The Work of Artur Avila

Artur Avila has made outstanding contributions to dynamical systems, analysis, and other areas, in many cases proving decisive results that solved long-standing open problems. A native of Brazil who spends part of his time there and part in France, he combines the strong mathematical cultures and traditions of both countries. Nearly all his work has been done through collaborations with some 30 mathematicians around the world. To these collaborations Avila brings formidable technical power, the ingenuity and tenacity of a master problem-solver, and an unerring sense for deep and significant questions.

Avila's achievements are many and span a broad range of topics; here we focus on only a few highlights. One of his early significant results closes a chapter on a long story that started in the 1970s. At that time, physicists, most notably Mitchell Feigenbaum, began trying to understand how chaos can arise out of very simple systems. Some of the systems they looked at were based on iterating a mathematical rule such as $3x(1-x)$. Starting with a given point, one can watch the trajectory of the point under repeated applications of the rule; one can think of the rule as moving the starting point around over time. For some maps, the trajectories eventually settle into stable orbits, while for other maps the trajectories become chaotic.

Out of the drive to understand such phenomena grew the subject of discrete dynamical systems, to which scores of mathematicians contributed in the ensuing decades. Among the central aims was to develop ways to predict long-time behavior. For a trajectory that settles into a stable orbit, predicting where a point will travel is straight-forward. But not for a chaotic trajectory: Trying to predict exactly where an initial point goes after a long time is akin to trying to predict, after a million tosses of a coin, whether the million-and-first toss will be a head or a tail. But one can model coin-tossing probabilistically, using *stochastic* tools, and one can try to do the same for trajectories. Mathematicians noticed that many of the maps that they studied fell into one of two categories: "regular", meaning that the trajectories eventually become stable, or "stochastic", meaning that the trajectories exhibit chaotic behavior that can be modeled stochastically. This dichotomy of regular vs. stochastic was proved in many special cases, and the hope was that eventually a more-complete understanding would emerge. This hope was realized in a 2003 paper by Avila, Welington de Melo, and Mikhail Lyubich, which brought to a close this long line of research. Avila and his co-authors considered a wide class of dynamical systems—namely, those arising from maps with a parabolic shape, known as unimodal maps—

and proved that, if one chooses such a map at random, the map will be either regular or stochastic. Their work provides a unified, comprehensive picture of the behavior of these systems.

Another outstanding result of Avila is his work, with Giovanni Forni, on *weak mixing*. If one attempts to shuffle a deck of cards by only cutting the deck—that is, taking a small stack off the top of the deck and putting the stack on the bottom—then the deck will not be truly mixed. The cards are simply moved around in a cyclic pattern. But if one shuffles the cards in the usual way, by interleaving them—so that, for example, the first card now comes after the third card, the second card after the fifth, and so on—then the deck will be truly mixed. This is the essential idea of the abstract notion of mixing that Avila and Forni considered. The system they worked with was not a deck of cards, but rather a closed interval that is cut into several subintervals. For example, the interval could be cut into four pieces, ABCD, and then one defines a map on the interval by exchanging the positions of the subintervals so that, say, ABCD goes to DCBA. By iterating the map, one obtains a dynamical system called an "interval exchange transformation".

Considering the parallel with cutting or shuffling a deck of cards, one can ask whether an interval exchange transformation can truly mix the subintervals. It has long been known that this is impossible. However, there are ways of quantifying the degree of mixing that lead to the notion of "weak mixing", which describes a system that just barely fails to be truly mixing. What Avila and Forni showed is that almost every interval exchange transformation is weakly mixing; in other words, if one chooses at random an interval exchange transformation, the overwhelmingly likelihood is that, when iterated, it will produce a dynamical system that is weakly mixing. This work is connected to more-recent work by Avila and Vincent Delecroix, which investigates mixing in regular polygonal billiard systems. Billiard systems are used in statistical physics as models of particle motion. Avila and Delecroix found that almost all dynamical systems arising in this context are weakly mixing.

In the two lines of work mentioned above, Avila brought his deep knowledge of the area of analysis to bear on questions in dynamical systems. He has also sometimes done the reverse, applying dynamical systems approaches to questions in analysis. One example is his work on quasi-periodic Schrödinger operators. These are mathematical equations for modeling quantum mechanical systems. One of the emblematic pictures from this area is the Hofstadter butterfly, a fractal pattern named after Douglas Hofstadter, who first came upon it in 1976. The Hofstadter butterfly represents the energy spectrum of an electron moving under an extreme magnetic field.

Physicists were stunned when they noticed that, for certain parameter values in the Schrödinger equation, this energy spectrum appeared to be the Cantor set, which is a remarkable mathematical object that embodies seemingly incompatible properties of density and sparsity. In the 1980s, mathematician Barry Simon popularized the "Ten Martini Problem" (so named by Mark Kac, who offered to buy 10 martinis for anyone who could solve it). This problem asked whether the spectrum of one specific Schrödinger operator, known as the almost-Mathieu operator, is in fact the Cantor set. Together with Svetlana Jitomirskaya, Avila solved this problem.

As spectacular as that solution was, it represents only the tip of the iceberg of Avila's work on Schrödinger operators. Starting in 2004, he spent many years developing a general theory that culminated in two preprints in 2009. This work establishes that, unlike the special case of the almost-Mathieu operator, general Schrödinger operators do not exhibit critical behavior in the transition between different potential regimes. Avila used approaches from dynamical systems theory in this work, including renormalization techniques.

A final example of Avila's work is a very recent result that grew out of his proof of a regularization theorem for volume-preserving maps. This proof resolved a conjecture that had been open for thirty years; mathematicians hoped that the conjecture was true but could not prove it. Avila's proof has unblocked a whole direction of research in smooth dynamical systems and has already borne fruit. In particular, the regularization theorem is a key element in an important recent advance by Avila, Sylvain Crovisier, and Amie Wilkinson. Their work, which is still in preparation, shows that a generic volume-preserving diffeomorphism with positive metric entropy is an ergodic dynamical system.

With his signature combination of tremendous analytical power and deep intuition about dynamical systems, Artur Avila will surely remain a mathematical leader for many years to come.

References

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Biography

Born in Brazil in 1979, Artur Avila is also a naturalized French citizen. He received his PhD in 2001 from the Instituto Nacional de Matemática Pura e Aplicada (IMPA) in Rio de Janeiro, where his advisor was Welington de Melo. Since 2003 Avila has been researcher in the Centre National de la Recherche Scientifique and became a Directeur de recherche in 2008; he is attached to the Institut de Mathématiques de Jussieu-Paris Rive Gauche. Also, since 2009 he has been a researcher at IMPA. Among his previous honors are the Salem Prize (2006), the European Mathematical Society Prize (2008), the Grand Prix Jacques Herbrand of the French Academy of Sciences (2009), the Michael Brin Prize (2011), the Prêmio of the Sociedade Brasileira de Matemática (2013), and the TWAS Prize in Mathematics (2013) of the World Academy of Sciences.