ and $k = 2$. The arrangement is generic: the corresponding affine matroid $(\mathcal{M}, g)$ is uniform, with $\mathcal{M}\backslash g \cong C^{5,3}$. The poset diagram of $\mathcal{U}(\mathcal{M}, g))$ can be obtained from Figure 3 by directing its graph away from the vertex that corresponds to $\mathcal{M}$. This vertex corresponds to the extension of Figure 1; it is marked by an arrow in Figure 3.

For this arrangement $\mathcal{U}(\mathcal{M}, g) = \mathcal{U}_{\subseteq}(\mathcal{M}, g)$ is not a lattice. In fact, denote the vertex set of $\mathbf{X}$ by V, and let v, v', v'', w', w'' be the vertices marked in Figure 5. Then the atoms $\{v'\}$ and $\{w''\}$ do not have a join: their minimal upper bounds are $V\backslash\{v\}$ and $\{v, v', v'', w', w''\}$.

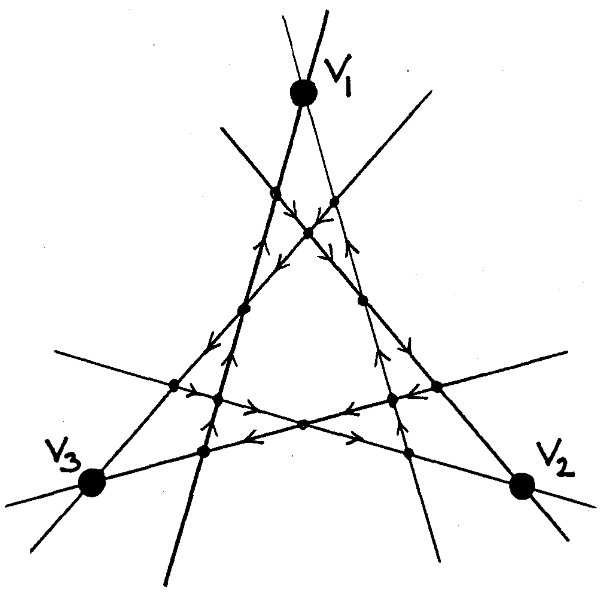

Figure 6: An affine arrangement in $\mathbb{R}^2$ for which $\mathcal{U}(\mathcal{M}, g)$ is not bounded.

Example 6.2. Let $\mathbf{X}$ be the affine arrangement in $\mathbb{R}^2$ sketched in Figure 6. We have $d = 2$, $r = 3$, $n = 6$ and $k = 3$. The arrangement is generic: the corresponding affine matroid $(\mathcal{M}, g)$ is uniform, with $\mathcal{M}\backslash g \cong C^{6,3}$.

For this arrangement $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$ is not bounded and not ordered by inclusion, and $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$ is not graded. In fact, denote the vertex set of $\mathbf{X}$ by V, and let $\{v_1, v_2, v_3\}$ be the vertices marked in Figure 6. The vertex set $\{v_1, v_2, v_3\}$ is consistent. Now let $S \supset \{v_1, v_2, v_3\}$ be consistent. The directed arcs in Figure 6 indicate that if the vertex at the tail end is in S, then the vertex at the head end has to be in S as well. From this it is easy to see that $S = V$: there is no consistent set S with $\{v_1, v_2, v_3\} \subset S \subset V$. In this case $\{v_1, v_2, v_3\} \subset V$ are two different maximal elements of $\mathcal{U}(\mathcal{M}, g)$. Hence we have $\ell(\mathcal{U}_{\subseteq}(\mathcal{M}, g)) = |V| = 15$, while every maximal chain of $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$ that contains $\{v_1, v_2, v_3\}$ is of the form $\hat{0} = \emptyset \subset \{v_i\} \subset \{v_i, v_j\} \subset \{v_1, v_2, v_3\} \subset V = \hat{1}$ and has length 4.

Lemma 4.6 can be applied to $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$. The minimal non-empty vertex sets are given by $A = \{\{v_1\}, \{v_2\}, \{v_3\}, V \backslash \{v_1, v_2, v_3\}\}$, with $a = |A| = 4$. Any union of these sets is a vertex set in $\mathcal{U}(\mathcal{M}, g)$. Thus Lemma 4.6 yields $\mathcal{U}(\mathcal{M}, g) \simeq S^2$, in contrast to Theorem 5.2.

In general, it is not clear how much can be said about the structure of $\mathcal{U}(\mathcal{M}, g)$ or of $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$. There is a close connection between the gradedness problem for $\mathcal{U}(\mathcal{M}, g)$ and "strong euclideanness" [StZ]. For example, we have the following result, which implies Theorem 4.1(G), and also has a similar proof (which we omit).

Proposition 6.3. *If the extension of $\mathcal{M} \backslash g$ by g is lexicographic and if $(\widetilde{\mathcal{M}}, g, f)$ is euclidean for every uniform extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$, then every element of $\mathcal{U}(\mathcal{M}, g)$ is contained in a maximal chain of length $|V| = \binom{n}{k+1}$.*

The search for affine oriented matroids without long chains in $\mathcal{U}(\mathcal{M}, g)$ is related to Las Vergnas' problem about the existence of mutations: if $(\mathcal{M}, g)$ is a uniform affine matroid for which g is not contained in a mutation of $\mathcal{M}$, then no vertex set in $\mathcal{U}(\mathcal{M}, g)$ has size 1, hence $\mathcal{U}(\mathcal{M}, g)$ has no chain of length $|V| = \binom{n}{k+1}$. Also, there is clearly relation between the extension space problem for $\mathcal{M}_0$ and the structure of $\mathcal{U}(\mathcal{M}, g)$.

7. Enumeration

The enumerative combinatorics of the k-analogues $A(n,k)$ is largely unexplored. Denote the size of $A(n,k)$ by $a(n,k)$, and the size of $B(n,k)$ by $b(n,k)$. Tables 1 and 2 list these numbers for small n and k.

k\n	1	2	3	4	5	6
1	1	2	6	24	120	720
2		1	2	16	768	292864
3			1	2	112	?
4				1	2	?
5					1	2
6						1
7						

Table 1: Values for $a(n,k) = |A(n,k)|$.

k\n	1	2	3	4	5	6	7
1	1	2	6	24	120	720	5040
2		1	2	8	62	908	?
3			1	2	10	148	?
4				1	2	12	338
5					1	2	14
6						1	2
7							1

Table 2: Values for $b(n,k) = |B(n,k)|$.

Proposition 7.1. *The following formulas hold for the sizes $a(n,k) := |A(n,k)|$ and $b(n,k) := |B(n,k)|$:*

$a(n,1) = b(n,1) = n!$ *is the size of* $\mathcal{S}_n$,

$a(n,2) = \frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-i}}$,

$b(n,n) = a(n,n) = 1, \quad a(n,n-1) = b(n,n-1) = 2,$

$b(n,n-2) = 2n.$

$b(n,n-3) = 2^n + n2^{n-2} - 2n.$

Proof. The result for $a(n,2)$ is Stanley's [Sta] formula for the number of maximal chains in the weak order of $\mathcal{S}_n$. All the others are trivial except for the size of $B(n,n-3)$, which counts antipodal paths through the poset of regions in the 2-dimensional affine cyclic arrangements. □

There is no explicit formula known for either $a(n,k)$ or for $b(n,k)$. It would also be of interest if one could count both admissible orders and consistent sets with respect to their number of inversions, that is, to determine $a(n,k;q) := \sum_{\rho \in A(n,k)} q^{|\text{inv}(\rho)|}$ and $b(n,k;q) := \sum_{U \in A(n,k)} q^{|U|}$, where $b(n,k;q)$ is also the rank generating function of $B(n,k)$. The answers corresponding to the cases above are given by the following proposition, which uses the notation $(n)_q := 1+q+\ldots+q^{n-1}$, $(n)_q! := (1)_q(2)_q\cdots(n)_q$, $\binom{n}{i}_q := \frac{(n)_q!}{(i)_q!(n-i)_q!}$ and $(2^i)_q := (1+q)\cdot\ldots\cdot(1+q^i)$ for the q-analoga of n, $n!$, $\binom{n}{i}$ and 2^i.

Proposition 7.2.

$a(n,1;q) = b(n,1;q) = (n)_q!$

$b(n,n;q) = a(n,n;q) = 1, \quad a(n,n-1;q) = b(n,n-1;q) = 1+q = (2)_q$

$b(n,n-2;q) = (2)_q(n)_q.$

$$b(n,n-3;q) = (2^{n-1})_q + \sum_{i=1}^{n-2} \Big\{ \big(q^i + q^{\binom{n}{2}-i(n-i)}\big) \left[\tfrac{n-1}{i}\right]_q \; + \\ \big(q^{i(n-i)} + q^{n-1-i}\big)(2^{i-1})_q \cdot (2^{n-2-i})_q \; - \; \big(q^{i(n-i)} + q^{\binom{n}{2}-i(n-i)}\big) \Big\}.$$

Proof. The first formula is well-known. All the others are trivial (using the descriptions of Section 4), except for the last one, which is the q-analogue of the formula for $b(n,n-3)$ in Proposition 7.1. □

There is probably not much hope for a nice general answer. We only note
$a(5,2;q) = 12+14q+38q^2+108q^3+142q^4+140q^5+142q^6+108q^7+38q^8+14q^9+12q^{10}$,
$a(5,3;q) = 12+4q+40q^2+40q^3+4q^4+12q^5$,
$b(5,2;q) = 1+3q+5q^2+9q^3+9q^4+8q^5+9q^6+9q^7+5q^8+3q^9+q^{10}$,
which were computed by computer enumeration. They show that in general the generating polynomials are not unimodal and they do not factor. It would also be of interest to determine the asymptotic behavior of $a(n,k)$ and of $b(n,k)$, in view of their relation with the enumeration of oriented matroids.

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References

[Bj1] A. BJÖRNER: *Orderings of Coxeter groups,* in: Combinatorics and Algebra (C. Greene, ed.), *Contemporary Math.*, **34**, Amer. Math. Soc., 1984, pp. 175–195.

[Bj2] A. BJÖRNER: *Topological methods,* in: Handbook of Combinatorics (R. Graham, M. Grötschel, L. Lovász, eds.), North Holland, to appear.

[BEZ] A. BJÖRNER, P.H. EDELMAN & G.M. ZIEGLER: *Hyperplane arrangements with a lattice of regions, Discrete Comput. Geometry* **5** (1990), 263–288.

[BLSWZ] A. BJÖRNER, M. LAS VERGNAS, B. STURMFELS, N. WHITE, & G.M. ZIEGLER: *Oriented Matroids,* Cambridge University Press, 1991, to appear.

[Ed1] P.H. EDELMAN: *A partial order on the regions of* $\mathbb{R}^n$ *dissected by hyperplanes,* Trans. *Amer. Math. Soc.* **283** (1984), 617–631.

[EdW] P.H. EDELMAN & J.W. WALKER: *The homotopy type of hyperplane posets, Proc. Amer. Math. Soc.* **94** (1985), 329–332.

[EdM] J. EDMONDS & A. MANDEL: *Topology of oriented matroids,* Ph.D. Thesis of A. Mandel, University of Waterloo, 1982.

[GoP] J.E. GOODMAN, R. POLLACK: *Semispaces of configurations, cell complexes of arrangements, J. Combinatorial Theory,* Ser. A, **37** (1984), 257–293.

[KaV] M.M. KAPRANOV & V.A. VOEVODSKY: *Combinatorial-geometric aspects of polycategory theory: Pasting schemes and higher Bruhat order,* Cahiers de Topologie et de la Géometrie Differentielle Catégorique, to appear.

[MaS] YU.I. MANIN & V.V. SCHECHTMAN: *Arrangements of hyperplanes, higher braid groups and higher Bruhat orders,* in: Algebraic Number Theory – *in honor of K. Iwasawa* (J. Coates et al., eds.) *Advanced Studies in Pure Mathematics* **17** (1989), Kinokuniya Company/Academic Press, pp. 289–308.

[MaS1] YU.I. MANIN & V.V. SCHECHTMAN: *Higher Bruhat orders, related to the symmetric group,* Funct. *Anal. Appl.* **20** (1986), 148–150.

[Ric] J. RICHTER-GEBERT: *On the realizability problem for combinatorial geometries – decision methods,* Dissertation, Darmstadt, 1991.

[Sta] R. P. STANLEY *On the number of reduced decompositions of elements of Coxeter groups,* European J. Combinatorics **5** (1984), 359–372.

[StZ] B. STURMFELS & G.M. ZIEGLER: *Extension spaces of oriented matroids,* preprint 1991.

[YaO] T. YANAGIMOTO & M. OKAMOTO: *Partial orderings of permutations and monotonicity of a rank correlation statistic,* Ann. Inst. Statist. Math., **21** (1969), 489–506.

[Zie] G. M. ZIEGLER: *A new local criterion for the lattice property,* preprint.