The Work of Manjul Bhargava

Manjul Bhargava's work in number theory has had a profound influence on the field. A mathematician of extraordinary creativity, he has a taste for simple problems of timeless beauty, which he has solved by developing elegant and powerful new methods that offer deep insights.

When he was a graduate student, Bhargava read the monumental *Disquisitiones Arithmeticae*, a book about number theory by Carl Friedrich Gauss (1777-1855). All mathematicians know of the *Disquisitiones*, but few have actually read it, as its notation and computational nature make it difficult for modern readers to follow. Bhargava nevertheless found the book to be a wellspring of inspiration. Gauss was interested in *binary quadratic forms*, which are polynomials $ax^2+bxy+cy^2$, where a, b, and c are integers. In the *Disquisitiones*, Gauss developed his ingenious *composition law*, which gives a method for composing two binary quadratic forms to obtain a third one. This law became, and remains, a central tool in algebraic number theory. After wading through the 20 pages of Gauss's calculations culminating in the composition law, Bhargava knew there had to be a better way.

Then one day, while playing with a Rubik's cube, he found it. Bhargava thought about labeling each corner of a cube with a number and then slicing the cube to obtain 2 sets of 4 numbers. Each 4-number set naturally forms a matrix. A simple calculation with these matrices resulted in a binary quadratic form. From the three ways of slicing the cube, three binary quadratic forms emerged. Bhargava then calculated the discriminants of these three forms. (The discriminant, familiar to some as the expression "under the square root sign" in the quadratic formula, is a fundamental quantity associated to a polynomial.) When he found the discriminants were all the same, as they are in Gauss's composition law, Bhargava realized he had found a simple, visual way to obtain the law.

He also realized that he could expand his cube-labeling technique to other polynomials of higher degree (the degree is the highest power appearing in the polynomial; for example, x^3-x+1 has degree 3). He then discovered 13 new composition laws for higher-degree polynomials. Up until this time, mathematicians had looked upon Gauss's composition law as a curiosity that happened only with binary quadratic forms. Until Bhargava's work, no one realized that other composition laws existed for polynomials of higher degree.

One of the reasons Gauss's composition law is so important is that it provides information about quadratic *number fields*. A number field is built by extending the rational numbers to include non-rational roots of a polynomial; if the polynomial is quadratic, then one obtains a quadratic number field. The degree of the polynomial and its discriminant are two basic quantities associated with the number field. Although number fields are fundamental objects in algebraic number theory, some basic facts are unknown, such as how many number fields there are for a fixed degree and fixed discriminant. With his new composition laws in hand, Bhargava set about using them to investigate number fields.

Implicit in Gauss's work is a technique called the "geometry of numbers"; the technique was more fully developed in a landmark 1896 work of Hermann Minkowski (1864-1909). In the geometry of numbers, one imagines the plane, or 3-dimensional space, as populated by a lattice that highlights points with integer coordinates. If one has a quadratic polynomial, counting the number of integer lattice points in a certain region of 3-dimensional space provides information about the associated quadratic number field. In particular, one can use the geometry of numbers to show that, for discriminant with absolute value less than X, there are approximately X quadratic number fields. In the 1960s, a more refined geometry of numbers approach by Harold Davenport (1907-1969) and Hans Heilbronn (1908-1975) resolved the case of degree 3 number fields. And then progress stopped. So a great deal of excitement greeted Bhargava's work in which he counted the number of degree 4 and degree 5 number fields having bounded discriminant. These results use his new composition laws, together with his systematic development of the geometry of numbers, which greatly extended the reach and power of this technique. The cases of degree bigger than 5 remain open, and Bhargava's composition laws will not resolve those. However, it is possible that those cases could be attacked using analogues of his composition laws.

Recently, Bhargava and his collaborators have used his expansion of the geometry of numbers to produce striking results about *hyperelliptic curves*. At the heart of this area of research is the ancient question of when an arithmetic calculation yields a square number. One answer Bhargava found is strikingly simple to state: A typical polynomial of degree at least 5 with rational coefficients never takes a square value. A hyperelliptic curve is the graph of an equation of the form $y^2 =$ a polynomial with rational coefficients. In the case where the polynomial has degree 3, the graph is called an *elliptic curve*. Elliptic curves have especially appealing properties and have been the subject of a great deal of research; they also played a prominent role in Andrew Wiles's celebrated proof of Fermat's Last Theorem.

A key question about a hyperelliptic curve is how one can count the number of points that have rational coordinates and that lie on the curve. It turns out that the number of rational points is closely related to the degree of the curve. For curves of degree 1 and 2, there is an effective way of finding all the rational points. For degree 5 and higher, a theorem of Gerd Faltings (a 1986 Fields Medalist) says that there are only finitely many rational points. The most mysterious cases are those of degree 3–namely, the case of elliptic curves–and of degree 4. There is not even any algorithm known for deciding whether a given curve of degree 3 or 4 has finitely many or infinitely many rational points.

Such algorithms seem out of reach. Bhargava took a different tack and asked, what can be said about the rational points on a *typical* curve? In joint work with Arul Shankar and also with Christopher Skinner, Bhargava came to the surprising conclusion that a positive proportion of elliptic curves have only one rational point and a positive proportion have infinitely many. Analogously, in the case of hyperelliptic curves of degree 4, Bhargava showed that a positive proportion of such curves have no rational points and a positive proportion have infinitely many rational points. These works necessitated counting lattice points in unbounded regions of high-dimensional space, in which the regions spiral outward in complicated "tentacles". This counting could not have been done without Bhargava's expansion of the geometry of numbers technique.

Bhargava also used his expansion of the geometry of numbers to look at the more general case of higher degree hyperelliptic curves. As noted above, Faltings' theorem tells us that for curves of degree 5 or higher, the number of rational points is finite, but the theorem does not give any way of finding the rational points or saying exactly how many there are. Once again, Bhargava examined the question of what happens for a "typical" curve. When the degree is even, he found that the typical hyperelliptic curve has no rational points at all. Joint work with Benedict Gross, together with follow-up work of Bjorn Poonen and Michael Stoll, established the same result for the case of odd degree. These works also offer quite precise estimates of how quickly the number of curves having rational points decreases as the degree increases. For example, Bhargava's work shows that, for a typical degree 10 polynomial, there is a greater than 99% chance that the curve has no rational points.

A final example of Bhargava's achievements is his work with Jonathan Hanke on the so-called "290-Theorem". This theorem concerns a question that goes back to the time of Pierre de Fermat (1601-1665), namely, which quadratic forms represent all integers? For example, not all integers are the sum of two squares, so $x^2 + y^2$ does not represent all integers. Neither does the sum of three squares, $x^2 + y^2 + z^2$. But, as Joseph-Louis Lagrange (1736-1813) famously established, the sum of four squares, $x^2 + y^2 + z^2 + w^2$, *does* represent all integers. In 1916, Srinivasa Ramanujan (1887-1920) gave 54 more examples of such forms in 4 variables that represent all integers. What other such "universal" forms could be out there? In the early 1990s, John H. Conway and his students, particularly William Schneeberger and Christopher Simons, looked at this question a different way, asking whether there is a number c such that, if a quadratic form represents integers less than c, it represents all integers. Through extensive computations, they conjectured that c could perhaps be taken as small as 290. They made remarkable progress, but it was not until Bhargava and Hanke took up the question that it was fully resolved. They found a set of 29 integers, up to and including 290, such that, if a quadratic form (in any number of variables) represents these 29 integers, then it represents all integers. The proof is a feat of ingenuity combined with extensive computer programming.

In addition to being one of the world's leading mathematicians, Bhargava is an accomplished musician; he plays the Indian instrument known as the tabla at a professional level. An outstanding communicator, he has won several teaching awards, and his lucid and elegant writing has garnered a prize for exposition.

Bhargava has a keen intuition that leads him unerringly to deep and beautiful mathematical questions. With his immense insight and great technical mastery, he seems to bring a "Midas touch" to everything he works on. He surely will bring more delights and surprises to mathematics in the years to come.

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Biography

Born in 1974 in Canada, Manjul Bhargava grew up primarily in the USA and also spent much time in India. He received his PhD in 2001 from Princeton University, under the direction of Andrew Wiles. Bhargava became a professor at Princeton in 2003. His honors include the Merten M. Hasse Prize of the Mathematical Association of America (2003), the Blumenthal Award for the Advancement of Research in Pure Mathematics (2005), the SASTRA Ramanujan Prize (2005), the Cole Prize in Number Theory of the American Mathematical Society (2008), the Fermat Prize (2011), and the Infosys Prize (2012). He was elected to the U.S. National Academy of Sciences in 2013.