47.

A DEMONSTRATION OF THE THEOREM THAT EVERY HOMOGENEOUS QUADRATIC POLYNOMIAL IS REDUCIBLE BY REAL ORTHOGONAL SUBSTITUTIONS TO THE FORM OF A SUM OF POSITIVE AND NEGATIVE SQUARES.

[*Philosophical Magazine*, IV. (1852), pp. 138—142.]

IT is well known that the reduction of any quadratic polynomial

$$(1, 1)\, x^2 + 2\,(1, 2)\, xy + (2, 2)\, y^2 + \ldots + (n, n)\, t^2$$

to the form $a_1\zeta^2 + a_2\eta^2 + \ldots + a_n\theta^2$, where $\zeta, \eta \ldots \theta$ are linear functions of $x, y \ldots t$, such that $x^2 + y^2 + \ldots + t^2$ remains identical with $\zeta^2 + \eta^2 + \ldots + \theta^2$ (which identity is the characteristic test of orthogonal transformation), depends upon the solution of the equation

$$\begin{vmatrix} (1, 1) + \lambda, & (1, 2) \ldots\ldots\ldots (1, n) \\ (2, 1), & (2, 2) + \lambda \ldots (2, n) \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ (n, 1), & (n, 2) \ldots\ldots\ldots (n, n) + \lambda \end{vmatrix} = 0.$$

The roots of this equation give $a_1, a_2 \ldots a_n$; and if they are real, it is easily shown that the connexions between $x, y \ldots t$; $\zeta, \eta \ldots \theta$, are also real. M. Cauchy has somewhere given a proof of the theorem*, that the roots of λ in the above equation must necessarily always be real; but the annexed demonstration is, I believe, new; and being very simple, and reposing upon a theorem of interest in itself, and capable no doubt of many other applications, will, I think, be interesting to the mathematical readers of this *Magazine*.

* Jacobi and M. Borchardt have also given demonstrations; that of the latter consists in showing that Sturm's functions for ascertaining the total number of real roots expressed by my formulæ (many years ago given in this *Magazine*) are all, in the case of $f(\lambda)$, representable as the sums of squares, and are therefore essentially positive.

Let

$$f(\lambda) = \begin{vmatrix} (1,1)+\lambda, & (1,2) \ldots\ldots & (1,n) \\ (2,1), & (2,2)+\lambda \ldots\ldots & (2,n) \\ (3,1), & (3,2), \quad (3,3)+\lambda \ldots & (3,n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ (n,1), & (n,2) \ldots\ldots & (n,n)+\lambda \end{vmatrix},$$

is easily proved that $f(\lambda) \times f(-\lambda)$

$$= \begin{vmatrix} [1,1]-\lambda^2, & [1,2] \ldots\ldots & [1,n] \\ [2,1], & [2,2]-\lambda^2 \ldots & [2,n] \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ [n,1], & [n,2] \ldots\ldots & [n,n]-\lambda^2 \end{vmatrix},$$

where $[\iota, \epsilon] = (\iota, 1) \times (1, \epsilon) + (\iota, 2) \times (2, \epsilon) + \ldots + (\iota, n) \times (n, \epsilon)$.

If, now, for all values of r and s, $(r, s) = (s, r)$, that is, if $f(0)$ becomes the complete determinant to a symmetrical matrix, then every term $[r, s]$ in the derived matrix becomes a sum of squares, and is essentially positive, and $(-1)^n f(\lambda) \times f(-\lambda)$ assumes the form

$$(\lambda^2)^n - F(\lambda^2)^{n-1} + G(\lambda^2)^{n-2} + \ldots \pm L,$$

where $F, G, \ldots L$ will evidently be all positive; for it may be shown that F will be the sum of the squares of the separate terms, that is, of the last minor determinants of the given matrix, G the sum of the squares of the last but one minors, and so on, L being the square of the complete determinant. For instance, if

$$f(\lambda) = \begin{vmatrix} a+\lambda, & \gamma, & \beta \\ \gamma, & b+\lambda, & \alpha \\ \beta, & \alpha, & c+\lambda \end{vmatrix}$$

$$-f(\lambda) \times f(-\lambda) = \lambda^6 - F\lambda^4 + G\lambda^2 - H,$$

where

$$F = a^2 + b^2 + c^2 + 2\alpha^2 + 2\beta^2 + 2\gamma^2,$$

$$G = (ab - \gamma^2)^2 + (bc - \alpha^2)^2 + (ac - \beta^2)^2$$
$$+ 2(a\alpha - \beta\gamma)^2 + 2(b\beta - \gamma\alpha)^2 + 2(c\gamma - \alpha\beta)^2,$$

$$H = \begin{vmatrix} a, & \gamma, & \beta \\ \gamma, & b, & \alpha \\ \beta, & \alpha, & c \end{vmatrix}^2.$$

Hence it follows immediately that $f(\lambda) = 0$ cannot have imaginary roots; for, if possible, let $\lambda = p + q\sqrt{(-1)}$, and write

$$a + p = a', \quad b + p = b', \quad c + p = c', \quad \lambda + p = \lambda',$$

$f(\lambda)$ becomes
$$\begin{vmatrix} a'+\lambda', & \gamma, & \beta \\ \gamma, & b'+\lambda', & \alpha \\ \beta, & \alpha, & c'+\lambda' \end{vmatrix},$$
or say $\phi(\lambda')$, and the equation $\phi(\lambda')\times\phi(-\lambda')=0$ will be of the form
$$\lambda'^6 - F'\lambda'^4 + G'\lambda'^2 - H' = 0,$$
where F', G', H' are all essentially positive. Hence, by Descartes' rule, no value of λ'^2 can be negative, that is, $(\lambda-p)^2$ cannot be of the form $-q^2$; that is to say, it is impossible for any of the roots of $f(\lambda)=0$ to be imaginary, or, as was to be demonstrated, all the roots are real.

I may take this occasion to remark, that by whatever linear substitutions, orthogonal or otherwise, a given polynomial be reduced to the form $\Sigma A_1\zeta^2$, the number of positive and negative coefficients is invariable: this is easily proved. If now we proceed to reduce the form (expressed under the umbral notation) $(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2$ to the form
$$A_1\zeta_1^2 + A_2\zeta_2^2 + \ldots + A_{n-1}\zeta^2_{n-1} + A_n\zeta_n^2,$$
by first driving out the mixed terms in which x_1 enters, then those in which x_2 enters, and so forth until eventually only x_n of the original variables is left, it may readily be shown that
$$A_1 = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix},\quad A_2 = \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \div \begin{pmatrix} a_1 \\ a_1 \end{pmatrix},\quad A_3 = \begin{pmatrix} a_1a_2a_3 \\ a_1a_2a_3 \end{pmatrix} \div \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \ldots\ldots$$
$$\ldots A_n = \begin{pmatrix} a_1a_2\ldots a_n \\ a_1a_2\ldots a_n \end{pmatrix} \div \begin{pmatrix} a_1a_2\ldots a_{n-1} \\ a_1a_2\ldots a_{n-1} \end{pmatrix}.$$
It follows, therefore, that in whatever order we arrange the umbræ $a_1a_2\ldots a_n$, the number of variations and of continuations of sign in the series
$$1,\ \begin{pmatrix} a_1 \\ a_1 \end{pmatrix},\ \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix}\ldots\begin{pmatrix} a_1a_2\ldots a_n \\ a_1a_2\ldots a_n \end{pmatrix},$$
will be invariable, and in fact will be the same as the number of positive and negative roots in the generating function in λ above treated of, that is, since all the roots are real, will be the same as the number of variations and continuations in the series formed by the coefficients of the several powers of λ, that is
$$1,\ \Sigma\begin{pmatrix} a_1 \\ a_1 \end{pmatrix},\ \ \Sigma\begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix}\ldots\begin{pmatrix} a_1a_2\ldots a_n \\ a_1a_2\ldots a_n \end{pmatrix}.$$

The first part of this theorem admits of an easy direct demonstration; for by my theory of compound determinants, given in this *Magazine**, we know that
$$\begin{array}{cc} \overline{a_1a_2\ldots a_{r-1}a_r|} & \overline{a_1a_2\ldots a_{r-1}a_{r+1}|} \\ a_1a_2\ldots a_{r-1}a_r & a_1a_2\ldots a_{r-1}a_{r+1} \end{array}$$
$$= \begin{pmatrix} a_1a_2\ldots a_{r-1} \\ a_1a_2\ldots a_{r-1} \end{pmatrix} \times \begin{pmatrix} a_1a_2\ldots a_{r-1}a_ra_{r+1} \\ a_1a_2\ldots a_{r-1}a_ra_{r+1} \end{pmatrix}.$$

[* Cf. pp. 241, 252 above.]

The first member of this equation is equivalent to

$$\begin{pmatrix} a_1 a_2 \ldots a_{r-1} a_r \\ a_1 a_2 \ldots a_{r-1} a_r \end{pmatrix} \times \begin{pmatrix} a_1 a_2 \ldots a_{r-1} a_{r+1} \\ a_1 a_2 \ldots a_{r-1} a_{r+1} \end{pmatrix} - \begin{pmatrix} a_1 a_2 \ldots a_{r-1} a_r \\ a_1 a_2 \ldots a_{r-1} a_{r+1} \end{pmatrix}^2.$$

Hence it follows, that if the two factors on the right-hand side of the equation have the same sign,

$$\begin{pmatrix} a_1 a_2 \ldots a_{r-1} a_r \\ a_1 a_2 \ldots a_{r-1} a_r \end{pmatrix} \text{ and } \begin{pmatrix} a_1 a_2 \ldots a_{r-1} a_{r+1} \\ a_1 a_2 \ldots a_{r-1} a_{r+1} \end{pmatrix}$$

have also the same sign *inter se*, and consequently the two triads

$$\begin{bmatrix} a_1 a_2 \ldots a_{r-1} \\ a_1 a_2 \ldots a_{r-1} \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \ldots a_{r-1} a_r \\ a_1 a_2 \ldots a_{r-1} a_r \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \ldots a_{r-1} a_r a_{r+1} \\ a_1 a_2 \ldots a_{r-1} a_r a_{r+1} \end{bmatrix},$$

and

$$\begin{bmatrix} a_1 a_2 \ldots a_{r-1} \\ a_1 a_2 \ldots a_{r-1} \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \ldots a_{r-1} a_{r+1} \\ a_1 a_2 \ldots a_{r-1} a_{r+1} \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \ldots a_{r-1} a_{r+1} a_r \\ a_1 a_2 \ldots a_{r-1} a_{r+1} a_r \end{bmatrix},$$

will in all cases present the same number of changes and continuations, which proves that the contiguous umbræ, a_r, a_{r+1}, may be interchanged without affecting the number of variations and continuations in the entire series; but, as is well known, any one order of elements is always convertible into any other order by means of successive interchanges of contiguous elements, which demonstrates that, in whatever order the elements $a_1, a_2 \ldots a_n$ be arranged, the number of continuations and variations in

$$1, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} a_1 a_2 \\ a_1 a_2 \end{pmatrix} \ldots \begin{pmatrix} a_1 a_2 \ldots a_n \\ a_1 a_2 \ldots a_n \end{pmatrix},$$

is invariable. But that the same thing is true (as we know it to be), for the relation between any one of these unsymmetrical series and the symmetrical series (resulting from the method of orthogonal transformation)

$$1, \Sigma \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \Sigma \begin{pmatrix} a_1 a_2 \\ a_1 a_2 \end{pmatrix}, \ldots \begin{pmatrix} a_1 a_2 \ldots a_n \\ a_1 a_2 \ldots a_n \end{pmatrix},$$

is by no means so easily demonstrable in the general case by a direct method, and the attention of algebraists is invited to supply such direct method of demonstration. My knowledge of the fact of this equivalence is, as I have stated, deduced from that remarkable but simple law to which I have adverted, which affirms the invariability of the number of the positive and negative signs between all linearly equivalent functions of the form $\Sigma \pm c_r x^r$ (subject, of course, to the condition that the equivalence is expressible by means of equations into which only real quantities enter); a law to which my view of the physical meaning of quantity of matter inclines me, upon the ground of analogy, to give the name of the Law of Inertia for Quadratic Forms, as expressing the fact of the existence of an invariable number inseparably attached to such forms.