A STABLE DECOMPOSITION OF $\Omega^n S^n X$

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If X is a countable, connected CW complex it is well known (e.g. [**4**; §2.1]) that the reduced product spaces, X_∞, of [**1**] and their combinatorial extension properties imply the existence of a homotopy decomposition

$$S\Omega SX \simeq SX_\infty \simeq \bigvee_{j=1}^{\infty} S\left(\bigwedge_1^j X\right).$$

If X is a connected, compactly generated Hausdorff space, the spaces, $C_n X$, constructed in [3] give a weak homotopy approximation to $\Omega^n S^n X$. Using these and their stable combinatorial extension properties we derive a decomposition of $S^\infty C_n X$ similar to those of [**2**; §§3.6, 3.7] in the case $n = \infty$.

In §1 we derive the stable decomposition from the existence of weak homotopy equivalences constructed in §3 and summarised in Theorems 3.1, 3.2. In §2 we recall the spaces, $C_n Y$, and define related spaces, $D_n(Y, A)$, where A is a closed subspace of Y. The $D_n(Y, A)$ are slightly larger than the $C_n Y$ and hence more convenient for the construction of maps into them. However, just as there are evaluation maps $S^n C_n X \to S^n X$ and $S^n \Omega^n S^n X \to S^n X$ so there is an evaluation map

$$S^n D_n(Y, A) \to S^n(Y/A).$$

This last map is defined and proved continuous in Proposition 2.1. The construction of the weak homotopy equivalences of §3 is made using the composition of the evaluation map with suitable suspension of combinatorially constructed maps into the $D_n(Y, A)$ spaces.

1. Let A and B be spaces. By a *stable weak homotopy equivalence of A with B* we mean an element

$$x \in [S^\infty A, S^\infty B] = \varinjlim\, [S^n A, S^n B]$$

which induces isomorphisms of all stable homotopy groups

$$\pi_i(S^\infty A) = [S^\infty S^i, S^\infty A] \to [S^\infty S^i, S^\infty B] = \pi_i(S^\infty B).$$

The spaces, $C_n X$, are filtered by closed subspaces, $F_k C_n X \subset F_{k+1} C_n X$, $(k \geqslant 0)$.

THEOREM 1.1. *Let X be a connected compactly generated Hausdorff space. There is a stable weak homotopy equivalence,*

$$x \in \left[S^\infty C_n X,\ S^\infty\left(\bigvee_{j=1}^{\infty} (F_j C_n X / F_{j-1} C_n X)\right)\right].$$

Proof. Put $B_j = F_j C_n X / F_{j-1} C_n X$. From Theorem 3.2, there is a filtered space, $\tilde{C}_n X$, which is homotopy equivalent to $C_n X$ as a filtered space, and there are

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weak homotopy equivalences,

$$\left\{\alpha_k : S^{l_k} F_k \tilde{C}_n X \to \bigvee_{j=1}^{k} S^{l_k} B_j ; k \geq 1\right\},$$

with the following properties. The integers l_k monotonically increase with k and if $r = l_k - l_{k-1}$ then the restriction of α_k to $S^{l_k} F_{k-1} \tilde{C}_n X$ is homotopic to $S^r(\alpha_{k-1})$. Let i_k denote the canonical inclusion of the wedge,

$$\bigvee_{j=1}^{k} B_j \subset \bigvee_{j=1}^{\infty} B_j.$$

Hence $\varprojlim_k (S^{l_k}(i_k) \circ \alpha_k)$ determines an element of

$$\varprojlim_k \left[S^\infty F_k C_n X, S^\infty \left(\bigvee_{j=1}^{\infty} B_j\right)\right].$$

Choose x to be any inverse image of this element under the epimorphism

$$\left[S^\infty C_n X, S^\infty \left(\bigvee_{j=1}^{\infty} B_j\right)\right] \to \varprojlim_k \left[S^\infty F_k C_n X, S^\infty \left(\bigvee_{j=1}^{\infty} B_j\right)\right].$$

2. Let T be the category of based, compactly generated Hausdorff spaces and based maps. The base point, $*$, is required to be non-degenerate in the sense that $(X, *)$ is a strong NDR pair and, in the notation of [3; p. 162], $u^{-1}[0, \frac{1}{2}]$ is contractible. Let I^n denote the unit n-cube and J^n its interior. An (open) *little n-cube* is a linear embedding, f, of J^n, with parallel axes. Thus $f = f_1 \times \ldots \times f_n$ where $f_i : J \to J$ is a linear function, $f_i(t) = (y_i - x_i).t + x_i$ with $0 \leq x_i < y_i \leq 1$. A little n-cube and its image in J^n will be denoted by the same symbol. Let $\mathscr{C}_n(j) \in T$ be the set of those j-tuples, $\langle c_1, \ldots, c_j\rangle$, of little n-cubes such that the images, c_r, are pairwise disjoint. Denote by ${}^jJ^n$ the disjoint union of j copies of J^n. Regard $\langle c_1, \ldots, c_j\rangle$ as a map ${}^jJ^n \to J$ and topologise $\mathscr{C}_n(j)$ as a subspace of the space of all continuous functions ${}^jJ^n \to J^n$. Write $\mathscr{C}_n(0) = \langle\ \rangle$, a point. Let $\alpha = (\frac{1}{4}, \ldots, \frac{1}{4}) \in J^n$ and $\beta = (\frac{3}{4}, \ldots, \frac{3}{4}) \in J^n$; then $(c_1(\alpha), c_1(\beta), \ldots, c_j(\alpha), c_j(\beta)) \in J^{2nj}$. Considering $\mathscr{C}_n(j)$ as a subspace of J^{2nj} in this way gives the same topology [3; §4.2]. Denote by $\mathscr{D}_n(j)$ the set of j-tuples, $\langle c_1, \ldots, c_j\rangle$, (without the disjoint images requirement) topologised in either manner. Hence $\mathscr{C}_n(j) \subset \mathscr{D}_n(j)$, $(j > 0)$, and $\mathscr{D}_n(0) = \mathscr{C}_n(0)$.

Let $Y, A \in T$, with A a closed subspace of Y. Define $C_n(Y)$ and $D_n(Y, A)$ in the following manner. Form, in T, the disjoint union $Z_n = \bigcup_{j \geq 0} \mathscr{D}_n(j) \times Y^j$ and let $\approx$ be the equivalence relation, on Z_n, generated by

(i) $$(\langle c_1, \ldots, c_m\rangle, y_1, \ldots, y_{i-1}, *, y_{i+1}, \ldots, y_m) \approx (\langle c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m\rangle, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m),$$

and

(ii) $$(\langle c_1, \ldots, c_m\rangle, y_1, \ldots, y_m) \approx (\langle c_{\sigma(1)}, \ldots, c_{\sigma(m)}\rangle, y_{\sigma(1)}, \ldots, y_{\sigma(m)})$$

for $\sigma \in \Sigma_m$, the symmetric group on m letters.

Put

$$C_n Y = \{[\langle c_1, \ldots, c_m\rangle, y_1, \ldots, y_m] \in Z_n/\approx \text{ such that } \langle c_1, \ldots, c_m\rangle \in \mathscr{C}_n(m)\}$$

and

$D_n(Y, A) = \{[\langle c_1, \ldots, c_m\rangle, y_1, \ldots, y_m] \in Z_n/\approx$ such that if $y_{i_1}, \ldots, y_{i_k}$ are all the co-ordinates not in A then $\langle c_{i_1}, \ldots, c_{i_k}\rangle \in \mathscr{C}_n(k)\}$,

with subspace topologies, ([ ] $\equiv$ equivalence class). These are filtered spaces with m-th filtrations, $F_m C_n Y$ and $F_m D_n(Y, A)$, given by the equivalence classes of points

$$\bigcup_{j=0}^{m} \mathscr{D}_n(j) \times Y^j.$$

Let $S^n Y$ denote the n-th suspension, $S^n Y = (I^n/\partial I^n) \wedge Y$.

PROPOSITION 2.1. *The evaluation map,* eval: $S^n D_n(Y, A) \to S^n(Y/A)$ *defined by*

$$\text{eval}\,\big[t[\langle c_1, \ldots, c_m\rangle y_1, \ldots, y_m]\big] = \begin{cases} [u, y_j] \text{ if } c_j(u) = t \text{ and } y_j \notin A, \\ \quad * \quad \text{if no such } j \text{ and } u \text{ exist} \end{cases}$$

is continuous.

Proof. Y may be assumed compact.

(a) Suppose $\text{eval}\big[t[\langle c_1, \ldots, c_m\rangle, y_1, \ldots, y_m]\big] = [u, y_j] \neq *$. If $t \in c_s$ only for $s = i_p$, $(1 \leqslant p \leqslant k)$, take neighbourhoods $N(y_j) \subset (Y - A)$, $N(t)$ and $N(\langle c_1, \ldots, c_m\rangle)$ such that $\langle c_1', \ldots, c_m'\rangle \in N(\langle c_1, \ldots, c_m\rangle)$ implies

$$N(t) \subset \bigcap_{p=1}^{k} c'_{i_p}.$$

If $\big[t'[\langle c_1', \ldots, c_m'\rangle, y_1', \ldots, y_m']\big]$ is an element of

$$[N(t) \times N(\langle c_1, \ldots, c_m\rangle) \times Y \times N(y_j) \times \ldots \times Y] \cap S^n D_n(Y, A)$$

then $\text{eval}\big[t'[\langle c_1', \ldots\rangle \ldots y_m']\big] = [(c_j')^{-1}(t'), y_j']$.

(b) Suppose $\text{eval}\big[t[\langle c_1, \ldots,\rangle \ldots y_m]\big] = *$ and t does not belong to any c_i. For any neighbourhood $N(\partial I^n) \subset I^n$ it is possible to find neighbourhoods $N(t)$, $N(\langle c_1, \ldots, c_m\rangle)$ such that, for $\langle c_1', \ldots, c_m'\rangle \in N(\langle c_1, \ldots, c_m\rangle)$, $N(t) \cap c_i' = \varnothing$ or $N(t) \subset c_i'(N(\partial I^n))$ for $1 \leqslant i \leqslant m$.

(c) Suppose $\text{eval}\big[t[\langle c_1, \ldots\rangle \ldots y_m]\big] = *$ and $t \in c_s$ only for $s = i_p$, $(1 \leqslant p \leqslant k)$. For any neighbourhoods, $N(\partial I^n)$ and $N(A)$, it is possible to find neighbourhoods $N(y_{i_p}) \subset N(A)$, $(1 \leqslant p \leqslant k)$, and $N(\langle c_1, \ldots, c_m\rangle)$ such that for

$$\langle c_1', \ldots, c_m'\rangle \in N(\langle c_1, \ldots, c_m\rangle),$$

$c_i \cap c_j = \varnothing$ and $c_i' \cap c_j' \neq \varnothing$ together imply that $c_i' \cap c_j' \subset N(\partial I^n)$.

3. Let $(X, *) \in T$ and $u : X \to I$ be a function such that $u^{-1}(0) = *$, $((X, *)$ is an NDR pair, cf. [3; Appendix]). Let (Y, A) be a pair in T. Define $\tilde{C}_n X \subset C_n X \times [0, \infty)$ as the telescope $\bigcup_{i=1}^{\infty} F_i C_n X \times [i-1, i)$ with the filtration

$$F_m \tilde{C}_n X = \bigcup_{i=1}^{m} F_i \times [i-1, \infty).$$

It is clear from [3; Appendix] that $\tilde{C}_n X$ and $C_n X$ are homotopy equivalent as filtered spaces. Let $v : \tilde{C}_n X \to C_n X$ be the homotopy equivalence which collapses the telescope and $\pi : Y \to Y/A$ be the canonical projection.

In this section we prove the following result.

THEOREM 3.1. *Let* $f: F_m C_n X \to Y$ *be a map such that* $f(F_{m-1} C_n X) = *$ *and* $f[\langle c_1, \ldots, c_m\rangle x_1, \ldots, x_m] \in A$ *if and only if* $\min u(x_i) \leqslant \frac{1}{2}$. *There exists a canonical family of maps, a stable combinatorial extension of* f,

$$\{G_k : S^{l_k} F_k \tilde{C}_n X \to S^{l_k}(Y/A)\} \text{ each extending } S^{l_k}(\pi \circ f \circ \nu)$$

with $G_k | S^{l_k}(F_{k-1} \tilde{C}_n X) \simeq S^r(G_{k-1})$, $(r = l_k - l_{k-1})$.

Before proving Theorem 3.1 we make the following application.

THEOREM 3.2. (cf. [4; §2.1]). *If* $\{l_k\}$ *is the set of integers in Theorem* 3.1 *there exists a family of weak homotopy equivalences*

$$\left\{\alpha_k : S^{l_k} F_k \tilde{C}_n X \to \bigvee_{j=1}^{k} S^{l_k}(F_j C_n X / F_{j-1} C_n X)\right\}$$

such that $\alpha_k | S^{l_k}(F_{k-1} \tilde{C}_n X) \simeq S^r(\alpha_{k-1})$, $(r = l_k - l_{k-1})$.

Proof. Apply Theorem 3.1 to the canonical projection

$$f_m : F_m C_n X \to F_m C_n X / F_{m-1} C_n X$$

with A as the image of

$$A' = \{[\langle c_1, \ldots, c_m\rangle x_1, \ldots, x_m] \mid \min u(x_i) \leqslant \tfrac{1}{2}\}$$

to produce maps $\beta_m : S^{l_k} F_k \tilde{C}_n X \to S^{l_k}(F_m C_n X / A')$, $(1 \leqslant m \leqslant k)$, such that $\beta_m | S^{l_k}(F_m \tilde{C}_n X) = S^{l_k}(\pi \circ f_m \circ \nu)$. Since $(X, *)$ is a strong NDR pair, (cf. [3; §2, p. 162]) $A'/F_{m-1} C_n X$ is contractible through itself and

$$F_m C_n X / F_{m-1} C_n X \simeq F_m C_n X / A'.$$

For $(t_1, t_2, \ldots) \in I^{l_k}$, $z \in F_k \tilde{C}_n X$ define

$$\alpha_k[t_1, t_2, \ldots, z] = \begin{cases} \beta_m[2^m . t_1 - 1, t_2, \ldots, z], & \text{if } t \in [(\frac{1}{2})^m, (\frac{1}{2})^{m-1}] \\ * & \text{otherwise.} \end{cases}$$

Filtering the wedge in the canonical manner makes α_k into a map of filtered spaces which is clearly a homotopy equivalence on successive quotients.

In order to construct the stable combinatorial extension of Theorem 3.1 we need some preliminary notation.

3.3. If $U_1, \ldots, U_k$ are subsets of J^n and $0 \leqslant z_i \leqslant 1$, $\Sigma z_i = 1$ then $\Sigma z_i . U_i \subset J^n$ is the subset, $\{y \in J^n \mid y = \Sigma z_i . y_i; y_i \in U_i\}$.

For n-cubes, $c_i = U_i$, $\Sigma z_i . c_i \subset J^n$ is not in general an n-cube as it may not have parallel axes. However an n-cube can be constructed as follows. If $\hat{c}_i$ is the centre of c_i there is a well-defined n-cube, c, centre $\Sigma z_i . \hat{c}_i$ which is maximal subject to $c \subset \Sigma z_i . c_i$. When $z_i = 0$ $(i \neq j)$ then $c = c_j$. Henceforth the cube, c, will also be denoted by $\Sigma z_i . c_i$ when no ambiguity can occur.

3.4. For $k \geqslant m$ let $W_k = (w_1, \ldots, w_p)$, $(p = {}_kC_m$, binomial coefficient), be the ordered set of subsequences of $(1, \ldots, k)$ with exactly m elements, ordered as in [**1**]. That is, if $w_r = (\alpha_1, \ldots, \alpha_m)$ and $w_s = (\beta_1, \ldots, \beta_m)$ are elements of W_k then $r < s$ if and only if there exists $u \leqslant m$ such that $\alpha_v = \beta_v (v > u)$ and $\alpha_u < \beta_u$. For $z = (\alpha_1, \ldots, \alpha_m) \in W_k$ write c_z for $\langle c_{\alpha_1}, \ldots, c_{\alpha_m} \rangle$ and x_z for $(x_{\alpha_1}, \ldots, x_{\alpha_m}) \in X^m$.

3.5. Define integers, q_k $(k \geqslant m)$, by $q_k = n$ $(k = m, m+1)$ and $q_k = m.n$ $(k > m+1)$. Further, define l_k $(k \geqslant m)$ by $l_m = n$ and $l_j = l_{j-1} + 1 + q_j$, $(j \geqslant m+1)$.

3.6. The proof of Theorem 3.1 is divided into four steps which we sketch as follows.

STEP 1. For $k \geqslant m$, construct $\phi_k : \mathscr{C}_n(k) \to \mathscr{C}_{q_k}(p)$, $(p = {}_kC_m)$ such that $\phi_k \langle c_1, \ldots, c_k \rangle$ is a set of little q_k-cubes with disjoint closures.

STEP 2. Inductively construct

$$\psi_k : \mathscr{C}_n(k) \times X^k \to \mathscr{D}_{l_k}(p), \qquad (p = {}_kC_m),$$

with the property that if

$$\psi_k(\langle c_1, \ldots, c_k \rangle x_1, \ldots, x_k) = \langle c_1', \ldots, c_p' \rangle$$

and $w_{i_1}, \ldots, w_{i_t} \in W_k$ are the only elements with $\min\limits_{\alpha \in w} u(x_\alpha) > \frac{1}{2}$ then the closures of $c_{i_1}', \ldots, c_{i_t}'$ are disjoint. ψ_k is constructed by a process of topologically joining ψ_{k-1} and ϕ_k.

STEP 3. Use ψ_k and the given map, f, to construct a map

$$\mathscr{C}_n(k) \times X^k \to \mathscr{D}_{l_k}(p) \times Y^p$$

which induces $\Psi_k : F_k \tilde{C}_n X \to D_n(Y, A)$.

STEP 4. Use the evaluation map and Ψ_k to construct G_k.

Proof of Theorem 3.1.

STEP 1. Take for ϕ_m the constant map to $\langle (\frac{1}{4}, \frac{3}{4})^n \rangle$. Define

$$\phi_{m+1} \langle c_1, \ldots, c_{m+1} \rangle = \langle c_{m+1}', c_m', \ldots, c_1' \rangle,$$

where c_j' is obtained from c_j by shrinking it linearly towards its centre to reduce its diameter by half. Let $\gamma \langle c_1, \ldots, c_m \rangle = (z_1, \ldots, z_m) \in (J^n)^m$ when z_j is the centre of c_j. For $k > m+1$ map $\mathscr{C}_n(k)$ into $\mathscr{C}_{m.n}(p)$ by mapping $\langle c_1, \ldots, c_k \rangle$ to $\langle c_1', \ldots, c_p' \rangle$, where, if $w_j = (\alpha_1, \ldots, \alpha_m) \in W_k$, c_j' has centre $\gamma(c_{w_j})$ and where each side of each little q_k-cube, c_j', is of length d and cubes at these centres with all sides of length $2.d$ would "just touch". Note that a permutation of $(1, \ldots, k)$ (or $c_1, \ldots, c_k$) induces a permutation of $c_1, \ldots, c_p$ which acts on these in the same way as the induced permutation acts on W_k.

STEP 2. Define ψ_m as the composition of ϕ_m with the projection onto $\mathscr{C}_n(m)$.

Suppose ψ_{k-1} is constructed, $k \geqslant m+1$ and $0 < e < \frac{1}{4}$. From this data the tilde construction on a little s-cube, c, is $\tilde{c} = c \times (\frac{1}{2} - e, \frac{1}{2} + e)^{1+q_k}$ if $s = l_{k-1}$ and $\tilde{c} = (\frac{1}{2} - e, \frac{1}{2} + e)^{l_{k-1}} \times (\frac{3}{4} - e, \frac{3}{4} + e) \times c$ if $s = q_k$. The tilde construction produces little l_k-cubes from cubes of smaller dimension. The map ψ_k will be constructed by joining l_k-cubes produced from ψ_{k-1} and ϕ_k by the tilde construction.

If $(\langle c_1, \ldots, c_k \rangle x_1, \ldots, x_k) \in \mathscr{C}_n(k) \times X^k$ and $1 \leqslant j \leqslant p$ define $\chi_j \in \mathscr{D}_{l_{k-1}}(1)$ in the following manner. If not all the $u(x_g)$ are zero ($g \notin w_j \in W_k$) define

$$\chi_j = \big(1/\Sigma u(x_g)\big).\textstyle\sum_g u(x_g).\chi_{j,g}$$

where the sum is taken over $g \notin w_j$ and $\chi_{j,g}$ is the r-th cube in

$$\psi_{k-1}(\langle c_1, \ldots, c_{g-1}, c_{g+1}, \ldots, c_k\rangle x_1, \ldots, x_{g-1}, x_{g+1}, \ldots, x_k),$$

w_j being the r-th m-tuple in the W_{k-1} formed from $(1, \ldots, g-1, g+1, \ldots, k)$. If $u(x_g) = 0$, $(g \notin w_j)$ put χ_j as the common value of the r-th entry of any $\psi_{k-1}(\langle d_1, \ldots, d_{k-1}\rangle z_1, \ldots, z_{k-1})$, where $z_{\alpha_i} = x_{\alpha_i}$ if $w_j = (\alpha_1, \ldots, \alpha_m)$ and z_s is the base point otherwise. This common value is the image of $(\frac{1}{4}, \frac{3}{4})^n$ under successive tilde operations and can be arranged, in the inductive step, to be $(\frac{1}{4}, \frac{3}{4})^n \times (\frac{1}{8}, \frac{5}{8})^{l_{k-1}-n}$, (cf. the penultimate sentence of *Step* 2).

Now define $\psi_{e,k}(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k) \in \mathscr{D}_{l_k}(p)$ to have j-th entry,

$$v_j.(\tilde{c}_j') + (1 - v_j).\tilde{\chi}_j,$$

where $\phi_k\langle c_1, \ldots, c_k\rangle = \langle c_1', \ldots, c_p'\rangle$ and $v_j = \min\big(1, 2\min\limits_{g \notin w_j} u(x_g)\big)$.

For fixed e it is straightforward, although tedious, to check that $\psi_{e,k}$ is continuous. However, $\psi_{e,k}$ does not necessarily satisfy the disjoint-closure condition required of the ψ_k in *Step* 2. We proceed to amend $\psi_{e,k}$ to produce a ψ_k satisfying this condition.

For each $(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k)$ put $E(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k)$ equal to the supremum of those e such that $\psi_{e,k}(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k)$ does satisfy the disjoint-closure condition. The function E is not necessarily continuous. However, since ψ_{k-1} and ϕ_k satisfy disjoint-closure conditions, E is strictly positive. Also for $S \subset (1, \ldots, k)$ E is continuous on $\mathscr{C}_n(k) \times Q_S$ where $Q_S \subset X^k$ is $\{(x_1, \ldots, x_k) | u(x_j) \leqslant \frac{1}{2}$ if and only if $j \in S\}$. Hence, using a canonical partition of unity on I^k, there exists a continuous function, $e' : \mathscr{C}_n(k) \times X^k \to (0, \frac{1}{4}]$ such that $e' \leqslant E/2$. Produce a function, e, invariant under the action of the symmetric group by defining

$$e(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k) = \min_{\sigma \in \Sigma_k} e'(\langle c_{\sigma(1)}, \ldots, c_{\sigma(k)}\rangle x_{\sigma(1)}, \ldots, x_{\sigma(k)}).$$

Using $e(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k)$ at the point $(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k)$ in the formula for $\psi_{e\ k}$ defines the function ψ_k.

To ensure the inductive definition of the $\{\chi_j\}$, as described, we need to insist that $e(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k) = \frac{1}{8}$ if all but m of the $\{x_i\}$ are the base point. If $\psi_k(\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k) = \langle c_1', \ldots, c_p'\rangle$ note that a permutation of $(1, \ldots, k)$ induces a permutation of $(c_1', \ldots, c_p')$ which acts in the same manner as the induced permutation on W_k.

STEP 3. Using the maps of *Step* 2 and $f : F_m C_n X \to Y$ define

$$\Psi_k : F_k \tilde{C}_n X \to D_{l_k}(Y, A)$$

as follows.

Define
$$\Psi_k(F_{m-1}\tilde{C}_n X) = *,$$

$$\Psi_k([\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k]\, t) = \Psi_k([\langle c_1, \ldots, c_k\rangle x_1, \ldots, x_k]\, k-1)$$

if $t \geqslant k-1$,
and for $(m \leqslant i \leqslant k-1)$, $s = {}_iC_m$, $t \in [i-1, i]$, $W_s = (w_1, \ldots, w_s)$ put

$$\Psi_k([\langle c_1, \ldots, c_i\rangle x_1, \ldots, x_i]\, t)$$

equal to

$$[\langle (i-t).c' + (1+t-i)\,c'' \rangle f\,[c_{w_1}, x_{w_1}], \ldots, f\,[c_{w_s}, x_{w_s}]]$$

where $\psi_i(\langle c_1, \ldots, c_i \rangle\, x_1, \ldots, x_i) \times (0,1)^{l_k - l_i}) = \langle c' \rangle$ and the j-th cube, c_j'', in $\langle c'' \rangle$ is the j-th cube in $\psi_{i+1}(\langle c_1, \ldots, c_i, c_{i+1} \rangle\, x_1, \ldots, x_i, *) \times (0,1)^{l_k - l_{i+1}}$. Note that c_j'' is independent of c_{i+1}. This map is well defined since $\psi_k(\langle c_1, \ldots, c_k \rangle\, x_1, \ldots, x_k)$ behaves well under permutations of $(1, \ldots, k)$ as remarked in *Step* 2. Also the maps induced by ψ_k and $\psi_{k-1} \times (0,1)^{l_k - l_{k-1}}$ from $F_{k-1}\, C_n X$ to $D_{l_k}(Y, A)$ are homotopic (by the linear homotopy featured in the definition of Ψ_k) since when one of the x_i is the base point ψ_k is $\tilde{\psi}_{k-1}$ in the coordinates for which $f\,[c_{w_j}, x_{w_j}] \neq *$. Hence if

$$\sigma : D_{l_k}(Y, A) \to D_{l_{k+1}}(Y, A)$$

is the map induced by sending a little cube, c, to $c \times (0,1)^{l_{k+1} - l_k}$ then

$$\Psi_{k+1} | F_k\, \tilde{C}_n X \simeq \sigma \circ \Psi_k.$$

STEP 4. Put G_k equal to the composition $(\mathrm{eval}) \circ S^{l_k}(\Psi_k)$. The homotopy relation between G_k and G_{k+1} follows from the behaviour of Ψ_k, Ψ_{k+1} and σ remarked in *Step* 3.

Remark 3.6. Put $t_m = n+1$, $t_{m+1} = 2n+3$ and $t_j = t_{j-1} + 1 + m.(n+1)$, $(j > m+1)$. For $1 \leqslant j \leqslant n+1$, the map which sends a little n-cube, $c = f_1 \times \ldots \times f_n$ to

$$f_1 \times \ldots \times f_{j-1} \times (0,1) \times f_j \times \ldots \times f_n$$

induces maps $\sigma_j : F_m\, C_n\, X/A' \to F_m\, C_{n+1}\, X/A'$ and $\sigma_j : F_k\, \tilde{C}_n X \to F_k\, \tilde{C}_{n+1} X$. For $g : S^{l_k} V \to S^{l_k} Z$ let $S^a(g)$, $(a = t_k - l_k)$ be the a-fold suspension of g using as suspension coordinates in I^{t_k} those in positions $j, t_m + j + 1, t_k + 1 + j + r.(n+1)$, $(0 \leqslant r \leqslant m-1;\, k \geqslant m+1)$.
With this notation a linear homotopy between little cubes yields a homotopy commutative diagram

$$\begin{array}{ccc}
S^{t_k} F_k\, \tilde{C}_n X & \xrightarrow{S^a(\beta_m)} & S^{t_k}(F_m\, C_n\, X/A') \\
S^{t_k}(\sigma_j) \downarrow & & \downarrow S^{t_k}(\sigma_j) \\
S^{t_k} F_k\, \tilde{C}_{n+1} X & \xrightarrow[\beta_m]{} & S^{t_k}(F_m\, C_{n+1}\, X/A'),
\end{array}$$

where β_m is as in Theorem 3.2 (proof).

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