SMOOTH MANIFOLDS AND THEIR APPLICATIONS IN HOMOTOPY THEORY

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Introduction

The main aim of the present work is the homotopy classification of maps of the $(n+k)$-sphere Σ^{n+k} into the n-sphere S^n; this problem is however only solved here for $k = 1, 2$. The method described here in detail was published earlier in notes [1, 2]. It was used by V. A. Rohlin [3] also to solve the problem for $k = 3$, but there has been no success in the meantime in obtaining results for $k > 3$ by this method. The method depends on the study of certain properties of smooth or, what comes to the same thing, differentiable manifolds of dimensions k and $k+1$. After the appearance of the works [1–3] there appeared a series of works by French mathematicians [4] in which the classification of maps of spheres into spheres of lower dimension was advanced very considerably. The methods of the French school of topology are quite different from those applied here.

Smooth manifolds are the principal and perhaps even the unique tools of the investigation, so Chapter 1 of the work is devoted to an independent study of them in full, the study being carried out somewhat more comprehensively than is required for subsequent applications. Besides the fundamental definitions Chapter 1 contains a proof, somewhat simpler than Whitney's [5], of the theorem on embedding n-dimensional smooth manifolds in $(2n+1)$-dimensional Euclidean space, and also a statement and study of the problem of typical singular points of smooth maps of an n-dimensional manifold into Euclidean space of dimension less than $2n+1$.

In Chapter 2 the method of applying smooth manifolds to the solution of the homotopy problem is expounded. First of all it is established that, for the homotopy classification of maps of one smooth manifold into another, one may restrict attention to smooth maps and smooth deformations. Then the method of applying smooth manifolds to the homotopy classification of maps of the sphere Σ^{n+k} into the sphere S^n is described.

A smooth closed manifold M^k of dimension k, situated in Euclidean space Σ^{n+k} of dimension $n+k$, is said to be *framed* and is denoted by (M^k, U) if at each point $x \in M^k$ there is given a system $U(x) = \{u_1(x), \cdots, u_n(x)\}$ of n linearly independent vectors orthogonal to M^k and depending smoothly on x. By adjoining a point at infinity q' to Euclidean space we obtain the sphere Σ^{n+k}. Further let $e_1, \cdots, e_n$ be a system of linearly independent vectors tangent to the sphere S^n at its north pole p. It turns out that there exists a smooth map f of Σ^{n+k} into S^n such that $f^{-1}(p) = M^k$ and that the map f_x, obtained by linearizing the map f at the point $x \in M^k$, transforms the vectors $u_1(x), \cdots, u_n(x)$ into the vectors $e_1, \cdots, e_n$

spectively. The homotopy class of a map f possessing these properties is uniquely determined by the framed manifold (M^k, U). For each homotopy class of maps of Σ^{n+k} into S^n there exists a framed manifold such that the corresponding map f belongs to the given homotopy class. Two framed manifolds (M_0^k, U_0) and (M_1^k, U_1) determine the same homotopy class of maps of Σ^{n+k} into S^n if and only if they are *homologous* in the following sense. Let $E^{n+k} \times E^1$ be the direct product of the Euclidean space E^{n+k} and the real line E^1 parametrized by the variable t. We will suppose the framed manifold (M_0^k, U_0) situated in the space $E^{n+k} \times 0$ and (M_1^k, U_1) situated in $E^{n+k} \times 1$. The framed manifolds (M_0^k, U_0) and (M_1^k, U_1) will be regarded as homologous if there is a smooth framed manifold (M^{k+1}, U) lying in the strip $0 \le t \le 1$, whose boundary consists of the manifolds M_0^k and M_1^k and whose frame U coincides on the boundary with the given frames U_0 and U_1.

The construction described makes possible the reduction of the problem of the homotopy classification of maps of Σ^{n+k} into S^n to the homology classification of framed k-dimensional manifolds. The role played by k-dimensional and $(k+1)$-dimensional manifolds is evident here. The homology classification of 0-dimensional framed manifolds is trivial and it is correspondingly easy to classify maps of Σ^n into S^n. The homology classification of 1-dimensional and 2-dimensional manifolds also presents no particular difficulties and leads to the homotopy classification of maps of Σ^{n+k} into S^n for $k = 1, 2$. Chapter 4 of this work is devoted to this. The homology classification of 3-dimensional manifolds already presents considerable difficulty. It has been obtained by V. A. Rohlin [3].

To achieve the homology classification of framed manifolds in this work, use is made of homology invariants. With a framed submanifold (M^k, U) of Euclidean space E^{n+k} we associate a homology invariant which appears simultaneously as a homotopy invariant of the corresponding map of Σ^{n+k} into S^n. For $n = k+1$ there is the well-known Hopf invariant γ of maps of Σ^{2k+1} into S^{k+1}. It is easy to interpret the invariant γ as a homology invariant of framed manifolds. In Chapter a definition of the invariant γ is given which depends on the theory of smooth manifolds, and it is interpreted as a homology invariant of a framed manifold. For $k = 1$ the Hopf invariant turns out to be the unique invariant; this fact is proved (by familiar methods) in Chapter 4. For the cases $k = 1, 2$; $n \geq 2$ an invariant δ is constructed in Chapter 4. This invariant is a residue class mod 2. From its existence it follows that the number of homotopy classes of maps of Σ^{n+k} into S^n for $k = 1, 2$; $n \geq 2$ cannot be less than 2. That this invariant is unique for all cases except $k = 1, n = 2$ is proved on the basis of the uniqueness of the invariant γ for $k = 1$.

CHAPTER I

Smooth manifolds and smooth maps

§1. Smooth manifolds

Here at the outset the definition is given of a smooth manifold—or, equivalently, a differentiable manifold of finite class—and the simplest ideas connected with them are introduced; next, concepts playing an important role in the theory of smooth manifolds are considered, namely, submanifolds of smooth manifolds, manifolds of line elements of a smooth manifold, direct products of smooth manifolds, and manifolds of vector subspaces of a given dimension of some vector space. Together with differentiable manifolds of finite class it is possible to define also infinitely differentiable manifolds, where all the functions considered are infinitely differentiable and, in the same way, analytic manifolds, where all the functions considered are analytic. In the present work infinitely differentiable and analytic manifolds play no role and so will not be considered.

The notion of a smooth manifold. A) Let E^k be Euclidean space of dimension k with Cartesian coordinates $x^1, \cdots, x^k$. By the *half-space* of the space E^k we shall understand the set E_0^k, given by the relation

$$x^1 \leq 0. \tag{1}$$

By the *boundary* of the half-space E_0^k we shall understand the hyperplane E^{k-1}, given by the relation

$$x^1 = 0. \tag{2}$$

By a *region* of the half-space E_0^k we shall understand an open subset of E_0^k (which may or may not be open in E^k). Points of a region of E_0^k, belonging to its boundary E^{k-1}, will be called *boundary-points* of the region. A topological space M^k with a countable base, each of whose points a has a neighbourhood U^k homeomorphic to some region W^k of the half-space E_0^k or the space E^k, will be called a *topological manifold*. (Evidently, each region of E^k is homeomorphic to some region of E_0^k, but for the introduction of coordinate systems it is convenient to consider regions of both spaces.) If the point a corresponds to a boundary point of the region W^k, it is called a *boundary-point* of the manifold M^k and of its neighbourhood U^k. It is known that the notion of boundary point is topologically invariant. Manifolds possessing boundary points we shall sometimes call manifolds *with boundary*; and manifolds not possessing boundary points we shall sometimes call manifolds *without boundary*. Compact manifolds without boundary will be called *closed*. It is easy to verify that the collection of all boundary points of the manifold M^k is a $(k-1)$-dimensional manifold without boundary.

Definition 1. Let M^k be a topological manifold of dimension k and let U^k be a neighbourhood in M^k, homeomorphic to a region W^k of E_0^k or E^k. The choice of a definite homeomorphism between U^k and W^k is equivalent to the introduction ıto U^k of a definite system X of coordinates $x^1, \cdots, x^k$, corresponding to the oordinates of the Euclidean space E^k. Then two different systems of coordinates ⁄ and Y in U^k are always connected by a one-one bicontinuous transformation

$$y^j = y^j(x^1, \cdots, x^k), \quad j = 1, \cdots, k. \tag{3}$$

/e choose once and for all a fixed number m and suppose that the functions (3) are ot only continuous, but m times continuously differentiable in the region U^k and hat the Jacobian $\left|\frac{\partial y^j}{\partial x^i}\right|$ does not vanish. By this condition we associate each of he systems X and Y with one and the same *smoothness class* of order m. It is vident that different classes do not intersect and that each class is determined ·y an arbitrary coordinate system belonging to that class. If, among all these lasses, a particular class has been distinguished, the neighbourhood U^k will be alled m times continuously *differentiable*. On the strength of this two m times ontinuously differentiable neighbourhoods U^k, V^k of the manifold M^k always in-uce in their common part two classes of coordinate system; if these classes co-ncide we say that the neighbourhoods U^k and V^k are differentiably *compatible*. f all neighbourhoods of some basis of neighbourhoods of the manifold M^k are m imes continuously differentiable and moreover pairwise compatible, the manifold 1^k will be said to be m times continuously *differentiable*, or *smooth* of class m— or sometimes simply smooth without indicating the number m, which, however, will ılways be assumed sufficiently large for our purposes. [Analogously, if the func-ions (3) are analytic, the manifold is called *analytic*.]

As is evident from the given definition, the property of a manifold of being dif-erentiable is determined by properties of the neighbourhoods of some basis. If vith the help of two bases in the manifold two differential structures are defined, :hey are considered to be the same if and only if the union of the two bases again satisfies the conditions of definition 1. Actually, to determine a differential struc-ure in a manifold it is sufficient to describe it in each neighbourhood of some cov-ring of the manifold. Such a covering also determines, of course, the topology of the manifold.

If only connected neighbourhoods are admitted—which is always possible—then in each neighbourhood all distinguished coordinate systems fall into two classes, in each of which the transitions (3) are realized by means of transforma-:ions with positive Jacobian. Each of these classes will be called an *orientation* of the given neighbourhood. It is evident that a smooth manifold is orientable if and only if it is possible to choose compatible orientations of its neighbourhoods. To each such choice corresponds a definite orientation of the manifold.

B) The boundary M^{k-1} of a smooth manifold M^k may be itself regarded in a natural way as a smooth manifold of the same class, by means of the following

construction. Let U^k be a neighbourhood in M^k, with distinguished coordinate system X, for which the intersection $U^{k-1} = U^k \cap M^{k-1}$ is non-empty. The equation of U^{k-1} in U^k evidently has the form $x^1 = 0$, and so it is natural to take $x^2, \cdots, x^k$ as distinguished coordinates in U^{k-1}. If V^k is another neighbourhood in M^k (perhaps coinciding with U^k), with distinguished coordinate system Y, for which the intersection $V^{k-1} = V^k \cap M^{k-1}$ is non-empty, then in the common part of U^k and V^k we have

$$y^j = y^j(x^1, \cdots, x^k), \quad j = 1, \cdots, k, \tag{4}$$

whence for $x^1 = 0$ we obtain

$$y^j = y^j(0, x^2, \cdots, x^k), \quad j = 2, \cdots, k. \tag{5}$$

From the differentiability of the relations (4) follows the differentiability of the relations (5). Further, from the relation $y^1(0, x^2, \cdots, x^k) = 0$ it follows that in $U^{k-1} \cap V^{k-1}$:

$$\frac{\partial(y^2, \cdots, y^k)}{\partial(x^1, \cdots, x^k)} = \frac{\partial y^1}{\partial x^1} \cdot \frac{\partial(y^2, \cdots, y^k)}{\partial(x^2, \cdots, x^k)}, \tag{6}$$

and since the left-hand side is non-zero we obtain $\frac{\partial(y^2, \cdots, y^k)}{\partial(x^2, \cdots, x^k)} \neq 0$. If the system X is used to orient U^k, the system $x_2, \cdots, x_k$ will be taken as an orienting coordinate system of the neighbourhood U^{k-1}. Because $\frac{\partial y^1}{\partial x^1} > 0$, the fact that $\frac{\partial(y^1, \cdots, y^k)}{\partial(x^1, \cdots, x^k)}$ is positive implies that $\frac{\partial(y^2, \cdots, y^k)}{\partial(x^2, \cdots, x^k)}$ is positive. In this way, the boundary of a smooth oriented manifold acquires a natural orientation.

C) Let a be a point of a smooth manifold M^k. Each coordinate system, defined in some neighbourhood U^k of the point a and belonging to the distinguished class is called a *local system of coordinates* at the point a. It is evident that each point a of the manifold M^k may be taken as origin of some local coordinate system. A (contravariant) *vector* on the manifold M^k at the point a is a function associating with each local coordinate system at a a system of k real numbers—*components* of the vector relative to this coordinate system—in such a way that the components $u^1, \cdots, u^k$ and $v^1, \cdots, v^k$ of one and the same vector relative to two local coordinate systems $x^1, \cdots, x^k$ and $y^1, \cdots, y^k$ are always connected by the relation

$$v^j = \sum_{i=1}^{k} \frac{\partial y^j(a)}{\partial x^i} u^i, \quad j = 1, \cdots, k. \tag{7}$$

It is evident that a vector is uniquely determined by its components relative to an arbitrary local coordinate system. By defining a linear operation on vectors as an operation on their components, we turn the set of all vectors on the manifold M^k at the point a into a k-dimensional vector space R^k_a, which is said to be *tangent* to the smooth manifold M^k at the point a. To each local coordinate system at a corresponds, clearly, a definite basis in the tangent space, the components of any vector with respect to this basis being the components with respect to the coordinate system. If the point a belongs to the boundary M^{k-1} of the manifold M^k it determines, besides the tangent space R^k_a, a space R^{k-1}_a tangent to M^{k-1}. We take as local coordinates in M^{k-1} the parameters $x^2, \cdots, x^k$ (see B)) and identify the vector of R^{k-1}_a with components $u^2, \cdots, u^k$ with the vector of R^k_a with components $0, u^2, \cdots, u^k$; then we obtain a natural embedding of R^{k-1}_a in R^k_a.

Smooth maps. D) Let M^k and N^l be two smooth manifolds of class m, and let ϕ be a continuous map from the first to the second. We choose a local coordinate system X at the point $a \in M^k$ and a local coordinate system Y at $b = \phi(a) \in N^l$; then in the neighbourhood of the point a the map ϕ takes the form

$$y^j = \phi^j(x^1, \cdots, x^k), \quad j = 1, \cdots, l. \tag{8}$$

If the functions ϕ^j are n times continuously differentiable, $n \leq m$, they will be as often differentiable for any other choice of local coordinate system; thus we may say that the map ϕ itself is of smoothness class n. In what follows, in talking of smooth maps, we will always suppose that n is sufficiently large for our purposes. If the matrix $\|\frac{\partial \phi^j}{\partial x^i}\|$ has rank k at a, we will say that the map ϕ is *regular* at a. It is easy to see that if the point a belongs to the boundary M^{k-1} of the manifold M^k, then from the regularity of ϕ at a follows the regularity of $\phi|M^{k-1}$ at a. If ϕ is regular at each point $a \in M^k$ it is called *regular*. It is easy to verify that if ϕ is regular at a, then it is regular and homeomorphic in some neighbourhood of a. A regular homeomorphic map is called a *smooth embedding*. The map ϕ is called *proper* at $a \in M^k$ if the rank of the matrix $\|\frac{\partial \phi^j}{\partial x^i}\|$, $j = 1, \cdots, l$; $i = 1, \cdots, k$, is l. It is easy to see that the set of all improper points of the map ϕ is closed in M^k. The point $b \in N^l$ is called a *proper* point of the map ϕ if ϕ is proper at each point of the set $\phi^{-1}(b)$. The point a is called a *singular* point of ϕ if it is neither regular nor proper; that is, if the rank of the matrix $\|\frac{\partial \phi^j}{\partial x^i}\|$ is less than each of k and l.

E) Every smooth map ϕ of the smooth manifold M^k in the smooth manifold N^l induces at each point $a \in M^k$ a definite linear transformation ϕ_a of the vector space R^k_a, tangent to M^k at a, into the vector space R^l_b, tangent to N^l at $b = \phi(a)$. Namely, if local coordinate systems X and Y are chosen at the points a and b respectively, then to the vector $u \in R^k_a$ with components $u^1, \cdots, u^k$ relative to the system X corresponds the vector $v = \phi_a(u) \in R^l_b$ with components

$$v^j = \sum_{i=1}^{k} \frac{\partial \phi^j(a)}{\partial x^i} u^i, \quad j = 1, \cdots, l \tag{9}$$

relative to the system Y. It is not difficult to see that this definition is unambiguous; that is, for arbitrary choice of local coordinate systems it leads to the same definition of ϕ_a. If the map ϕ is regular at a, the map ϕ_a is one-one and may be used to embed R^k_a in R^l_b. If the map ϕ is proper at a, then $\phi_a(R^k_a) = R^l_b$.

Definition 2. A smooth map ϕ of class n of a smooth manifold M^k of class m onto a smooth manifold N^k of class m, $m \geq n$, is called a *smooth homeomorphism* if it is homeomorphic and regular. It is evident that if the map ϕ is a smooth homeomorphism of class n, then the inverse map ϕ^{-1} is also a smooth homeomorphism of class n. Tow manifolds are called *smoothly homeomorphic* if there exists a smooth homeomorphism of one onto the other.

Some ways of constructing smooth manifolds. F) Let P^r be a subset of the smooth manifold M^k of class m, defined near each of its points by a system of $k-r$ independent equations. This means that for each point $a \in P^r$ there exists a neighbourhood U^k, in the manifold M^k, with local system X, such that the intersection $P^r \cap U^k$ consists of all points whose coordinates satisfy the equations

$$\psi^j(x^1, \cdots, x^k) = 0, \quad j = 1, \cdots, k-r. \tag{10}$$

Further we assume that the functions ψ^j are m times continuously differentiable and that the functional matrix $\|\frac{\partial \psi^j(a)}{\partial x^i}\|$, $j = 1, \cdots, k-r$; $i = 1, \cdots, k$, has rank $k-r$; if, indeed a is a boundary point of the manifold M^k, then we assume that even the truncated functional matrix $\|\frac{\partial \psi^j(a)}{\partial x^i}\|$, $j = 1, \cdots, k-r$; $i = 2, \cdots, k$, has rank $k-r$. Under these conditions the set P^r assumes in a natural way the structure of an r-dimensional manifold of class m, smoothly embedded in M^k. We will call this manifold P^r a *submanifold* of the manifold M^k. In addition it turns out that the boundaries P^{r-1} and M^{k-1} of the manifolds P^r and M^k satisfy the relation

$$P^{r-1} = P^r \cap M^{k-1}, \tag{11}$$

and, if $a \in P^{r-1}$ and R^k_a, R^{k-1}_a, R^r_a, R^{r-1}_a are the tangent spaces to the manifolds M^k, M^{k-1}, P^r, P^{r-1} at the point a, then

$$R^{r-1}_a = R^r_a \cap R^{k-1}_a. \tag{12}$$

Here the spaces R^{k-1}_a, R^r_a, R^{r-1}_a are considered as subspaces of the space R^k_a (see C) and E)).

To prove that P^r is an r-dimensional manifold and to give it a differential structure we renumber, if necessary, the coordinates in such a way that the Jacobian $|\frac{\partial \psi^j(a)}{\partial x^i}|$, $j = 1, \cdots, k-r$; $i = r+1, \cdots, k$, is non-zero; but in the case of a boundary point we do not alter the number of the coordinate x^1. Then the system (10) admits a unique solution with respect to the variables $x^{r+1}, \cdots, x^k$:

$$x^i = f^i(x^1, \cdots, x^r), \quad i = r+1, \cdots, k. \tag{13}$$

In the case of a boundary point the coordinate x^1 figures among the independent variables. The functions f^i are defined and m times continuously differentiable in any region W^r of the half-space E_0^r of the variables $x^1, \cdots, x^r$ and determine a homeomorphism of this region onto some neighbourhood U^r of the point a in P^r. Thus P^r is an r-dimensional manifold. The differential structure in the neighbourhood U^r is defined by means of the coordinates $x^1, \cdots, x^r$.

The natural inclusion of the manifold P^r in the manifold M^k is given in U^r by the relations

$$x^i = x^i,\ i = 1, \cdots, r;\ x^i = f^i(x^1, \cdots, x^r), \quad i = r+1, \cdots, k, \tag{14}$$

where the parameters $x^1, \cdots, x^r$ on the right are regarded as coordinates in U^r and the parameters $x^1, \cdots, x^k$ on the left are coordinates in U^k. The relation (11) is is evident. Now let $a \in P^{r-1}$; we prove relation (12). To the local system X there corresponds some basis $e_1, \cdots, e_k$ of R_a^k; a basis of the space R_a^{k-1} consists of the vectors $e_2, \cdots, e_k$; a basis of the space R_a^r consists of the vectors

$$e_i + \sum_{j=r+1}^{k} \frac{\partial f^j}{\partial x^i} e_j,\ i = 1, \cdots, r,$$

and a basis of R_a^{r-1} consists of all these vectors except the first. By considering these bases we easily convince ourselves of the truth of relations (12).

To establish the compatibility of the coordinate systems introduce into P^r, we consider, together with the point a, a point $b \in P^r$ with local system Y and neighbourhoods V^k and V^r, analogous to the neighbourhoods U^k and U^r. The relations analogous to (13) will be:

$$y^i = g^i(y^1, \cdots, y^r), \quad i = r+1, \cdots, k. \tag{15}$$

We suppose that U^r and V^r intersect; then U^k and V^k themselves intersect, so let

$$y^i = y^i(x^1, \cdots, x^k), \quad i = 1, \cdots, k; \tag{16}$$

$$x^i = x^i(y^1, \cdots, y^k), \quad i = 1, \cdots, k, \tag{17}$$

be the transformations from X to Y and conversely. Substituting for $x^{r-1}, \cdots, x^k$ from (13) in (16), we obtain for the first r variables y:

$$y^i = \overset{*}{y}{}^i(x^1, \cdots, x^r), \quad i = 1, \cdots, r. \tag{18}$$

In the same way, substituting for $y^{r-1}, \cdots, y^k$ from (15) in (17), we obtain

$$x^i = \overset{*}{x}{}^i(y^1, \cdots, y^r), \quad i = 1, \cdots, r. \tag{19}$$

The transformations (18) and (19) are m times continuously differentiable, and, as they are mutally inverse, their Jacobians are also mutually inverse and so non-zero.

Thus assertion (F) is completely proved.

G) Let M^k be a smooth manifold of class $m \geq 2$ and L^{2k} the set of all vectors tangent to M^k (see C)); thus L^{2k} consists of pairs (a, u) where $a \in M^k$, $u \in R_a^k$. By means of the following construction L^{2k} becomes in a natural way a smooth $2k$-dimensional manifold of class $m-1$. Let U^k be a neighbourhood in M^k with local system X. By U^{2k} we will denote the set of all pairs $(x, u) \in L^{2k}$ subject to the condition $x \in U^k$. The set U^{2k} constitutes a neighbourhood in L^{2k} and a distinguished coordinate system is defined in it as follows. Let $x^1, \cdots, x^k$ be the coordinates of the point x in the system X and let $u^1, \cdots, u^k$ be components of the vector u relative to the local system X; then for coordinates of the pair (x, u) we take the numbers

$$x^1, \cdots, x^k, u^1, \cdots, u^k. \tag{20}$$

If V^k is a neighbourhood in M^k (which may coincide with U^k) with distinguished system Y, such that $x \in V^k$, and if the coordinates of the pair (x, u) in the neighbourhood V^{2k}, generated by the system Y, are

$$y^1, \cdots, y^k, v^1, \cdots, v^k, \tag{21}$$

then the transition from the coordinates (20) to the coordinates (21) is evidently given by the relations

$$y^j = y^j(x^1, \cdots, x^k),\ j = 1, \cdots, k; \tag{22}$$

$$v^j = \sum_{i=1}^{k} \frac{\partial y^j}{\partial x^i} u^i,\ j = 1, \cdots, k, \tag{23}$$

(see (9)). These relations are $m-1$ times continuously differentiable and their Jacobian is equal to $\left|\frac{\partial y^j}{\partial x^i}\right|^2$ which is evidently positive. Since the neighbourhoods of the type U^{2k} cover L^{2k}, the given construction turns L^{2k} into a smooth manifold of class $m-1$.

H) Let R^k be a vector space of dimension k. By a ray u^* in R^k passing through the vector $u \neq 0$, we will understand the totality of all vectors tu where t is a positive real number. We fix a basis in R^k and denote by R_i^{k-1} the coordinate hyperplane $u^i = 0$. If the ray u^* does not lie in R_i^{k-1} there exists in u^* a unique vector u satisfying the condition $|u^i| = 1$; we call this vector *fundamental* with respect to the plane R_i^{k-1}. The totality of all rays for which the fundamental vector with respect to R_i^{k-1} satisfies the condition $u^i = +1$ or $u^i = -1$ we will denote U_{i1}^{k-1} or U_{i2}^{k-1} respectively. For coordinates of the ray $u^* \in U_{ip}^{k-1}$, $p = 1$ or 2, we take the components $u^1, \cdots, u^{i-1}, u^{i+1}, \cdots, u^k$ of the vector u of the ray which is fundamental with respect to the plane R_i^{k-1}. As the system of all sets U_{ip}^{k-1} covers the space S^{k-1} of all rays, so S^{k-1} becomes a smooth manifold, evidently homeomorphic to the $(k-1)$-dimensional sphere.

8) Let M^k be a smooth manifold of class m. By its *manifold of line elements* we understand the set L^{2k-1} of all pairs (x, u^*), where $x \in M^k$ and u^* is a ray of R_x^k, to which is assigned a natural differential structure by means of the following construction. Let U^k be a neighbourhood in M^k with distinguished system X. In the vector space R_x^k, tangent to M^k at the point x, there is defined a basis corresponding to the local system X; in this way regions $U_{ip,x}^{k-1}$ (see H)) are defined in

the set S_x^{k-1} of rays of the space R_x^k, with their coordinate systems. By U_{ip}^{2k-1} we denote the totality of all pairs (x, u^*) satisfying the condition $x \in U^k$, $u^* \in U_{ip,x}^{k-1}$, and for coordinates of the pair (x, u) in U_{ip}^{2k-1} we take the numbers

$$x^1, \cdots, x^k, u^1, \cdots, u^{i-1}, u^{i+1}, \cdots, u^k, \tag{24}$$

where $x^1, \cdots, x^k$ are the coordinates of the point x in the system X and $u^1, \cdots, u^{i-1}$, $u^{i+1}, \cdots, u^k$ are the coordinates of the ray u^* in $U_{ip,x}^{k-1}$. It is easy to verify that the system of neighbourhoods U_{ip}^{2k-1} covers L^{2k-1} and that the coordinate systems introduced into these neighbourhoods are compatible among themselves, so that L^{2k-1} is a $(2k-1)$-dimensional smooth manifold of class $m-1$.

K) Let M^k and N^l be two smooth manifolds of class m, M^k having no boundary points. The direct[1] product $P^{k+l} = M^k \times N^l$, that is, the set of all pairs (x, y), $x \in M^k$, $y \in N^l$, appears in a natural way as a smooth manifold of class m on the basis of the following construction. Let U^k and V^l be arbitrary coordinate neighbourhoods in the manifolds M^k and N^l and let X and Y be coordinate systems in them. The set $U^k \times V^l \subset M^k \times N^l$ we take as a coordinate neighbourhood in the manifold P^{k+l}, taking as coordinates of the point $(x, y) \in U^k \times V^l$ the numbers $(x^1, \cdots, x^k, y^1, \cdots, y^l)$ where $(x^1, \cdots, x^k)$ are the coordinates of the point x in the system X and $(y^1, \cdots, y^l)$ are the coordinates of the point y in the system Y. It is immediately clear that the system of coordinate neighbourhoods so constructed turns P^{k+l} into a smooth manifold of class m. If M^k and N^l are oriented and the systems X and Y correspond to the orientations of the manifolds then we regard the system X, Y as corresponding to the orientation of P^{k+l}. In this way, the direct product of oriented manifolds acquires a natural orientation. If N^{l-1} is the boundary of the manifold N^l, then $M^k \times N^{l-1}$ is the boundary of the manifold $M^k \times N^l$.

L) Let E^{k+l} be a vector space of dimension $k+l$ and let $G(k, l)$ the set of all its k-dimensional subspaces. The set $G(k, l)$ becomes a smooth (even analytic) manifold on the basis of the following construction. Let $E_0^k \in G(k, l)$ and let $(e_1, \cdots, e_k, f_1, \cdots, f_l)$ be a basis of E^{k+l} with the property that the vectors $e_1, \cdots, e_k$ lie in E_0^k. We designate by E^l the space spanned by the vectors $f_1, \cdots, f_l$, and we let U^{kl} stand for the set of all vector subspaces $E^k \in G(k, l)$ intersecting E^l in the zero vector only. If $E^k \in U^{kl}$, it possesses a basis $(e_1', \cdots, e_k')$ determined by the relations

$$e_i' = e_i + \sum_{j=1}^{l} x_i^j f_j, \qquad i = 1, \cdots, k,$$

where $\|x_i^j\|$ is a matrix of real numbers. The elements x_i^j, $i = 1, \cdots, k$; $j = 1, \cdots, l$ of this matrix may be taken as coordinates of the element E^k in the coordinate

1) Translator's note: i.e., topological product.

neighbourhood U^{kl}. It may be immediately verified that the totality of coordinate neighbourhoods of the form U^{kl} define an analytic structure in $G(k, l)$ so that $G(k, l)$ becomes an analytic manifold of dimension kl.

§ 2. Embedding smooth manifolds in Euclidean space

In the present section it will be shown that a compact k-dimensional smooth manifold of class $m \geq 2$ may be mapped regularly and homeomorphically into Euclidean space R^{2k+1} of dimension $2k+1$ and regularly into Euclidean space R^{2k} of dimension $2k$, the map itself being smooth of class m. This proposition in somewhat stronger form, namely for $m \geq 1$ and without the requirement of compactness, was first proved by Whitney [5]; the proof given here is considerably simpler.

In the proof an essential role is played by the very elementary Theorem 1.

Smooth maps of manifolds unto manifolds of higher dimension.

Theorem 1. *Let M^k and N^l be two smooth manifolds of dimensions k and l, where $k < l$, and let ϕ be a smooth map of class 1 of the manifold M^k into the manifold N^l. Then the set $\phi(M^k)$ is of the first category in N^l, that is, it may be represented as the union of countably many sets nowhere dense in N^l. In particular, if the manifold M^k is compact, the set $\phi(M^k)$ is also compact and $N^l - \phi(M^k)$ is*[1] *everywhere dense in N^l.*

Proof. Let $a \in M^k$, $b = \phi(a)$, let V_b^l be a coordinate neighbourhood of the point b in N^l and let U_a^k be a coordinate neighbourhood of a in M^k such that $\phi(U_a^k) \subset V_b^l$. We choose neighbourhoods U_{a1}^k, U_{a2}^k of a in M^k such that $\bar{U}_{a1}^k \subset U_a^k$, $\bar{U}_{a2}^k \subset U_{a1}^k$ and $\bar{U}_{a1}^k$ is compact. The regions U_{a2}^k, $a \in M^k$, cover the manifold M^k. A countable subcovering of this covering may be chosen and so it is sufficient, for the proof of this theorem, to show that, for an arbitrary choice of the point $a \in M^k$ the set $\phi(\bar{U}_{a2}^k)$ is nowhere dense in V_b^l. Since the region U_a^k is a homeomorphic image of a region of the Euclidean half-space E_0^k we may simply regard U_{a2}^k as a region of the half-space E_0^k. In the same way we may regard V_b^l as a region of the Euclidean half-space E_0^l. Thus ϕ may be regarded as a smooth map of class 1 of the region U_a^k into the Euclidean space E^l and it is sufficient for us to show that the set $\phi(\bar{U}_{a2}^k)$ is nowhere dense in E^l. This we now prove.

From the smoothness of the map ϕ and the compactness of the set $\bar{U}_{a1}^k$ follows immediately the existence of a positive constant c such that for any two points x and x' of $\bar{U}_{a1}^k$ the inequality

$$\rho(\phi(x), \phi(x')) < c\rho(x, x'), \tag{1}$$

is satisfied. We choose an ϵ-cube subdivision of the Euclidean half-space, that is, we divide the half-space E_0^k in a regular way into cubes of edge-length ϵ. We denote by Ω the totality of all closed cubes of the chosen subdivision which meet

1) Translator's note: The author uses '$A \setminus B$' for the complement of B in A.

$\bar{U}^k_{a2}$. Since the set $\bar{U}^k_{a2}$ is compact and so can be included in a sufficiently large cube, the number of cubes of the collection Ω does not exceed c_1/ϵ^k, where c_1 is a positive constant not depending on ϵ. Let δ be the distance between the sets $E^k_0 - U^k_{a1}$ and $\bar{U}^k_{a2}$. We suppose that the diameter $\epsilon\sqrt{k}$ of each cube of Ω is less than δ. Then each cube K_i of the collection Ω lies in U^k_{a1} and in the light of the inequality (1) the set $\phi(K_i)$ is contained in some cube L_i of the space E^l of edge-length $c\sqrt{k}\cdot\epsilon$, the volume of which is equal to $c^l \cdot k^{l/2} \cdot \epsilon^l$. In this way the whole $\phi(\bar{U}^k_{a2})$ is contained in a union of cubes L_i, the number of which does not exceed c_1/ϵ^k, and so the volume of the whole space $\phi(\bar{U}^k_{a2})$ does not exceed the number $c_1 c^l k^{l/2} \cdot \epsilon^{l-k}$. Since ϵ is arbitrarily small, it follows from this that the set $\phi(\bar{U}^k_{a2})$ does not contain a region and, being compact, it must therefore be nowhere dense in E^l.

In this way Theorem 1 is proved.

Projection operators in Euclidean space. In what follows an essential role vill be played by projection operators. Let C^r be a vector space and B^q a vector ubspace. Regarding the space C^r as an additive group and B^q as a subgroup, we btain a dissection of the space C^r into residue classes modulo the subspace B^q, nd moreover the residue classes themselves form a vector space A^p of dimension $= r - q$. By associating with each element $x \in C^r$ the residue class $\pi(x) \in A^p$ hich contains it, we obtain a linear map π of the space C^r on the space A^p, alled the *projection along the projecting subspace* B^q. More intuitively the space ^{p} may be realized as a linear subspace of dimension p of the space C^r, intersecting B^q only in the zero vector; then the operator π becomes the usual proection. If the space C^r is Euclidean then, defining B^q as the orthogonal complement of the given subspace $A^p \subset C^r$, we obtain an orthogonal projection π of the pace C^r on the subspace A^p.

A) Let ϕ be a smooth map of the smooth manifold M^k into the vector space ^{r}, regular at the point $a \in M^k$, and let π be the projection of the space C^r along e 1-dimensional subspace B^1 onto the space A^{r-1}. It turns out that the map $\pi\phi$ M^k into A^{r-1} fails to be regular at the point $a \in M^k$ (see § 1, D) if and only if e line $\phi(a) + B^1$, passing through $\phi(a)$ and parallel to B^1, touches $\phi(M^k)$ at the oint $\phi(a)$.

To prove this we choose in the neighbourhood of the point a a coordinate sysn $x^1, \cdots, x^k$, and we choose in C^r rectilinear coordinates $y^1, \cdots, y^r$ such at the last axis coincides with B^1. In the chosen coordinates the map ϕ assumes form: $y^j = \phi^j(x^1, \cdots, x^k)$, $j = 1, \cdots, r$, where, by virtue of the postulated regrity of ϕ at the point a, the rank of the matrix $\|\frac{\partial\phi^j}{\partial x^i}\|$, $j = 1, \cdots, r$; $i = 1, \cdots, k$, equal to k. To each vector u on M^k at the point a corresponds the vector

$\phi_a(u) = v \in C^r$, tangent to $\phi(M^k)$ at the point $\phi(a)$, with components $v^1, \cdots, v^r$ [determined by the relations (9), § 1, $l = r$]. If now the map $\pi\phi$ is not regular at the point a, the rank of the matrix $\|\frac{\partial\phi^j}{\partial x^i}\|$, $j = 1, \cdots, r-1$; $i = 1, \cdots, k$, is less than k, and so there exists a vector $u \neq 0$, such that the vector $v = \phi_a(u)$ has the property $v^1 = \cdots = v^{r-1} = 0, v^r \neq 0$, and this means that $v \in B^1$. If conversely there exists a vector $v = \phi_a(u) \neq 0$, belonging to B^1, then the rank of the matrix $\|\frac{\partial\phi^j}{\partial x^i}\|$ $j = 1, \cdots, r-1$; $i = 1, \cdots, k$ is less than k; that is, the map $\pi\phi$ is not regular at the point a.

B) Let ϕ be a smooth regular map, of class 2, of the smooth manifold M^k into the vector space C^r of dimension $r > 2k$ and let $B^q \in G(q, r-q)$ be a projecting subspace of dimension $q \leq r - 2k$ of the space C^r onto the space A^p. The projection will be denoted by π. By Ω'_q we denote the set of all those projecting subspaces B^q for which $\pi\phi$ is not regular. It turns out that Ω'_q is a set of the first category in the manifold $G(q, r-q)$ of all projecting directions.

Let (x, u^*) be an arbitrary line element in the manifold M^k (see § 1, I) and let u be some non-zero vector of the ray u^*. To the vector u there corresponds, in the light of the relations (9) of § 1, a vector $v = \phi_x(u) \neq 0$. The ray v^* in the space C^r, defined by means of the vector v, depends only on the line element (x, u^*) and we set $v^* = \Phi(x, u^*)$. It is easy to see that the map Φ of the manifold L^{2k-1} (see § 1, I) into the manifold S^{r-1} (see § 1, H) is smooth of class 1, and so $\Phi(L^{2k-1})$ is of the first category in S^{r-1} (since $r - 1 > 2k - 1$, see Theorem 1). Thus, in the light of (A), (B) follows for $q = 1$.

Repeated application of this construction enables us to prove assertion (B) for arbitrary $q \leq r - 2k$.

C) Let ϕ be a one-one smooth map, of class 1, of the smooth manifold M^k into the vector space C^r and let $B^q \in G(q, r-q)$ be a projecting subspace of dimension $q \leq r - 2k - 1$. The projection will be denoted by π. By Ω''_q we denote the set of all those projecting subspaces B^q for which the map $\pi\phi$ is not one-one. It turns out that Ω''_q is a set of the first category in $G(q, r-q)$.

Let x and y be two distinct but arbitrary points of the manifold M^k. By $\Phi'(x, y)$ we denote the ray consisting of all vectors of the form $t(\phi(y) - \phi(x))$, where t is a positive number. In this way we obtain a map Φ' of the manifold M^{2k} of all ordered pairs (x, y), $x \neq y$, into the manifold S^{r-1} of all rays in the space C^r. A differential structure may be introduced in a natural way into the manifold M^{2k}, and it is easy to verify that the map Φ' is then smooth of class 1. Thus $\Phi'(M^{2k})$ turns out to be a set of the first category in S^{r-1} (see Theorem 1), from which (C) follows for $q = 1$. Applying this construction a sufficient number of times, we obtain a proof of assertion (C) for arbitrary $q \le r - 2k - 1$.

From (B) and (C) there immediately follows

D) Let ϕ be a one-one regular, smooth map, of class 2 of the smooth manifold M^k into the vector space C^r and let $B^q \in G(q, r-q)$ be a projecting subspace of dimension $q \le r - 2k - 1$. The projection will be denoted by π, and Ω_q will denote the set of all those projecting subspaces B^q for which the map $\pi\phi$ is not both one-one and regular. Since $\Omega_q = \Omega_q' \cup \Omega_q''$, Ω_q is a set of the first category in the manifold $G(q, r-q)$.

Embedding Theorem. E) Let $\phi_1, \cdots, \phi_n$ be smooth maps, of class m, of the smooth manifold M^k into the vector spaces $C_1, \cdots, C_n$ respectively. We denote by C the direct sum of the spaces $C_1, \cdots, C_n$, consisting of all systems $[u_1, \cdots, u_n]$ where $u_i \in C_i$. We define the direct sum ϕ of the maps $\phi_1, \cdots, \phi_n$ by setting $\phi(x) = [\phi_1(x), \cdots, \phi_n(x)]$, $x \in M^k$. It is easy to see that ϕ is a smooth map, of class m, of the manifold M^k into C. It is easy to verify that, so long as one of the maps $\phi_1, \cdots, \phi_n$ is regular at the point $a \in M^k$, then the map ϕ is also regular at a. Further, it is easy to verify that if two points a and b of M^k are transformed into distinct points by at least one of the maps $\phi_1, \cdots, \phi_n$, then they are transformed into distinct points by the map ϕ.

Theorem 2. *Let M^k be a smooth manifold, of class $m \ge 2$. There exists a smooth embedding, of class m, of the manifold M^k into Euclidean space of finite dimension.*

Proof. We denote by $K(t)$ an arbitrary real function of the real variable t, differentiable an arbitrary number of times, and satisfying the following conditions:

$$K(t) = 1 \text{ for } |t| \le \tfrac{1}{2}; \quad K(t) = 0 \text{ for } |t| \ge 1;$$

for $-1 \le t \le -\frac{1}{2}$ the function $K(t)$ is monotone increasing; for $\frac{1}{2} \le t \le 1$ the function $K(t)$ is monotone decreasing. It is easy to construct such a function. We put

$$K^i(t^1, t^2, \cdots, t^k) = t^i \cdot K(t^1) \cdot K(t^2) \cdots K(t^k), \qquad i = 1, \cdots, k,$$

$$K^{k+1}(t^1, t^2, \cdots, t^k) = K(t^1) \cdot K(t^2) \cdots K(t^k).$$

Let R^k be Euclidean space with Cartesian coordinates $t^1, \cdots, t^k$ and let R^{k+1} be Euclidean space with Cartesian coordinates $y^1, \cdots, y^{k+1}$. We designate by Q the cube in the space R^k given by the inequalities $|t^i| < 2$, by Q' the cube in the same space given by $|t^i| < 1$, and by Q'' the cube given by $|t^i| < \frac{1}{2}$. By Q_0 we designate the half cube of Q given by $t^1 < 0$. We now map the space R^k into the space R^{k+1} by the relations

$$y^j = K^j(t^1, \cdots, t^k), \qquad j = 1, \cdots, k+1. \tag{2}$$

It is easy to see that this map is arbitrarily often differentiable, and transforms the set $R^k - Q^1$ into the origin of coordinates in R^{k+1}, that it maps the cube Q' continuously and one-one, and finally that it maps the cube Q'' regularly.

Now let a be an arbitrary point of M^k, and let U_a^k be a coordinate neighbourhood of a with coordinate system X, having its origin at the point a; finally let ϵ be a sufficiently small positive number so that, under the map

$$t^i = \frac{x^i}{\epsilon}, \quad i = 1, \cdots, k \tag{3}$$

of the neighbourhood U_a^k into the space R^k, the image of U_a^k covers the whole of the cube Q if a is an interior point of M^k and covers the whole of the half cube Q_0 if a is a boundary point of M^k. Let Q_a', Q_a'' be the inverse images of Q', Q'' under this map.

We define a map ϕ_a of the manifold M^k into Euclidean space R^{k+1} by putting

$$y^j = K^j\left(\frac{x^1}{\epsilon}, \frac{x^2}{\epsilon}, \cdots, \frac{x^k}{\epsilon}\right)$$

for points $x \in U_a^k$ with coordinates $x^1, \cdots, x^k$, and $y^j = 0$ for points $x \in M^k - U_a^k$. It is easy to see that ϕ_a is a smooth map of class m of the manifold M^k into R^{k+1}, homeomorphic on Q_a' and regular on Q_a''.

By choosing from the neighbourhoods Q_a'' a finite covering $Q_{a_1}'', \cdots, Q_{a_n}''$ of the manifold M^k and forming the direct sum of the maps $\phi_{a_1}, \cdots, \phi_{a_n}$ corresponding to these cubes (see E), we obtain the required map ϕ of the manifold M^k into finite-dimensional Euclidean space.

From the proposition we have proved the theorem stated in the opening paragraph follows immediately. Indeed, the manifold M^k can be regularly and homeomorphically embedded in a vector space C^r of sufficiently high dimension (see Theorem 2). Further, in the space C^r there exists a projecting subspace B^{r-2k-1} such that the induced projection of the manifold M^k into the space A^{2k+1} is regular and homeomorphic (see D). In the same way, in the space C^r there exists a projecting subspace B^{r-2k} such that the projection of the manifold into A^{2k} is regular (see B). We prove here a stronger Theorem 3 asserting that for an arbitrary smooth map of the manifold M^k into Euclidean space C^{2k+1} there exists an arbitrarily near map of the manifold which is regular. For this formulation of Theorem 3 it is necessary to introduce the notion of proximity of maps of class m, taking into account all derivatives up to and including order m.

We remark first of all that if f is a smooth map of a region W^k of the Euclidean half-space E_0^k into the vector space C^r, the partial derivatives of the vector function $f(x) = f(x^1, \cdots, x^k)$ appear as vectors of the space C^r.

F) Let M^k be a smooth compact manifold of class m, let E^l be a vector space and let P be the set of all smooth maps of class m of the manifold M^k into the space E^l. We introduce a topology into P by assigning to it a metric, depending on a random choice of certain elements of construction. In fact, let U_s, V_s;

$s = 1, \cdots, n$ be finite collections of coordinate regions in the manifold M^k such that the regions U_s; $s = 1, \cdots, n$, cover the manifold M^k and satisfy the inclusions $\overline{U}_s \subset V_s$; $s = 1, \cdots, n$; and moreover in each region V_s let a coordinate system X_s be chosen. Further let Y be a Cartesian coordinate system in the space E^l. We define the distance $\rho(f, g)$ between two maps f and g in P in such a way which depends on the choice of regions U_s, V_s, coordinate systems X_s; $s = 1, \cdots, n$, and coordinate system Y. For this we write the maps f and g, restricted to V_s, in coordinate form by the rule

$$y^j = f_s^j(x) = f_s^j(x^1, \cdots, x^k), \qquad (4)$$

$$y^j = g_s^j(x) = g_s^j(x^1, \cdots, x^k). \qquad (5)$$

Let $i_1, \cdots, i^k$ be a set of non-negative whole numbers whose sum does not exceed m. We set

$$\omega_s^j(x; i_1, \cdots, i_k) = \left| \frac{\partial^{i_1+\cdots+i_k}(f_s^j(x) - g_s^j(x))}{(\partial x^1)^{i_1} \cdots (\partial x^k)^{i_k}} \right|.$$

The maximum of the function $\omega_s^j(x; i_1, \cdots, i_k)$ with respect to x for $x \in \overline{U}_s$ we designate by $\omega_s^j(i_1, \cdots, i_k)$, and the greatest of all the numbers $\omega_s^j(i_1, \cdots, i_k)$, when $i_1, i_2, \cdots, i_k, s, j$ vary over all admissible values, we take as the distance $\rho(f, g)$ between the maps f and g. It is easy to verify that the topological space P does not depend on the random choice of systems of regions U_s, V_s; $s = 1, \cdots, n$, and coordinate systems X_s; $s = 1, \cdots, n$; Y. The topological space P is called *the space of maps of class* m of the manifold M^k into the space E^l. The assertion that, in arbitrary 'proximity of class m' to a map f, there exists a map possessing a certain property A, means that in an arbitrary neighbourhood of the point f of the space P there exists a map possessing the property A.

Theorem 3. *Let M^k be a smooth compact k-dimensional manifold of class $m \geq 2$, let A^p be a vector space of dimension p, and let P be the space of maps of class m of the manifold M^k into the space A^p. We denote the totality of all regular maps in the set P by Π' and the totality of all maps in P which are both regular and homeomorphic we denote by Π. It turns out that the sets Π' and Π are always regions in the space P. Further, if $p \geq 2k$, the region Π' is everywhere dense in P, and, if $p \geq 2k + 1$, the region Π is everywhere dense in P.*

Proof. We show first that the sets Π' and Π are everywhere dense in the space P for the values of p indicated in the theorem. Let $f \in P$ and let e be a smooth, regular, homeomorphic map of class m of the manifold M^k into a vector space B^q of sufficiently large dimension (see Theorem 2). We denote the direct sum of A^p and B^q by C^r and we regard A^p and B^q as linear subspaces of the space C^r. The map h, direct sum of the maps f and e (see E) is regular and homeomorphic and the projection of h onto A^p along B^q coincides with the map f. By virtue of

propositions B and D it follows that, arbitrarily close to the projecting direction B^q there exists a projecting direction B_1^q such that the projection g of the map h is regular if $p \geq 2k$ and both regular and homeomorphic if $p \geq 2k + 1$. Thus, arbitrarily close to the map f there exists a map g possessing the required properties.

We show that Π' is a region. Let $f \in \Pi'$. Since the map f is regular at the point $x \in U_s$, the rank of the matrix $\left\| \frac{\partial f_s^j}{\partial x^i} \right\|$ at this point is equal to k (see F). From this it follows that the rank of a matrix near to $\left\| \frac{\partial f_s^j}{\partial x^i} \right\|$ is also k. Thus there exists a sufficiently small positive number ϵ' such that for $\rho(f, g) < \epsilon'$ the map g is regular at the point x. Since the first derivatives of the functions $f_s^j(x)$ are continuous and the sets $\overline{U}_s$ are compact and a finite number of them cover the manifold M^k, there exists a sufficiently small positive number ϵ such that for $\rho(f, g) < \epsilon$ the map g is regular at every point of M^k.

To prove that Π is a region we first note the following fact.

a) In the space Q of all linear maps of the Euclidean vector space E^k into the Euclidean vector space A^p we introduce a metric by means of coordinate systems X and Y in these spaces. Let ϕ and ψ be elements of Q with coordinate expressions:

$$y^j = \sum_{i=1}^{k} \phi_i^j x^i, \qquad j = 1, \cdots, p;$$

$$y^j = \sum_{i=1}^{k} \psi_i^j x^i, \qquad j = 1, \cdots, p.$$

We define the distance $\rho(\phi, \psi)$ as the greatest of the numbers $|\phi_i^j - \psi_i^j|$. It turns out that for each compact set F of non-degenerate maps there exists a sufficiently small positive number δ such that for $\rho(F, \psi) < \delta$ we have

$$|\psi(x)| > \delta \cdot |x|,$$

where x is an arbitrary vector of E^k.

This proposition is easily proved by reductio ad absurdum from considerations of continuity.

Let $f \in \Pi$. It turns out that there exist small numbers δ and ϵ such that for $\rho(f, g) < \epsilon$ (see F) we have the inequality

$$\rho(g(a), g(x)) \geq \delta \rho(f(a), f(x)), \qquad (6)$$

where a and x are two arbitrary points of M^k. Indeed, when $\rho(f(a), f(x)) < \alpha$, where α is some positive constant, the maps f and g are sufficiently well represented near the point a by linear maps and moreover uniformly with respect to the point $a \in M^k$. In this case the inequality (6) follows easily from proposition (a). In the case when $\rho(f(a), f(x)) \geq \alpha$ the inequality (6) follows for sufficiently small

ϵ from the one-one property of the map f. From the inequality (6) and the one-one property of the map f follows the one-one property of maps g sufficiently close to f.

Thus, Theorem 3 is completely proved.

§3. Improper points of smooth maps

We recall first the definition of improper points of a map (see §1, D). Let ϕ be a smooth map of the manifold M^k into the manifold N^l. The point a of the manifold M^k is called an improper point of the map ϕ if the functional matrix of the map ϕ at the point a has rank less than l. The point b of the manifold N^l is called an improper point of the map ϕ, if the counterimage $\phi^{-1}b$ of this point contains at least one improper point of the map ϕ. Thus it is necessary to distinguish improper points of the map ϕ lying in M^k from improper points of the map ϕ lying in N^l. If F is the set of all improper points of the map lying in the manifold M^k, then $\phi(F)$ is the set of all improper points of ϕ lying in N^l. Theorem 4 following, due to Dubovitskiĭ [6], asserts that $\phi(F)$ is a set of the first category in N^l, that is, it may be represented as the countable sum of compact nowhere dense subsets of N^l. From this it follows that the set $N^l - \phi(F)$ of all proper points of the map ϕ lying in N^l is of the second category in N^l, that is, "it is big enough" and, in any case, everywhere dense. This fact can be expressed in somewhat informal terms by saying that the points of the manifold N^l are, in general, proper. Theorem 4 plays a very important role in the theory of smooth manifolds; from it may be deduced a whole series of results about what, in general terms, can occur under this or that hypothesis. To deduce each of these results it is necessary to select properly the manifolds M^k and N^l and the map ϕ. This choice can be described with the help of the following very general and so somewhat vague proposition (A).

Reduction to general position. A) Let Q be some smooth manifold and let P be some collection of functions on it, also forming a smooth manifold. For a given function $p \in P$ on the manifold Q some point $q \in Q$ may be regarded as special[1)] in a sense which can be precisely explained. Pairs (p, q), $p \in P$, $q \in Q$, are called *distinguished* if the point q is special with respect to the function p. It is supposed that the set of all distinguished pairs (p, q) forms a smooth submanifold M^k of the manifold $P \times Q$ (see §1; K, E). To each point $(p, q) \in M^k$ may be set in correspondence the point $\phi(p, q) = p$. In this way there arises a map ϕ of the manifold M^k into the manifold $N^l = P$. If the point $p_0 \in P$ is a proper point of the map ϕ, then each point $q \in Q$ which is special with respect to the function p_0 may be regarded as in some sense *typical*, and the collection Q_0 of all points q of the manifold Q which are special with respect to the function p_0 consists of typical special points.

1) Translator's note: The reader may prefer the word 'singular' to the word 'special' here—either is a legitimate translation from the Russian.

The construction (A) has numerous applications, some of which will be described in §4. A very simple application of construction (A), of an illustrative character, I give here in the form of proposition (B).

B) Let A^r and B^s be two smooth submanifolds of a vector space E^n. The manifolds A^r and B^s are said to be *in general position* at the point $a \in A^r \cap B^s$ if the tangents at this point to the manifolds A^r and B^s have an intersection of dimension $r + s - n$. The manifolds A^r and B^s are said to be *in general position* if they are in general position at each point of their intersection. It may readily be verified that if A^r and B^s are in general position their intersection $A^r \cap B^s$ is a submanifold of the space E^n of dimension $r + s - n$. Let $p \in E^n$. We denote by A^r_p the manifold consisting of all points of the form $p + x$, where $x \in A^r$. Thus the manifold A^r_p is obtained from the manifold A^r by displacement along the vector p. It turns out that the set of all vectors $p \in E^n$ for which the manifolds A^r_p and B^s are in general position is of the second category in E^n so that there exist arbitrarily small displacements p for which the manifolds A^r_p and B^s are in general position.

To prove proposition (B) we make use of construction (A) by putting $Q = A^r \times B^s$, $P = E^n$ and regarding the point $q = (a, b) \in A^r \times B^s$ as special with respect to the function $p \in E^n$ if $p + a = b$. The collection M^k of all distinguished pairs (p, q), where $p \in E^n$, $q = (a, b) \in A^r \times B^s$ is thus defined by the relation $p = b - a$, that is, the pair (p, q) is uniquely determined by the point $q = (a, b)$, and so there arises in a natural way a smooth homeomorphism of the manifolds M^k and $A^r \times B^s$, by means of which we can identify these manifolds. The map ϕ of the manifold $M^k = A^r \times B^s$ into the manifold $P = E^n$ is defined by the formula $\phi(a, b) = b - a$. A simple calculation shows that the point $q = (a, b) \in M^k$ is a proper point of the map ϕ if and only if the manifolds A^r_{b-a} and B^s are in general position at the point b of their intersection. Thus the point $p_0 \in E^n$ is a proper point of the map ϕ if and only if A^r_{p0} and B^s are in general position. From this and from Theorem 4 proved below the assertion of proposition (B) follows.

The theorem of Dubovitskiĭ. In the formulation due to Dubovitskiĭ himself the smoothness class of the map ϕ of the manifold M^k into the manifold N^l is given by the formula $m = k - l + 1$ and not by formula (1). In this sense Theorem 4 is weaker than the theorem of Dubovitskiĭ. Since the precise value of the smoothness class is not essential in what follows, I give here the crude value (1) which makes it possible to simplify the proof.

Theorem 4. *Let M^k and N^l be two smooth manifolds of positive dimensions k and l and let ϕ be a map of M^k into N^l of smoothness class*

$$m = m(k, l) = 2 + \frac{(k-l)(k-l+1)}{2}. \qquad (1)$$

It turns out that the set of all improper points of the map ϕ lying in the manifold N^l is of the first category in N^l. In fact, if the manifold M^k is compact, the complement of this set is an everywhere dense region in the manifold N^l.

Proof. We consider first the case when the manifold M^k is without boundary. Let $a \in M^k$, $b = \phi(a)$, let V_b^l be a coordinate neighbourhood of the point b in the manifold N^l, and let U_a^k be a coordinate neighbourhood of the point a in the manifold M^k such that $\phi(U_a^k) \subset V_b^l$. We choose neighbourhoods U_{a1}^k, U_{a2}^k of the point a in M^k such that $\bar{U}_{a1}^k < U_a^k$, $\bar{U}_{a2}^k \subset U_{a1}^k$, and such that the set $\bar{U}_{a1}^k$ is compact. The regions u_{a2}^k, $a \in M^k$ cover the manifold M^k. From this a countable subcovering can be chosen and so for the proof of the theorem it is sufficient for us to establish the assertion of the theorem for the map ϕ of the manifold $U_{a2}^k \subset M^k$ into the manifold V_b^l. Since the region U_a^k is the homeomorphic image of a region of Euclidean space E^k, we may simply suppose that U_a^k is itself a region of the space E^k. In exactly the same way we will suppose that V_b^l is a region of Euclidean space E^l. By this device the map ϕ becomes a smooth map of class m of the region U_a^k into Euclidean space E^l and it is sufficient for us to demonstrate that the set of improper points is of the first category in E^l. We prove this.

We fix the point a and drop the index a from the notation for its neighbourhoods. The map ϕ of the region U^k of Euclidean space E^k into Euclidean space E^l may be expressed in Cartesian coordinates:

$$y^j = \phi^j(x) = \phi^j(x^1, \cdots, x^k), \qquad j = 1, \cdots, l. \qquad (2)$$

Here the functions ϕ^j are m times continuously differentiable. We denote by F_0 the set of all points $x \in U_2^k$ at which the functional matrix $\left\| \frac{\partial \phi^j}{\partial x^i} \right\|$, $i = 1, \cdots, k$; $j = 1, \cdots, l$, has rank less than l. If $K < l$ Theorem 4 is a restatement of the already proved Theorem 1 so we suppose that $k \geq l$. We put $s = k - l + 1$. The function ϕ^l will play a special role in what follows. From (1) it follows that $m > s$ and so the function ϕ^l is still $(s + 1)$ times continuously differentiable. Let r be a natural number not exceeding s. We denote by F_r the collection of all points of F_0 at which all the partial derivatives of ϕ^l of orders $1, 2, \cdots, r$ vanish. Evidently we have:

$$F_0 \supset F_1 \supset \cdots \supset F_s.$$

It will be shown that the images of each of the sets $F_0 - F_1$, $F_1 - F_2$, $\cdots$, $F_{s-1} - F_s$, F_s under the map ϕ are of the first category in E^l. Thus it will be shown that the set $\phi(F_0)$ of improper points of the map ϕ is itself of the first category in E^l.

First we shall concern ourselves with the set F_s. The expansion of the function ϕ^l in a Taylor series at the point $p \in F_s$ does not contain terms of degree

$1, 2, \cdots, s$. From this and from the compactness of the set $\bar{U}_1$ may be deduced the existence of a positive constant c such that for $p \in F_s$, $x \in \bar{U}_1$ we have;

$$|\phi^l(x) - \phi^l(p)| < c \cdot (\rho(p, x))^{s+1}. \qquad (3)$$

The remaining functions ϕ^j; $j = 1, \cdots, l - 1$ satisfy the inequality:

$$|\phi^j(x) - \phi^j(p)| < c\rho(p, x), \qquad (4)$$

on account of the continuity of the first derivatives and the compactness of the set $\bar{U}_1$. The constant c in the inequalities (3) and (4) has been given a common value for all the functions ϕ^j; $j = 1, 2, \cdots, l$. We choose in E^k an ϵ-cube subdivision, that is, we divide the space E^k in a regular way into cubes of edge-length ϵ, and we denote by Ω the totality of all closed cubes of the chosen subdivision which intersect F_s. Since the set $\bar{F}_s$ is compact, the number of cubes of Ω does not exceed $c_1/_\epsilon k$, where c_1 is a positive constant independent of ϵ. Let δ be the distance between the sets $E^k - U_1^k$ and $\bar{U}_2$. We will suppose that $\epsilon < \delta/\sqrt{k}$; then each cube K_q of Ω is contained in U_1^k. From this, together with the fact that the cube K_q contains a point $p \in F_s$ and from the inequalities (3), (4) it follows that the set $\phi(K_q)$ is contained in a rectangular parallelepiped L_q of the space E^l, one edge-length if which is equal to $2c\sqrt{k} \cdot \epsilon^{s+1}$ and the remaining $(l - 1)$ edge-lengths of which are equal to $2c\sqrt{k} \cdot \epsilon$. The volume of the parallelepiped L_q is equal to $2^l c^l k^{l/2} \epsilon^{l+s}$. The compact set $\phi(\bar{F}_s)$ is contained in the sum of closed parallelepipeds L_q, the number of which does not exceed c_1/ϵ^k. From this it follows that the volume of the set $\phi(\bar{F}_s)$ does not exceed $c_2\epsilon^{l+s-k} = c_2\epsilon$ (c_2 does not depend on ϵ), and since ϵ is arbitrarily small the compact set $\phi(\bar{F}_s)$ does not contain a region of the space E^l and so is nowhere dense in E^l.

If $k = 1$, then in view of the hypothesis $k \geq l \geq 1$ we have $l = 1$, $s = 1$. In this case $F_s = F_0$ and from what has been proved the assertion of the theorem follows for $k = 1$. This gives us the basis for an induction on k. We suppose the theorem true when the manifold being mapped has dimension less than k and prove it for manifolds of dimension k.

We prove that for $0 \leq r < s$ the set $\phi(F_r - F_{r+1})$ is of the first category in the space E^l. In fact this part of the proof of the theorem will be carries out inductively. Let $p \in F_r - F_{r+1}$. Since p does not belong to F_{r+1} there exists a partial derivative of order $(r + 1)$ of the function ϕ^l which does not vanish at p. The value of this derivative at the point $x \in U^k$ we denote by $\omega_1(x)$. Since $\omega_1(x)$ is a derivative of order $r + 1$, it follows that $\omega_1(x) = \partial\omega(x)/\partial x^i$, where $\omega(x)$ is a derivative of order r if $r > 0$ and the actual function $\phi^l(x)$ if $r = 0$. We will suppose for the sake of definiteness that $i = k$. We set

$$z^i = x^i, \ i = 1, \cdots, k - 1; \ z^k = \omega(x) = \omega(x^1, \cdots, x^k). \qquad (5)$$

Since $\partial\omega(p)/\partial x^k \neq 0$ it follows that the fundamental determinant of the transformation (5) is non-zero at the point p and so this transformation introduces new coordinates $z^1, \cdots, z^k$ into some neighbourhood W^k_p of the point p. We will suppose that W^k_p does not intersect F_{r+1} and we choose a neighbourhood W^k_{p1} of the point p whose closure is compact and contained in W^k_p. By varying the point p we can cover the set $F_r - F_{r+1}$ by a countable system of neighbourhoods of the form W^k_{p1}. Thus to prove that $\phi(F_r - F_{r+1})$ is of the first category, it is sufficient for us to establish that $\phi(F_r \cap \overline{W}^k_{p1})$ is nowhere dense in E^l. We now concern ourselves with the proof of this fact.

We fix the point p and drop the index p from the notation for its neighbourhoods. Substituting for $x^1, \cdots, x^k$ their expressions in terms of $z^1, \cdots, z^k$, we obtain from (2) an expression for the map ϕ relative to the coordinates $z^1, \cdots, z^k$ in the region W^k. Let this expression be:

$$y^j = \phi^j(x) = \psi^j(z^1, \cdots, z^k). \tag{6}$$

Here $z^1, \cdots, z^k$ are the new coordinates of the point x. We will consider the region W^k with coordinates $z^1, \cdots, z^k$ as a smooth manifold. From the relations (5) it follows that the map ϕ of the smooth manifold W^k into the space E^l given by the relations (6) has smoothness class $m(k, l) - r$. For $r = 0$ the smoothness class of the map ϕ, so considered, is equal to $m(k, l) = m(k-1, l-1)$ (see (1)). By the choice for $r > 0$ of the least favourable value for the smoothness class, obtained for $r = s - 1 = k - l$, we see that for $r > 0$ the smoothness class of ϕ (as a map of W^k) is equal to $m(k, l) - (k - l) = m(k-1, l)$ (see (1)). The set $H \subset W^k$ of all improper points of the map ϕ of the manifold W^k is determined by the equality $H = W^k \cap F_0$. This follows from the fact that the transformation (5) is not degenerate in W^k. We designate by W^{k-1}_t the submanifold of the manifold W^k, given by the equation $z^k = t$. We remark that the smoothness class of the map of the manifold W^{k-1}_t into E^l is equal to $m(k-1, l-1)$ for $r = 0$ and to $m(k-1, l)$ for $r > 0$. We now examine separately the cases $r = 0$ and $r > 0$.

Let $r = 0$. Then $\omega(x) = \phi^l(x) = z^k$. Thus the expression (6) for the map ϕ assumes the special form:

$$y^j = \psi^j(z^1, \cdots, z^k);\ j = 1, \cdots, l-1;\ y^l = z^k. \tag{7}$$

We denote by E^{l-1}_t the linear subspace of the space E^l determined by the equation $y^l = t$. It follows from the relations (7) that $\phi(W^{k-1}_t) \subset E^{l-1}_t$. We denote by $H_t \subset W^{k-1}_t$ the set of all improper points of the map ϕ of the manifold W^{k-1}_t into E^{l-1}_t. From the relations (7) it follows that $H_t = H \cap W^{k-1}_t$. If the set $\phi(F_0 \cap \overline{W}^k_1)$ contained a region, then there would exist a value of t for which the intersection $\phi(F_0 \cap \overline{W}^k_1) \cap E^{l-1}_t$ would contain a region of E^{l-1}_t. This however is impossible since

$$\phi(F_0 \cap \overline{W}^k_1) \cap E^{l-1}_t \subset \phi(H) \cap E^{l-1}_t = \phi(H \cap W^{k-1}_t) = \phi(H_t),$$

and the set $\phi(H_t)$ is, by the inductive hypothesis, of the first category in E^{l-1}_t. Thus the set $\phi(F_0 \cap \overline{W}^k_1)$ is nowhere dense in E^l and the case $r = 0$ is disposed of.

Now let $r > 0$; then $\omega(x)$ is a derivative of order r of the function ϕ^l, and so $\omega(x) = 0$ for $x \in F_r$. Since, in the neighbourhood W^k, we have $\omega(x) = z^k$, it follows that

$$F_r \cap W^k \subset W^{k-1}_0.$$

Let $H' \subset W^{k-1}_0$ be the set of all improper points of the map ϕ of the manifold W^{k-1}_0 into the space E^l. It is easy to see that $H \cap W^{k-1}_0 \subset H'$ (see (6)), and since $F_r \cap W^k_1 \subset H$ it follows from this and from (8) that $F_r \cap W^k_1 \subset H'$. In view of the inductive hypothesis the set $\phi(H')$ is of the first category in E^l and since $F_r \cap W^k_1 \subset H'$, the set $\phi(F_r \cap W^k_1)$ is nowhere dense in E^l. This completes the analysis of the case $r > 0$.

Thus Theorem 4 is proved for a manifold M^k without boundary.

Finally let the manifold M^k possess a boundary M^{k-1}. Let $F' \subset M^{k-1}$ be the set of all improper points of the manifold M^{k-1} into the manifold N^l and let $F \subset M^k$ be the set of all improper points of the map ϕ of the manifold M^k into N^l. It is easy to see that

$$F \cap M^{k-1} \subset F'.$$

Thus

$$F \subset (F - M^{k-1}) \cup F'.$$

The set $F - M^{k-1}$ may be regarded as the collection of all improper points of the map ϕ of the manifold $M^k - M^{k-1}$, which is without boundary. In exactly the same way the set F' may be regarded as the totality of all improper points of the map ϕ of the manifold M^{k-1}, which is without boundary. Thus both the sets $\phi(F - M^{k-1})$ and $\phi(F')$ are of the first category in N^l. The set $\phi(F)$ is contained in their sum and so is also of the first category in N^l.

Thus Theorem 4 is completely proved.

§4. Non-degenerate singular points of smooth maps

Let f be a smooth map of the manifold M^k into the manifold N^l. Let $a \in M^k$ and $b = f(a) \in N^l$ be interior points of the manifolds M^k and N^l. In the neighbourhoods of the points a and b we introduce local coordinates $x^1, \cdots, x^k$ and $y^1, \cdots, y^l$, taking these points as origin of coordinates. Let

$$y^j = f^j(x) = f^j(x^1, \cdots, x^k)$$

be the expressions for the map f in the chosen coordinates.

We suppose that a is a regular point of the map f, that is, that the rank of the matrix $\left\| \frac{\partial f^j(a)}{\partial x^i} \right\|$, $j = 1, \cdots, l$; $i = 1, \cdots, k$ is equal to k, and we will assume for definiteness that the determinant $\left| \frac{\partial f^j(a)}{\partial x^i} \right|$, $i, j = 1, \cdots, k$, is non-zero. With these hypotheses the relations

$$\xi^i = f^i(x^1, \cdots, x^k), \quad i = 1, \cdots, k$$

can be used to introduce new coordinates $\xi^1, \cdots, \xi^k$ for the point x in the neighbourhood of the point a. Let

$$y^j = \xi^j, \quad j = 1, \cdots, k; \; y^j = \phi^j(\xi^1, \cdots, \xi^k), \quad j = k+1, \cdots, l$$

be the expressions for the map f in these new coordinates. We introduce new coordinates $\eta^1, \cdots, \eta^l$ into the neighbourhood of the point b by setting

$$\eta^j = y^j, \quad j = 1, \cdots, k; \; \eta^j = y^j - \phi^j(y^1, \cdots, y^k), \quad j = k+1, \cdots, l.$$

In the coordinates $\xi^1, \cdots, \xi^k, \eta^1, \cdots, \eta^l$ the map f assumes the form

$$\eta^j = \xi^j, \quad j = 1, \cdots, k; \; \eta^j = 0, \quad j = k+1, \cdots, l. \tag{1}$$

We now suppose that the point a is proper, that is, that the rank of the matrix $\left\| \frac{\partial f^j(a)}{\partial x^i} \right\|$, $j = 1, \cdots, l$; $i = 1, \cdots, k$ is equal to l, and we assume for definiteness that the determinant $\left| \frac{\partial f^j(a)}{\partial x^i} \right|$, $i, j = 1, \cdots, l$, is non-zero. Then the relations

$$\xi^i = f^i(x^1, \cdots, x^k), \quad i = 1, \cdots, l; \; \xi^i = x^i, \quad i = l+1, \cdots, k$$

can be used to introduce new coordinates $\xi^1, \cdots, \xi^k$ for the point x in the neighbourhood of the point a. If moreover we introduce the identity transformation

$$\eta^j = y^j, \quad j = 1, \cdots, l,$$

we see that, in the coordinates $\xi^1, \cdots, \xi^k, \eta^1, \cdots, \eta^l$, the map f assumes the form

$$\eta^j = \xi^j, \quad j = 1, \cdots, l. \tag{2}$$

Thus, if the manifold M^k is closed and if $b \in N^l$ is a proper point of the map f, then $f^{-1}(b)$ is a smooth $(k-1)$-dimensional submanifold of M^k with local coordinates $\xi^{k+1}, \cdots, \xi^k$ in the neighbourhood of the point a. If the manifolds M^k and N^l are oriented and the orientations are given by the coordinate systems $\xi^{l+1}, \cdots, \xi^k, \xi^1, \cdots, \xi^l$ and $\eta^1, \cdots, \eta^l$ respectively, then the manifold $f^{-1}(b)$ acquires a natural orientation, given by the coordinate system $\xi^{l+1}, \cdots, \xi^k$.

We see that both in the case of a regular point a and in the case of a proper point a, the map has a very simple expression in properly chosen coordinate systems (see (1), (2)).

In §2 it was shown that in any proximity to an arbitrary smooth map of the manifold M^k into the vector space A^{2k} there exists a regular map and that all maps which are sufficiently close to a regular map are regular (see Theorem 3). In this sense singular points (see §1, D) of maps of the manifold M^k into the space A^{2k} are unstable – being destroyed by small movements of the map; regular points, on the other hand, are stable. For maps of the manifold M^k into the vector space A^{2k-1} the situation is quite different: here the property of being a singular point

is, in general, stable – not being destroyed by small movements. In this case the problem arises of describing typical stable singular points. This problem was solved by Whitney, and a new and much simpler proof of his theorem (see Theorem 6) is presented here. This theorem is not used in the present work. The problem of typical singular points is also solved here for maps of the manifold M^k into the one-dimensional vector space A^1, that is, into a straight line (see Theorem 5; in the sequel it is applied to the homotopy theory of maps, see §14). Thus the problem of typical singular points of maps is solved for maps of manifolds of dimension k into spaces of dimension $(2k-1)$ and 1. For the remaining dimensions it is not solved and presents considerable interest.

A regular map of the manifold M^k into the vector space A^{2k} is, in general, non-homeomorphic – it can have self-intersections which can turn out to be unstable under small movements of the map. The problem of typifying self-intersections is also solved here (see (A) and (B); these propositions will be used in the sequel.

For the proof of Theorems 5 and 6 and also of proposition (B) essential use is made of the construction (A) of the preceding paragraph and of Theorem 4.

Typical points of self-intersection for maps of a manifold M^k into the vector space E^{2k}.

A) Let f be a smooth regular map of class $m \geq 1$ of the closed manifold M^k into the vector space A^{2k} and let a and b be two distinct points of M^k which are transformed by f into the same point $f(a) = f(b)$ of the space A^{2k}. Further let U and V be neighbourhoods of the points a and b in M^k such that f is homeomorphic on each of them, and let T^k_a and T^k_b be tangent at $f(a)$ and $f(b)$ to the manifolds $f(U)$ and $f(V)$. We will say that the map f is *typical* at the pair of self-intersections[1] a, b if the tangent spaces T^k_a and T^k_b are in general position, that is, intersect only in one point $f(a) = f(b)$. It is evident that is this case, for sufficiently small neighbourhoods U and V, the manifolds $f(U)$ and $f(V)$ have a unique common point $f(a) = f(b)$ (Implicit Function theorem), and that for small movements of the map typical self-intersections are preserved. If the map f is typical at each pair of self-intersections and if moreover no three distinct points are transformed by f into the same point, then we will say that the map f is *typical*. Since the manifold M^k is closed it follows that if the map f is typical at each pair of self-intersections then there exist only finitely many pairs of self-intersections.

B) Let f be a smooth regular homeomorphism of the closed manifold M^k into the vector space C^{2k+1}. The collection P^{2k} of all pairs (x, y), where $x \in M^k$, $y \in M^k$, $x \neq y$, forms in a natural way a smooth manifold of dimension $2k$. With

1) Translator's note: A pair of self-intersections (of f) is simply a pair of points at which f takes the same value.

each point $(x, y) \in P^{2k}$ we associate the point $\sigma(x, y) = (f(y) - f(x))^* \in S^{2k}$, that is, the ray of the vector $f(y) - f(x)$ (see §1, H). Let e be an arbitrary non-zero vector of the space C^{2k+1}, and π_e the projection in the direction of the one-dimensional subspace e^{**} containing the vector e. It turns out that the regular map $\pi_e f$ is typical at each pair of self-intersections (see (A)) if and only if the map σ is proper at the point $e^* \in S^{2k}$. From this, in view of Theorem 4, it follows that, arbitrarily close to any one-dimensional projecting direction, there exists a one-dimensional projecting direction e^{**} such that the map $\pi_e f$ is typical at each pair of self-intersections. Further it turns out that, arbitrarily close to any one-dimensional projecting direction, there exists a direction e_0^{**} such that $\pi_{e_0} f$ is typical.

We prove proposition (B). Let $e_1, \cdots, e_{2k+1}$ be a basis of the vector space C^{2k+1}. We denote by W the collection of all vectors $u = \sum_{n=1}^{2k+1} u^n e_n$ of the space C^{2k+1} for which $u^{2k+1} > 0$, and by W^* the set of all rays u^* with $u \in W$. As coordinates of the ray $u^* \in W^*$ we take the numbers $u^{*n} = u^n/u^{2k+1}$, $n = 1, \cdots, 2k$; thereby local coordinates are introduced into the region W^* of the manifold S^{2k} (see §1, H). Now let a, b be two distinct points of the manifold M^k. We choose a basis $e_1, \cdots, e_{2k+1}$ with $e_{2k+1} = e = f(b) - f(a)$. In neighbourhoods of the points a and b of the manifold M^k we choose local coordinates $x^1, \cdots, x^k$ and $y^1, \cdots, y^k$ and let

$$u^n = f_a^n(x^1, \cdots, x^k) = f_a^n(x), \quad n = 1, \cdots, 2k+1; \tag{3}$$

$$u^n = f_b^n(y^1, \cdots, y^k) = f_b^n(y), \quad n = 1, \cdots, 2k+1 \tag{4}$$

be the coordinate form of the map f in the neighbourhoods of the points a and b respectively. By projecting in the direction of the vector $e = f(b) - f(a)$ the points b and a coalesce: $\pi_e f(a) = \pi_e f(b)$; and evidently the condition that the map $\pi_e f$ is typical at the pair of self-intersections a, b is implied by the condition that the determinant

$$\begin{vmatrix} \frac{\partial f_a^1(a)}{\partial x^i} & \cdots\cdots & \frac{\partial f_a^{2k}(a)}{\partial x^i} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_a^1(a)}{\partial x^k} & \cdots\cdots & \frac{\partial f_a^{2k}(a)}{\partial x^k} \\ \frac{\partial f_b^1(b)}{\partial y^1} & \cdots\cdots & \frac{\partial f_b^{2k}(b)}{\partial y^1} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_b^1(b)}{\partial y^k} & \cdots\cdots & \frac{\partial f_b^{2k}(b)}{\partial y^k} \end{vmatrix} \tag{5}$$

is non-zero. In the neighbourhood of the point (a, b) of the manifold P^{2k} we may

take for coordinates the numbers $x^1, \cdots, x^k, y^1, \cdots, y^k$, and the map σ assumes the coordinate form

$$u^{*n} = \frac{f_b^n(y) - f_a^n(x)}{f_b^{2k+1}(y) - f_a^{2k+1}(x)}, \quad n = 1, \cdots, 2k. \tag{6}$$

With these coordinates the functional determinant of the map σ at the point (a, b) evidently coincides to within sign with the determinant (5). Thus we have shown that the regular map $\pi_e f$ is typical at each pair of self-intersections if and only if the map σ is proper at the point e^*.

We now choose the ray e^* in such a way that the vector e is not parallel to any vector tangent to the manifold $f(M^k)$ and that the map σ is proper at the point $e^* \in S^{2k}$. In view of Theorems 1 and 4, the set of rays satisfying these conditions is everywhere dense in the manifold S^{2k}. We suppose that there exist three distinct points a, b, c of the manifold M^k satisfying the condition $\pi_e f(a) = \pi_e f(b) = \pi_e f(c)$. We introduce local coordinates $z^1, \cdots, z^k$ in the neighbourhood of the point c in the manifold M^k, and let

$$u^n = f_c^n(z^1, \cdots, z^k) = f_c^n(z), \quad n = 1, \cdots, 2k+1 \tag{7}$$

be the coordinate form of the map f in the neighbourhood of the point c, analogous to the expressions (3) and (4). If, now, x, y, z are three points of the manifold M^k, close to the points a, b, c respectively and such that $f(x), f(y), f(z)$ lie on one straight line, then we have

$$\frac{f_a^n(x) - f_c^n(z)}{f_a^{2k+1}(x) - f_c^{2k+1}(z)} = \frac{f_b^n(y) - f_c^n(z)}{f_b^{2k+1}(y) - f_c^{2k+1}(z)}, \quad n = 1, \cdots, 2k. \tag{8}$$

Here we have $2k$ equations. We will regard these equations as defining implicit functions $x^1, \cdots, x^k, y^1, \cdots, y^k$ of the independent variables $z^1, \cdots, z^k$. For initial values we take $x = a$, $y = b$ as solutions of $z = c$. With these initial values of the functions and the independent variables the functional determinant of the system (8) is non-zero since the determinant (5) is non-zero. Thus the system (8) satisfies the conditions of the Implicit Function theorem. From this it follows that the collection of point-triples x, y, z close to the triple a, b, c and satisfying the condition that the points $f(x), f(y), f(z)$ lie in a straight line forms a k-dimensional manifold. Hence it readily follows from Theorem 1 that in arbitrary proximity to the point e^* of the manifold S^{2k} there may be found a point e_0^* satisfying the conditions of proposition (B).

Typical critical points of a real-valued function on a manifold.

C) Let f be a smooth map of class $m \geq 2$ of the manifold M^k into the one-dimensional Euclidean space E^1 or, equivalently, into a straight line. By choosing a coordinate system on the line E^1, we may write the map f in the form $y^1 = f^1(x)$, $x \in M^k$, where f^1 is a real-valued function of class m defined on M^k. In the neighbourhood of some point $a \in M^k$ we introduce local coordinates $x^1, \cdots, x^k$ with origin at a and let

$$y^1 = f^1(x^1, \cdots, x^k)$$

be the expression for the map f in these coordinates. The point a is called a *critical point* of the function f^1 and the number $f^1(a)$ is called the *critical value* of the function f^1 at a if all first order derivatives of f^1 vanish at a, or, in other words, if a is a singular point of the map f (see §1, D). Expanding the functions f^1 in a Taylor series at the critical point a we obtain

$$f^1(x) = f^1(a) + \sum_{i,j} a_{ij}x^i x^j + \cdots. \tag{9}$$

If the determinant $|a_{ij}| \neq 0$, the critical (singular) point a is called *non-degenerate*. It may be seen by direct calculation that, at a critical point a of the function f, the elements of the matrix $\|a_{ij}\|$ transform, under arbitrary changes of coordinate system, as coefficients of a quadratic form. From this, indeed, it follows that the non-degeneracy of singular points is an invariant property, that is, it does not depend on the choice of coordinate system.

D) Let h be a smooth map of class $m \geq 2$ of the manifold M^k into the Euclidean vector space C^{q+1}. Let u be a non-zero vector in C^{q+1} and u^{**} the one-dimensional linear subspace containing the vector u. We will denote the orthogonal projection of the space C^{q+1} on to the line u^{**} by π_u. The set N^q of all pairs of the form (x, u^*), where $x \in M^k$ and u^* is a ray orthogonal to the manifold $h(M^k)$ at the point $h(x)$, may, in a natural way, be given the structure of a smooth q-dimensional manifold of class $(m-1)$. With each point $(x, u^*) \in N^q$ we associate the point $\nu(x, u^*) = u^* \in S^q$ (see §1, H). The map ν is a smooth map of class $(m-1)$ of the manifold N^q into S^q. It turns out that the point $a \in M^k$ is a singular point of the map $\pi_u h$ of the manifold M^k on to the line u^{**} if and only if the ray u^* is orthogonal to the manifold $h(M^k)$ at the point $h(a)$. Further, if the ray u^* is orthogonal to the manifold $h(M^k)$ at the point $h(a)$, then the singular point a of the map $\pi_u h$ is non-degenerate if and only if the point (a, u^*) is a proper point of the map ν.

We prove proposition (D). The scalar product of the vectors $u, v \in C^{q+1}$ will as usual be denoted by (u, v). Let $u \in C^{q+1}$ with $(u, u) = 1$. The real function $(u, h(x))$ of the variable $x \in M^k$, defined on M^k, corresponds to the map $\pi_u h$ of the manifold M^k on to the axis u^{**}. In local coordinates $x^1, \cdots, x^k$, defined near the point a, we have

$$\frac{\partial}{\partial x^i}(u, h(a)) = (u, \frac{\partial h(a)}{\partial x^i}), \qquad i = 1, \cdots, k. \tag{10}$$

The vanishing of the left-hand side of all the relations (10) shows that a is singular point of the map $\pi_u h$, while the vanishing of the right-hand side shows that the vector u is orthogonal to the manifold $h(M^k)$ at the point $h(a)$. In this way it

is proved that a is a singular point of the map $\pi_u h$ if and only if the ray u^* is orthogonal to $h(M^k)$ at the point $h(a)$.

To establish the criterion of non-degeneracy of the singular point a of the map $\pi_{u_0} h$ we choose an orthonormal basis $e_1, \cdots, e_{q+1}$ for the vector space C^{q+1} such that the vectors $e_1, \cdots, e_k$ are tangent to the manifold $h(M^k)$ at the point $h(a)$, and the vector e_{q+1} coincides with u_0. In the corresponding coordinates $y^1, \cdots, e^{q+1}$ of the space C^{q+1} the map h assumes near the point a the form

$$y^j = h^j(x) = h^j(x^1, \cdots, x^k), \qquad j = 1, \cdots, q+1. \tag{11}$$

Since the vectors $e_1, \cdots, e_k$ are tangent to the manifold $h(M^k)$ at the point $h(a)$ it readily follows that

$$\left|\frac{\partial h^j(a)}{\partial x^i}\right| \neq 0, \qquad i, j = 1, \cdots, k.$$

It then follows that the relations

$$\xi^i = h^i(x^1, \cdots, x^k), \qquad i = 1, \cdots, k,$$

may be used to introduce new coordinates $\xi^1, \cdots, \xi^k$ of the point x in the neighbourhood of the point a in M^k. In these coordinates the map h takes the form:

$$h(x) = \sum_{i=1}^{k} \xi^i e_i + \sum_{j=1}^{q+1-k} \phi^{k+j}(x) \cdot e_{k+j}. \tag{12}$$

Since the vectors $e_1, \cdots, e_k$ are tangent to the manifold $h(M^k)$ at the point $h(a)$ it follows that

$$\frac{\partial \phi^{k+j}(a)}{\partial \xi^i} = 0, \qquad i = 1, \cdots, k; \ j = 1, \cdots, q+1-k. \tag{13}$$

Let (x, u^*) be a point of the manifold N^q close to the point $(a, u_0^*) = (a, e_{q+1}^*)$. On the ray u^* we choose a vector u satisfying the condition

$$(u, e_{q+1}) = 1.$$

We denote the remaining q components of the vector u in the basis $e_1, \cdots, e_{q+1}$ by $u^1, \cdots, u^q$: $u^i = (u, e_i)$, $i = 1, \cdots, q$. The condition that u be orthogonal to $h(M^k)$ at the point $h(x)$ may now be expressed in the form

$$0 = (u, \frac{\partial h(x)}{\partial \xi^i}) = u^i + \sum_{j=1}^{q-k} u^{k+j} \frac{\partial \phi^{k+j}(x)}{\partial \xi^i} + \frac{\partial \phi^{q+1}(x)}{\partial \xi^i}, \quad i = 1, \cdots, k. \tag{14}$$

This relation shows that, as coordinates of the element (x, u^*) of the manifold N^q, we may take the coordinates $\xi^1, \cdots, \xi^k$ of the point x and the components $u^{k+1}, \cdots, u^q$ of the vector u. As coordinates of the ray u^* in the manifold S^q we take the first q components of the vector u, calling these components $v^1, \cdots, v^q$ to avoid confusion with the coordinates $u^{k+1}, \cdots, u^q$ of the element (x, u^*) in the manifold N^q. Since $v^i = u^i$, $i = 1, \cdots, q$, the map ν of the manifold N^q into the manifold S^q assumes in the chosen coordinates the form (see (14))

$$v^i = -\sum_{j=1}^{q-k} u^{k+j} \frac{\partial \phi^{k+j}(x)}{\partial \xi^i} - \frac{\partial \phi^{q+1}(x)}{\partial \xi^i}, \quad i = 1, \cdots, k,$$

$$v^{k+j} = u^{k+j}, \quad j = 1, \cdots, q-k.$$

An immediate calculation shows (see (13)) that the Jacobian of the map ν at the point (a, e^*_{q+1}) is equal to $(-1)^k \left| \frac{\partial^2 \phi^{q+1}(a)}{\partial \xi^i \cdot \partial \xi^\alpha} \right|$, $i, \alpha = 1, \cdots, k$. Thus the point (a, u^*_0) is a proper point of the map ν if and only if the relation

$$\left| \frac{\partial^2 \phi^{q+1}(a)}{\partial \xi^i \partial \xi^\alpha} \right| \neq 0 \tag{15}$$

is satisfied. Since the function $\phi^{q+1}(x)$ corresponds to the map $\pi_{u0} h$ of the manifold M^k on to the axis u_0^{**}, the condition (15) coincides with the condition that a be a non-degenerate singular point of the map $\pi_{u0} h$. In this way the proof of assertion (D) is completed.

Theorem 5. *Let M^k be a smooth compact manifold of class $m \geq 3$ with boundary M^{k-1} consisting of two closed manifolds M_0^{k-1} and M_1^{k-1}, each of which may be empty; and let f^1 be a real function of class m defined on M^k. We assume that the function f^1 takes the same value c_i at all points of the manifold M_i^{k-1}, $i = 0, 1$, where $c_0 < c_1$, and that for all interior points x of M^k the inequality $c_0 < f^1(x) < c$, is satisfied. We assume further that the critical points of the function f^1 do not lie on the boundary M^{k-1}. It turns out that in any proximity of class m (see §2, F) to the function f^1 there lies a function g^1, coinciding with f^1 in some neighbourhood of the boundary, such that all critical points of the function g^1 are non-degenerate and that the critical values at distince critical points are themselves distinct.*

Proof. With the function f^1 we associate the map f of the manifold M^k into the one-dimensional Euclidean vector space A^1. Let e be a regular homeomorphism of class m of M^k into the Euclidean vector space B^q (see Theorem 2). We denote by C^{q+1} the direct sum of the vector spaces A^1 and B^q and we will consider the spaces A^1 and B^q as orthogonal subspaces of the space C^{q+1}. We denote by h the direct sum of the maps f and e (see §2, E). The map h is a regular homeomorphism of class m of the manifold M^k into the Euclidean vector space C^{q+1} such that the orthogonal projection π of the map h on to the straight line A^1 coincides with $f: f = \pi h$. We show first that in arbitrary proximity to the line A^1 there exists a line, orthogonal projection on to which generates a function all of whose critical values are non-degenerate. The function g^1 described in the statement of the theorem will be obtained by a further modification.

Let N^q be the manifold of all normal elements (x, u^*) of the manifold $h(M^k)$, defined in proposition (D), and let ν be the map of the manifold N^q into the mani-

fold S^q, also defined in (D). We show that if $u^* \in S^q$ is a proper point of the map ν, then all singular points of the map $\pi_u h$ are non-degenerate. In fact, if a is a singular point of the map $\pi_u h$, the ray u^* is orthogonal to $h(M^k)$ at the point $h(a)$ and so $(a, u^*) \in N^q$. Since the map ν is proper at the point (a, u^*) of the manifold N^q the singular point a is non-degenerate (see (D)). Let ϵ be a given positive number and let u be a unit vector of the space C^{q+1} such that the function $h^1 = (u, h(x))$ lies in ϵ-proximity of class m to the function f^1 and such that $u^* \in S^q$ is a proper point of the map ν, so that all critical points of the function h^1 are non-degenerate. In view of Theorem 4 such a vector u exists.

Let δ be a positive number so small that for $f^1(x) < c_0 + 3\delta$ and also for $f^1(x) > c_1 - 3\delta$ the point x is not a critical point of the function f^1. The existence of such a number δ follows from the hypothesis of the theorem that on the boundary M^{k-1}, and consequently in some neighbourhood of it, there are no critical points of the function f^1. Further let $\chi(t)$ be a real function of class m of the real variable t, taking the value zero of $t \leq c_0 + \delta$ or $t \geq c_1 - \delta$ and the value one if $c_1 - 2\delta \geq t \geq c_0 + 2\delta$. We put

$$h^2(x) = f^1(x) + \chi(f^1(x))\,(h^1(x) - f^1(x)). \tag{16}$$

It is easy to see that if the positive number ϵ, on which the construction of $h^1(x)$ depends, is chosen sufficiently small then all the critical points of the function $h^2(x)$, defined by the relation (16), coincide with critical points of the function $h^1(x)$ and so are non-degenerate. Since the function $\chi(t)$ vanishes for $t \leq c_0 + \delta$ and $t \geq c_1 - \delta$, it follows that the functions $h^2(x)$ and $f^1(x)$ coincide in some neighbourhood of the boundary M^{k-1}.

We suppose now that at two distinct critical points a and b of the function $h^2(x)$ we have $h^2(a) = h^2(b)$. Let Q be a neighbourhood of a containing no critical point of $h^2(x)$ distinct from a. As the neighbourhood Q we may take a region which, in some local coordinates, has the form of a cube with centre at a. Let Q', Q'' be cubes, centre a, similar to the cube Q and having edge-lengths respectively a half and a quarter that of Q. It is easy to define on Q a smooth function $\kappa(x)$, vanishing on $Q - Q'$ and taking the value one on Q'' (compare the proof of Theorem 2). We may extend the function $\kappa(x)$ to the whole manifold M^k by giving it the value zero outside the region Q. We put

$$h^3(x) = h^2(x) + \alpha\kappa(x).$$

It is easy to see that for sufficiently small $\alpha \neq 0$ all critical points of the function $h^3(x)$ are non-degenerate; moreover the critical values $h^3(a)$ and $h^3(b)$ are distinct. By iterating this process a finite number of times we arrive at the required function $g^1(x)$. Thus Theorem 5 is proved.

Typical irregularities for maps of a manifold M^k into the vector space E^{2k-1}.

$$h(x) = \sum_{i=1}^{k} \xi^i e_i + \sum_{j=1}^{k} \phi^j(x) \cdot e_{k+j}, \tag{27}$$

where the functions $\phi^j(x)$ satisfy the conditions:

$$\frac{\partial \phi^j(x)}{\partial \xi^i} = 0, \qquad i, j = 1, \cdots, k. \tag{28}$$

Let (x, u^*) be an element of the manifold L^{2k-1} close to the element (a, u_0^*). The vector u is tangent to $h(M^k)$ at the point $h(x)$ and so may be expressed in the form

$$u = \sum_{i=1}^{k} u^i \cdot \frac{\partial h(x)}{\partial \xi^i} = \sum_{i=1}^{k} u^i e_i + \sum_{i,j=1}^{k} u^i \frac{\partial \phi^j(x)}{\partial \xi^i} e_{j+k}. \tag{29}$$

On the ray u^* we choose the vector u so that $u^1 = 1$; then the expression (29) takes the form

$$u = e_1 + \sum_{i=2}^{k} u^i e_i + \sum_{j=1}^{k} \frac{\partial \phi^j(x)}{\partial \xi^1} e_{k+j} + \sum_{j=1}^{k} \sum_{i=2}^{k} u^i \frac{\partial \phi^j(x)}{\partial \xi^i} e_{j+k}. \tag{30}$$

As coordinates of the elements (x, u^*) in the manifold L^{2k-1} we may take the numbers $u^2, \cdots, u^k, \xi^1, \cdots, \xi^k$. Since the first component of the vector u in the space C^{2k} is equal to one, the remaining components $v^2, \cdots, v^{2k}$ of the vector u may be taken as coordinates of the ray u^* in the manifold S^{2k-1}. In the chosen coordinates the map τ takes, in view of (30), the form

$$v^i = u^i, \quad i = 2, \cdots, k; \quad v^{k+j} = \frac{\partial \phi^j(x)}{\partial x^1} + \sum_{i=2}^{k} u^i \cdot \frac{\partial \phi^j(x)}{\partial x^i}, \quad j = 1, \cdots, k. \tag{31}$$

A direct calculation shows that the Jacobian of the map τ at the point (a, u_0^*) is equal to

$$\left|\frac{\partial^2 \phi^j(a)}{\partial \xi^1 \partial \xi^i}\right|, \qquad i, j = 1, \cdots, k. \tag{32}$$

We now consider the map $\pi_{u_0} h$. We will regard it as mapping M^k into the vector space A^{2k-1} with basis $e_2, \cdots, e_{2k}$ by means of projection along the line e_1^{**}. We then have (see (27))

$$f(x) = \pi_{u_0} h(x) = \sum_{i=2}^{k} \xi^i e_i + \sum_{\alpha=1}^{k} \phi^\alpha(x) e_{k+\alpha}. \tag{33}$$

Thus we obtain

$$\frac{\partial^2 f(a)}{\partial \xi^1 \partial \xi^i} = \sum_{\alpha=1}^{k} \frac{\partial^2 \phi^\alpha(a)}{\partial \xi^1 \partial \xi^i} \cdot e_{k+\alpha}, \qquad i = 1, \cdots, k,$$

$$\frac{\partial f(a)}{\partial \xi^j} = e_j, \qquad j = 2, \cdots, k.$$

It follows that, in the given case, the system of vectors (19) is linearly independent if and only if the Jacobian (32) is non-zero.

Thus, proposition (F) is proved.

Theorem 6. *Let f be a smooth map of class $m \geq 3$ of the compact manifold M^k of dimension k into the vector space A^{2k-1} of dimension $(2k-1)$. Then in arbitrary proximity of class m to the map f there exists a map g all of whose singular points are non-degenerate and do not lie on the boundary M^{k-1} of the manifold M^k.*

Proof. We will regard A^{2k-1} as a subspace of a vector space C^{2k} of dimension $2k$. Let B^1 be a one-dimensional subspace of C^{2k} not lying in A^{2k-1}. We denote by π the projection of the space C^{2k} on to A^{2k-1} in the direction B^1. Let a positive number ϵ be given and let h be a regular map of the manifold M^k into C^{2k} such that the map πh is in ϵ-proximity to f (see Theorem 3). Let L^{2k-1} be the manifold of line elements on the manifold $h(M^k)$ (see (F)); let L^{2k-2} be the submanifold of L^{2k-1} consisting of all elements of the form (x, u^*), where $x \in M^{k-1}$, and, finally, let τ be the map of the manifold L^{2k-1} into the sphere S^{2k-1} defined in (F). On the basis of proposition (F) it may readily be verified that if $u^* \in S^{2k-1}$ is not a singular point of the map τ and does not belong to the set $\tau(L^{2k-2})$, then all singular points of the map $\pi_u h$ are non-degenerate and do not belong to the boundary of the manifold M^k. In view of Theorems 4 and 1 these exists a vector u such that u^* satisfies the conditions stated above and the map $\pi_u h$ is in ϵ-proximity to the map πh. Thus there exists a map $g = \pi_u h$ in 2ϵ-proximity to the map f and satisfying the requirements of the theorem.

Thus, Theorem 6 is proved.

Canonical forms for typical critical points and typical non-regular points.

In propositions (C) and (E) certain singular points of maps of the manifold M^k into the vector spaces A^1 and A^{2k-1} respectively were designated as non-degenerate. In Theorems 5 and 6 it was shown that all degenerate singular points of the maps considered are unstable – being removable by small movements of the maps. It was not shown, however, that the singular points described as non-degenerate are stable – that they are preserved under small movements. The proof of this fact does not present difficulties but it will not be given here. Also the structure of the maps in the neighbourhood of a non-degenerate singular point will not be analysed. To do so in complete detail would not be a simple matter, and here I give only results without proofs.

G) Let a be a non-degenerate critical point of the real-valued function $f^1(x)$ defined on the manifold M^k. In proposition (A) it was remarked that the expansion of the function $f^1(x)$ in a Taylor series in the neighbourhood of the point a has the form (9). It turns out (see [7]) that by a change of coordinates in the neighbourhood of the point a this expansion can be put in the form

$$f^1(x) = f^1(a) + (x^1)^2 + \cdots + (x^s)^2 - (x^{s+1})^2 - \cdots - (x^k)^2, \tag{34}$$

where the number s of positive squares is an invariant of the point a, that is, it does not depend on the choice of coordinates in the neighbourhood of this point and is not altered by small changes of the map. Thus a function defined on k-dimensional manifolds can have $(k+1)$ distinct types of non-degenerate critical points

$(s = 0, \cdots, k)$. Since a map f of the manifold M^k into a straight line does not uniquely determine the function $f^1(x)$, points of distinct types for the function may prove to be of the same type for the map. In fact change of sign of the function $f^1(x)$ interchanges the roles of the numbers s and $k - s$, and so the corresponding critical points belong to the same type of singular point for maps.

One next remarks that the transition from the expression (9) to the expression (34) cannot be achieved, as may be shown, by a linear transformation of coordinates. Obviously a linear transformation provides only the first step in the transition from the expression (9) to the expression (34). Under a linear transformation terms of the third and higher orders are preserved, whereas in the expression (34) they are absent.

H) Let a be a non-degenerate singular point of the map f of the manifold M^k into the vector space A^{2k-1} (see (E)). It turns out (see [8]) that in the neighbourhood of the points a and $f(a)$ it is possible to transform (in general, non-linearly) the coordinate systems so that the map f appears near the point a in the coordinate form

$$y^1 = (x^1)^2,\ y^2 = x^1x^2,\ \cdots,\ y^k = x^1x^k,\ y^{k+1} = x^2,\ y^{k+2} = x^3,\ \cdots,\ y^{2k-1} = x^k. \quad (35)$$

Here the points a and $f(a)$ are the origins of coordinates.

Proposition (H) is a theorem requiring a somewhat involved proof.

The expressions (35) may be used to represent the geometrical character of the map f near the point a, particularly if $k = 2$.

CHAPTER II

Normally-framed manifolds

§5. Smooth approximations to continuous maps and deformations

In this section it is shown that, for the homotopy classification of maps of one smooth manifold into another, it is sufficient to consider only smooth maps and smooth homotopies of them. This follows from the following facts. Let M^k and N^l be two smooth closed manifolds of class m. It turns out that in each homotopy class of maps of N^l into M^k there exists a smooth map of class $(m - 1)$ and if two smooth maps of class $(m - 1)$ are homotopic then there exists a smooth homotopy of class $(m - 3)$ of one map into the other. Thus, for the study of maps of manifolds whose smoothness class is equal to m, one is involved in the consideration of homotopies of smoothness class $(m - 3)$. It would be possible, by using a certain amount of ingenuity, to avoid this reduction of the smoothness class, but since the results of this section will be applied only to the study of maps of spheres into spheres, and spheres are analytic manifolds, the decrease of three units in the smoothness is of no significance, and there would be no point in complicating the proofs in order to keep the smoothness class fixed.

The structure of the neighbourhood of a smooth submanifold.

The proposition below will only be used in the present section in applications to closed manifolds; in this case the proof may be considerably simplified as may readily be seen by reference to the actual proof. In the next section it is used in the general case.

A) Let E^{n+k} be a Euclidean space, in which a Cartesian coordinate system $y^1, \cdots, y^{n+k}$ is chosen; let E_0, E_1 be the two hyperplanes of the space E^{n+k} defined by the equations $y^{n+k} = c_0$ and $y^{n+k} = c_1$, where $c_0 < c_1$, and let E_*^{n+k} be the strip of E^{n+k} bounded by the hyperplanes, that is, the set of points satisfying the conditions $c_0 \le y^{n+k} \le c_1$. Further let M^k be a smooth compact submanifold (see §1, F) of E_*^{n+k} of class $m \ge 4$ (if M^k is closed it is sufficient to take $m \ge 2$) with boundary M^{k-1}. We denote by N_z the normal to M^k at the point z. It has the structure of an n-dimensional linear subspace of the Euclidean space E^{n+k}. With respect to the manifold M^k we will further assume that at each of its boundary points it is orthogonal to the boundary of the strip E_*^{n+k}; that is, that for $x \in M^{k-1}$ we have:

$$N_x \subset E_0 \cup E_1. \quad (1)$$

We designate by $H(z) = H_\delta(z)$ the open ball in the Euclidean space N_z with centre z and radius $\delta > 0$, and by $H_\delta(P)$, where $P \subset M^k$, the union of all balls $H_\delta(z)$ for $z \in P$. It turns out that, for δ sufficiently small, the balls $H_\delta(z)$ and $H_\delta(z')$ do not intersect if $z \neq z'$ and the set $W_\delta = H_\delta(M^k)$ constitutes a neighbourhood of the manifold M^k in E_*^{n+k}. By associating with each point $y \in W_\delta$ the unique point $z \in M^k$ such that $y \in H_\delta(z)$ we obtain a smooth map $y \to z = \pi(y)$ of the manifold W_δ into the manifold M^k; in the case of a closed manifold M^k this map has smoothness class $(m - 1)$.

We prove proposition (A). Let $a \in M^k$ and let $x^1, \cdots, x^k$ be a local coordinare system defined in the neighbourhood of a, which is their origin, and let

$$y^j = f^j(x) = f^j(x^1, \cdots, x^k), \qquad j = 1, \cdots, n + k \quad (2)$$

be the parametric equations defining the manifold M^k in the neighbourhood of the point a. The functions f^j are defined for values of the variables $x^1, \cdots, x^k$ satisfying the conditions

$$|x^i| < \epsilon, \qquad i = 1, \cdots, k$$

if a is an interior point of M^k, and the conditions (3) together with the conditions

$$x^1 \le 0$$

if a is a boundary point of M^k. Thus the functions f^j determine a map f_a of the open cube K_ϵ, given by the inequalities (3), or of the half cube K'_ϵ given by the inequalitites (3) and (4). In the case of a boundary point a, we extend the functions f^j to positive values of the variable x^1 by putting

$$f^j(x) = f^j(x^1, \cdots, x^k) = f^j(0, x^2, \cdots, x^k) + \frac{\partial}{\partial x^1} f^j(0, x^2, \cdots, x^k)\cdot x^1 +$$

$$+ \frac{\partial^2}{(\partial x^1)^2} f^j(0, x^2, \cdots, x^k)\cdot (x^1)^2, \ x^1 \geq 0.$$

The extended functions f^j determine a smooth regular homeomorphism f_a of the open cube K_ϵ (where ϵ is a sufficiently small positive number), for an arbitrary point $a \in M^{k-1}$.

The equation of the normal $N_{f_a(x)} = N_x$ to the manifold $f_a(K_\epsilon)$ at the point $f_a(x)$ may be written in vector form as

$$\left(\frac{\partial f_a(x)}{\partial x^i}, \ y - f_a(x)\right) = 0, \qquad i = 1, \cdots, k. \tag{5}$$

Here y is a vector ranging over the normal N_x. We will regard the system of relations (5) as a system of equations for the unknown functions $x^1, \cdots, x^k$ of the independent variables $y^1, \cdots, y^{n+k}$, the components of the vector y. For the initial values $y = f_a(0) = a$ the system (5) has the evident solution $x^i = 0, \ i = 1, \cdots, k$. The functional determinant of the system (5) for these values is equal to $(-1)^k$ $\left|\frac{\partial f_a(0)}{\partial x^i}, \frac{\partial f_a(0)}{\partial x^j}\right|, \ i, j = 1, \cdots, k$. This determinant is non-zero since the map f_a is regular at the point 0. Thus the system (5) is solvable. Let $x = \sigma(y)$ or, in coordinate form,

$$x^i = \sigma^i(y^1, \cdots, y^{n+k}), \qquad i = 1, \cdots, k \tag{6}$$

be its solution, defined for all points y belonging to some neighbourhood V_a of the point a in the space E^{n+k}. For $y \in V_a$ there exists then one and only one point $x \in K_\epsilon$ satisfying the condition $y \in N_x$; the point x is determined by the relation $x = \sigma(y)$. In other words, through each point $y \in V_a$ there passes a unique normal N_x, where $x \in K_\epsilon$. From the continuity of the function $\sigma(y)$ it follows easily that there exist sufficiently small positive numbers δ_a and ϵ_a such that for $\delta \leq \delta_a$, $\epsilon \leq \epsilon_a$ the set $H_\delta(f_a(K_\epsilon))$ is entirely contained in V_a and constitutes a neighbourhood of the point a in the space E^{n+k}.

We now show that for boundary points a there exist positive numbers δ_a' and ϵ_a' so small that for $\delta \leq \delta_a'$ and $\epsilon \leq \epsilon_a'$ the set $H_\delta(f_a(K_\epsilon'))$ is a neighbourhood of the point a in the strip E_*^{n+k}. For definiteness we will assume that $a \in E_1$; then we have

$$\sigma^1(y^1, \cdots, y^{n+k-1}, c_1) = 0. \tag{7}$$

Further it is immediately clear that

$$\frac{\partial \sigma^1(y^1, \cdots, y^{n+k-1}, c_1)}{\partial y^{n+k}} > 0. \tag{8}$$

From what has been said it follows that for a point y sufficiently near to a the sign of $\sigma^1(y^1, \cdots, y^{n+k})$ coincides with the sign of $y^{n+k} - c_1$, and this shows that for sufficiently small numbers δ and ϵ we have

$$H_\delta(f_a(K_\epsilon')) = H_\delta(f_a(K_\epsilon)) \cap E_*^{n+k}. \tag{9}$$

Since $H_\delta(f_a(K_\epsilon))$ is a neighbourhood of the point a in the space E^{n+k}, then, in view of (9), the set $H_\delta(f_a(K_\epsilon'))$ is a neighbourhood of the point a in the strip E_*^{n+k}.

For an interior point $a \in M^k$ we set $\delta_a' = \delta_a$, $\epsilon_a' = \epsilon_a$. The collection of all regions $U_a = f_a(K_{\epsilon_a'}) \cap M^k$, $a \in M^k$ covers the manifold M^k. Let $U_{a_1}, \cdots, U_{a_p}$ be a finite covering of M^k. There exists a number $\eta > 0$ so small that any two points of M^k whose distance apart does not exceed η belong, together, to one of the regions of the given finite covering. Now let δ be the smallest of the numbers δ_{a_n}', $n = 1, \cdots, p$ and $\eta/2$. Since $H_\delta(M^k) = H_\delta(U_{a_1}) \cup \cdots \cup H_\delta(U_{a_p})$, $H_\delta(M^k)$ is a neighbourhood of the manifold M^k in the strip E_*^{n+k}. Further, for two distinct points $z, z' \in M^k$ the balls $H_\delta(z)$ and $H_\delta(z')$ do not intersect. In fact, if $\rho(z, z') \leq \delta$ the points z and z' belong to the same region U_{a_n} and so, in view of what was proved earlier, the balls $H_\delta(z)$ and $H_\delta(z')$ cannot intersect. If, on the other hand, $\rho(z, z') > \delta$, then the balls cannot intersect since the distance between their centres is greater than the sum of their radii.

Thus, proposition (A) is proved.

Smooth approximations.

B) Let $f^1(x)$ be a *continuous* real-valued function defined on the smooth compact manifold M^k of class $m \geq 2$ and let ϵ be a positive number. There exists then a *smooth* real-valued function of class m, $g^1(x)$, defined on M^k and satisfying the condition $|g^1(x) - f^1(x)| < \epsilon$. In other words, it is possible to approximate to a function continuous on M^k arbitrarily closely by a smooth function.

We prove proposition (B). In view of Theorem 2 we may suppose the manifold M^k embedded in Euclidean space E^l of sufficiently large dimension. Let Q be a closed cube containing M^k. In view of the well-known theorem of Uryson (see [9]) it is possible to extend the function $f^1(x)$, defined on M^k, to a continuous function on Q. It is then possible to approximate to this function to within ϵ by a polynomial $g^1(x)$ in the Cartesian coordinates of the point $x \in Q$. Regarded as a function on M^k, $g^1(x)$ has the desired properties.

Theorem 7. *Let M^k be a smooth closed manifold of class $m \geq 2$, let N^l be a smooth compact manifold of class m, and let f be a map of N^l into M^k. We will regard M^k as a metric space. It turns out that for any positive ϵ there exists a smooth map h of class $(m-1)$ of N^l into M^k, satisfying the condition $\rho(f(x), h(x)) < \epsilon$, $x \in N^l$. In other words, a continuous map of N^l into M^k may be approximated arbitrarily closely by a smooth map.*

Proof. In view of Theorem 2, we may suppose that the manifold M^k is a submanifold of some Euclidean space E^{n+k}. Let δ be the number defined for this submanifold in proposition (A) and let $\epsilon' < \delta/\sqrt{n+k}$. We denote the components of the vector $f(x)$, $x \in N^l$, by $f^1(x), \cdots, f^{n+k}(x)$. In view of proposition (B) there exist real-valued functions $g^i(x)$, $i = 1, \cdots, n+k$ of smoothness class m, defined on N^l and satisfying the inequality $|f^i(x) - g^i(x)| < \epsilon$, $i = 1, \cdots, n+k$. We denote by $g(x)$ the vector with components $g^1(x), \cdots, g^{n+k}(x)$. The map g of the manifold N^l into E^{n+k} has smoothness class m and $g(N^l) \subset W_\delta$ (see (A)). For sufficiently small ϵ' the map $h = \pi g$ (see (A)) satisfies the requirements of the theorem.

C) A family of continuous map f_t, $0 \le t \le 1$ of the closed manifold N^l into the manifold M^k is called a *continuous family* or *deformation* of the map f_0 into the map f_1 if $f_t(x)$ is a continuous function of the pair of variables x, t. Let $N^l \times I$ be the direct product of the manifold N^l and the interval $I = [0, 1]$ (see §1, K). We put $f_*(x, t) = f_t(x)$. It is plain that the family f_t is continuous if and only if the map f_* is continuous. We will say that the family f_t is *smooth* of class m or that the deformation f_t is *smooth* of class m if the map f_* is smooth of class m. If the maps f_0 and f_1 are connected by a smooth deformation they may be described as *smoothly homotopic*. It is quite obvious that the relation of smooth homotopy is reflexive and symmetric. On the other hand the transitivity of the relation is not obvious and requires proof. This we provide.

Let f_{-1}, f_0, f_1 be three smooth maps of class m of the manifold N^l into the manifold M^k; let f_t, $-1 \le t \le 0$, be a smooth deformation of class m of f_{-1} into f_0 and let f_t, $0 \le t \le 1$, be a smooth deformation of class m of f_0 into f_1. The deformation f_t, $-1 \le t \le 1$, is evidently continuous, but for $t = 0$ it may fail to be smooth so it must be modified to achieve a smooth deformation of class m. Let n be an odd integer such that $n \ge m$. We put $g_t(x) = f_{t^n}(x)$. It is easy to see that g_t, $-1 \le t \le 1$, is a smooth deformation of class m of the map $g_{-1} = f_{-1}$. In this way, the transitivity of the relation of smooth homotopy is proved.

D) Let M^k and N^l be two smooth closed manifolds of class m, of which the manifold M^k is a metric space. Then there exists a sufficiently small positive ϵ such that if f_0 and f_1 are two smooth maps of class m of the manifold N^l into the manifold M^k whcih are distant apart less than ϵ, that is, such that $\rho(f_0(x), f_1(x)) < \epsilon$, $x \in N^l$, then there exists a smooth deformation of class $(m-1)$ of f_0 into f_1.

We prove proposition (D). In view of Theorem 2, we may assume that M^k is a submanifold of a Euclidean space E^{n+k} of sufficiently high dimension. Let δ be the number defined for $M^k \subset E^{n+k}$ in proposition (A). We will assume that the metric in the manifold M^k is given by the inclusion $M^k \subset E^{n+k}$, and we choose ϵ so small that for $\rho(x, x') < \epsilon$ the segment joining the points x and x' lies in W_δ. We put

$$f_t(x) = \pi(f_0(x)(1-t) + f_1(x)t).$$

It is clear that f_t, $0 \le t \le 1$, is a smooth deformation of class $(m-1)$ of the map f_0 into the map f_1 (see (A)).

Theorem 8. *Let f_t, $0 \le t \le 1$, be a continuous deformation of maps of the closed manifold N^l into the closed manifold M^k such that the maps f_0 and f_1 are smooth of class m. Then there exists a smooth deformation of class $(m-2)$ of the map f_0 into the map f_1. In other words, if two smooth maps can be connected by a continuous deformation then they can be connected by a smooth deformation.*

Proof. To the deformation f_t there corresponds a map f_* of the manifold $N^l \times I$ into M^k. In view of Theorem 7 the continuous map f_* can be approximated to within ϵ by a smooth map g_*, of class $(m-1)$ of $N^l \times I$ into M^k. To the smooth map g_* there corresponds a smooth deformation g_t, $0 \le t \le 1$, of maps of N^l into M^k. For sufficiently small ϵ the closeness of the maps f_i and g_i, $i = 0, 1$, ensures that they are connected by a smooth homotopy of class $(m-2)$ (see (D)). From all this, in view of the transitivity of the property of smooth homotopy, it follows that there exists a smooth homotopy of class $(m-2)$ between the maps f_0 and f_1.

Thus, Theorem 8 is proved.

§6. The basic method

In this section we associate, with each smooth map of an $(n+k)$-dimensional sphere Σ^{n+k} into an n-dimensional sphere S^n, a *smoothly-framed* submanifold M^k of Euclidean space E^{n+k}. The manifold M^k is framed in that at each of its points x there is given a system $U(x) = \{u_1(x), \cdots, u_n(x)\}$ of linearly independent vectors orthogonal to M^k such that the vector $u_i(x)$ depends continuously on $x \in M^k$. The framing is said to be smooth if the vectors $u_i(x)$ depend smoothly on x. The manifold M^k together with its frame is called a *normally-framed manifold* and denoted by (M^k, U). It turns out that each smoothly-framed manifold (M^k, U) corresponds to some map of the sphere Σ^{n+k} into the sphere S^n and that maps to which correspond identical smoothly-framed manifolds are homotopic. On the other hand it is possible that there may correspond to two homotopic maps two normally-framed manifolds which are not only not identical but even not homotopic to each other. In this connection we introduce the notion of a *homology* between two *normally-framed* manifolds (M_0^k, U_0) and (M_1^k, U_1), situated in the same Euclidean space E^{n+k}. Two normally-framed manifolds (M_0^k, U_0) and (M_1^k, U_1) are said to be homologous if in the product $E^{n+k} \times I$ of the space E^{n+k} with the segment $I = [0, 1]$ there exists a compact normally-framed submanifold (M^{k+1}, U) whose boundary consists of the given manifolds $M_0^k \times 0$ and $M_1^k \times 1$ and whose frame U coincides on the boundary with the frames $U_0 \times 0$ and $U_1 \times 1$ given on $M_0^k \times 0$ and $M_1^k \times 1$. It turns out that two maps of Σ^{n+k} into S^n are homotopic if and only if the associated smoothly-framed manifolds are homologous (the frame realising the homology is not assumed to be smooth). In this way the problem of homotopy classification of maps of spheres into spheres is reduced to the problem of homology classification of smoothly-framed manifolds. On the other hand it must be admitted that the problem of homology classification of framed manifolds in not a simple one.

Normally-framed manifolds[1]. Definition 2. Let E^{n+k} be a Euclidean space, $y^1, \cdots, y^{n+k}$ Cartesian coordinates in it, let E_0 and E_1 be two hyperplanes of E^{n+k}, given by the equations $y^{n+k} = c_0$ and $y^{n+k} = c_1$, $c_0 < c_1$, and let E_*^{n+k} be the strip consisting of all points of the space E^{n+k} for which $c_0 \leq y^{n+k} \leq c_1$. Further let M^k be a smooth compact submanifold of class m of the strip E_*^{n+k} with boundary M^{k-1} (*see* §1, F). If the manifold M^k is closed the hyperplanes E_0 and E_1 play no role and we will suppose that $E_*^{n+k} = E^{n+k}$. We will regard the normal N_x at the point $x \in M^k$ as a vector space with its zero at x and we suppose that

$$N_x \subset E_0 \cup E_1, \quad x \in M^{k-1},$$

that is, that the manifold M^k is orthogonal at points of its boundary to the boundary of the strip E_*^{n+k} (compare §5, A). In this way we will regard the manifold M^k situated in the strip E_*^{n+k} as *normally-framed* if in each vector space N_x a basis

$$u_1(x), \cdots, u_n(x),$$

is given such that the vector $u_i(x)$, considered as a vector of the space E^{n+k}, is a continuous function of the point $x \in M^k$. We will call the system $U(x) = \{u_1(x), \cdots, u_n(x)\}$ a *frame*[1] for the manifold M^k, and the manifold M^k together with the frame $U(x)$ will be denoted by $(M^k, U(x))$ and called a *normally-framed manifold*. We will call the frame $U(x)$ *orthonormal* if at each point $x \in M^k$ the basis $U(x)$ is orthonormal. The frame $U(x)$ will be called *smooth* of class m if each vector $u_i(x)$ is a smooth function of $x \in M^k$ of class m.

It should be remarked that every framed manifold[1] is orientable and admits a *natural orientation* if the Euclidean space E^{n+k} containing it is oriented. Indeed, let $e_1, \cdots, e_k$ be a linearly independent system of vectors tangent to the manifold M^k at some point x. We will regard the system $e_1, \cdots e_k$ as bestowing a *natural orientation* on the manifold M^k if the system $e_1, \cdots, e_k, u_1(x), \cdots, u_n(x)$ corresponds to the positive orientation of the space E^{n+k}.

In the following definition the notion of homology between two k-dimensional framed submanifolds of the Euclidean space E^{n+k} is made precise.

Definition 3. Let (M_0^k, U_0) and (M_1^k, U_1) be two smooth framed submanifolds of the Euclidean space E^{n+k}. Let $E^{n+k+1} = E^{n+k} \times E^1$, where E^1 is the real line parametrized by t. We put $E_t = E^{n+k} \times t$, $t = 0, 1$ and we denote by E_*^{n+k} the strip in the space E^{n+k+1} bounded by the hyperplanes E_0 and E_1. We will regard the framed manifolds (M_0^k, U_0) and (M_1^k, U_1) as *homologous* if there exists a framed submanifold (M^{k+1}, U) of the strip E_*^{n+k+1} such that

$$M^{k+1} \cap E_0 = M_0^k \times 0, \quad M^{k+1} \cap E_1 = M_1^k \times 1,$$

and such that the frame U coincides on $M_t^k \times t$ with the frame $U_t \times t$, $t = 0, 1$. The

1) Translator's note: Since the notion of a normal frame occurs very frequently we shall generally omit the word 'normal'.

framed manifold (M_0^k, U_0) is said to be *nullhomologous* if it is homologous to the framed manifold (M_1^k, U_1), where M_1^k is empty. In this case the framed manifold (M^{k+1}, U) realising the homology has as its boundary the manifold M^k. It turns out that the homology relation is reflexive, symmetric and transitive so that the set of all k-dimensional framed submanifolds of the Euclidean space E^{n+k} is divided into homology classes.

It is clear that the homology relation for framed manifolds is reflexive and symmetric. We prove that it is transitive. Let (M_{-1}^k, U_{-1}), (M_0^k, U_0) and (M_1^k, U_1) be three framed manifolds in the Euclidean space E^{n+k} satisfying the relations

$$(M_{-1}^k, U_{-1}) \sim (M_0^k, U_0), \sim (M_0^k, U_0) \sim (M_1^k, U_1).$$

Further let $E^{n+k+1} = E^{n+k} \times E^1$ be the direct product of the Euclidean space E^{n+k} with the real line E^1 parametrized by t; we pick out in it the strips E_{*i}, $i - 1 \leq t \leq i$, $i = 0, 1$. We put $E_* = E_{*0} \cup E_{*1}$. We will suppose that the homology $(M_{i-1}^k, U_{i-1}) \sim (M_i^k, U_i)$ is realised in the strip E_{*i} by the manifold (M_i^{k+1}, U_{*i}), $i = 0, 1$. Now let m be a sufficiently large odd number. We define a map ψ of the strip E_* into itself by setting $\psi(x, t) = (x, \sqrt[m]{t})$, $x \in E^{n+k}$; ψ is evidently homeomorphic, and regular at all points (x, t) where $t \neq 0$. It is easy to verify that $M^{k+1} = \psi(M_0^{k+1} \cup M_1^{k+1})$ is a smooth submanifold of the strip E_*. We denote the system of vector $U_{*i}(x, t)$ by $U_*(x, t)$; this should lead to no misunderstanding. Let N'_{xt} be the normal to the manifold $M_0^{k+1} \cup M_1^{k+1}$ at the point (x, t), $-1 \leq t \leq 1$, and let N_{xt} be the normal to the manifold M^{k+1} at the point $\psi(x, t)$. It is easy to convince oneself that the space N'_{xt} may be projected orthogonally on to the space N_{xt} without degeneracies. Thus by taking for $U(x, t)$ the orthogonal projection of the system $U_*(x, t)$ on N_{xt}, we obtain a framed manifold (M^{k+1}, U), realising the homology $(M_{-1}^k, U_{-1}) \sim (M_1^k, U_1)$ in the strip E_*. Thus the transitivity of the homology relation is proved.

Transition from maps to framed manifolds.

A) Let E^{r+1} be a Euclidean vector space. The sphere S^r in E^{r+1} of dimension r and radius ½ is given by the equation

$$(x, x) = \frac{1}{4}.$$

Let p and q be two diametrically opposite points of S^r, the first of which we will call the *north pole* and the second the *south pole*. Further let T_p and T_q be the tangent spaces to the sphere S^r at the points p and q, and let $e_1, \cdots, e_r$ be an orthonormal basis in the space T_p, giving a positive orientation to S^r. We obtain a basis for the space T_q by translating the vectors $e_1, \cdots, e_r$ from p to q. To the chosen bases correspond certain coordinate systems in T_p and T_q. We now introduce coordinates in the regions $S^r - q$, $S^r - p$, determined by the system $(p; e_1, \cdots, e_r)$. To this end, we denote by $\phi(x)$ the image of the point $x \in S^r - q$

under central projection from the centre q on th the space T_p and we take the coordinates $x^1, \cdots, x^r$ of the point $\phi(x)$ in T_p as coordinates of x in $S^r - q$. In exactly the same way we define the coordinates $y^1, \cdots, y^r$ of the point $x \in S^r - p$ by means of central projections from p on to T_q. It is easy to see that for $x \in S^r - (p \cup q)$ we have

$$y^i = \frac{x^i}{(x^1)^2 + \cdots + (x^r)^2}, \tag{1}$$

$$x^i = \frac{y^i}{(y^1)^2 + \cdots + (y^r)^2}. \tag{2}$$

In this way, S^r has the structure of an analytic manifold.

We now associate with each smooth map of the $(n + k)$-dimensional oriented sphere Σ^{n+k} into the n-dimensional oriented sphere S^n a closed normally-framed manifold (M^k, U) of dimension k situated in Euclidean space E^{n+k} of dimension $(n + k)$.

Definition 4. Let f be a smooth map of the oriented sphere Σ^{n+k} in the oriented sphere S^n. The north and south poles of Σ^{n+k} we denote by p' and q', the tangent space to Σ^{n+k} at p' by E^{n+k}, and central projection of the region $\Sigma^{n+k} - q'$ on to E^{n+k} from q' by ϕ. As north pole of the sphere S^n we choose an arbitrary proper point p of the map f, distinct from $f(q')$ (see Theorem 4). Let $e_1, \cdots e_n$ be an orthonormal system of vectors, tangent to S^n at the point p, giving an orientation to the sphere S^n. We denote the tangent space to S^n at p by T_p. Since p is a proper point of the map f, the set $f^{-1}(p)$ is a smooth k-dimensional submanifold of the manifold Σ^{n+k} (see §1, F). Since, moreover, the set $f^{-1}(p)$ does not contain the point q', it follows that $M^k = \phi f^{-1}(p)$ is a smooth closed submanifold of the Euclidean space E^{n+k}. The map $f\phi^{-1}$ of E^{n+k} into S^n is proper at each point $x \in M^k$. We denote the vector space tangent at x to the manifold E^{n+k} by E_x^{n+k} (see §1, C). Since E^{n+k} is Euclidean space, the space E_x^{n+k} may be identified with the space E^{n+k} by taking the point x as origin. We denote the normal and tangent to M^k at x by N_x and T_x respectively. The linear map of the vector space E_x^{n+k} on to the vector space $T_{p'}$, corresponding to the map $f\phi^{-1}$, we denote by f_x (see §1, E). Since the map $f\phi^{-1}$ is proper at x, we have $f_x(E^{n+k}) = T_{p'}$, and since $f\phi^{-1}(M^k) = p$, $f_x(T_x) = p$. It thus follows that the map f_x of the vector space N_x into the vector space T_p is a non-degenerate map on to T_p. We designate by $u_i(x)$ the counterimage in N_x of the vector e_i, under f_x. The system $U(x) = \{u_1(x), \cdots, u_n(x)\}$, $x \in M^k$, constitutes a smooth frame for the manifold M^k. We associate the framed manifold (M^k, U) with the map f: $f \to (M^k, U)$. The correspondence $f \to (M^k, U)$ depends on the arbitrary choice of the system $p, e_1, \cdots e_n$, so that the correspondence $f \to (M^k, U)$ is more precisely written in the form

$$(f;\ p, e_1, \cdots, e_n) \to (M^k, U).$$

The pole p' of the sphere Σ^{n+k} is to be regarded as fixed, that is, it is invariable on the study of all maps of the sphere Σ^{n+k} into the sphere S^n. The pole p of S^n must be a proper point of the map f distinct from $f(q')$ and so cannot be fixed from the start.

The following theorem, demonstrating that homotopic maps correspond to homologous framed manifolds, proves, in fact, the inessential nature of the choice of the system $p, e_1, \cdots, e_n$.

Subsequently it will be proved that if two framed manifolds are homologous then the corresponding maps are homotopic (see Theorem 10).

Theorem 9. *Let f_0 and f_1 be two smooth maps of the oriented sphere Σ^{n+k} into the oriented sphere S^n $(n \geq 2,\ k \geq 0)$ and suppose*

$$(f_0;\ p_0, e_{10}, \cdots, e_{n0}) \to (M_0^k, U_0),$$

$$(f_1;\ p_1, e_{11}, \cdots, e_{n1}) \to (M_1^k, U_1)$$

(see definition 4). It turns out that if the maps f_0 and f_1 are homotopic then the framed manifolds (M_0^k, U_0) and (M_1^k, U_1) are homologous.

Proof. Since the orientations of the sphere S^n determined by the tangential systems $e_{10}, \cdots, e_{n0}$ and $e_{11}, \cdots, e_{n1}$ coincide, there exists an isometric map θ of S^n on to itself, achieved by means of a continuous rotation and so homotopic to the identity, which transforms the system $p_0, e_{10}, \cdots, e_{n0}$ into the system $p_1, e_{11}, \cdots, e_{n1}$. We put $g_0 = f_0$, $g_1 = \theta^{-1} f_1$. Since θ is homotopic to the identity the maps g_0 and g_1 are homotopic. Moreover it is easy to see that

$$(g_0;\ p, e_1, \cdots, e_n) \to (M_0^k, U_0),$$

$$(g_1;\ p, e_1, \cdots, e_n) \to (M_1^k, U_1),$$

where

$$(p, e_1, \cdots, e_n) = (p_0, e_{10}, \cdots, e_{n0}).$$

Since the smooth maps g_0 and g_1 are homotopic these exsits a smooth homotopy g_t connecting them (see Theorem 8); corresponding to this deformation there is a smooth map g_* of $\Sigma^{n+k} \times I$ into S^n (see §5, C). We define a map ϕ_* of the manifold $(\Sigma^{n+k} - q') \times I$ on to the direct product $E^{n+k} \times I$ by putting

$$\phi_*(x, t) = (\phi(x), t), \tag{3}$$

and we will consider the product $E^{n+k} \times I$ as a strip E_*^{n+k+1} in the space $E^{n+k+1} = E^{n+k} \times E^1$, where E^1 is the real line. With regard to the map g_* we make the following assumption.

(a) The point p is a proper point of the map g_* of the manifold $\Sigma^{n+k} \times I$ and does not belong to the set $g_*(q' \times I)$.

In view of hypothesis (a), the set $M^{k+1} = \phi_* g_*^{-1}(p)$ is a smooth compact submanifold of the strip E_*^{n+k+1}. We denote by N_x the normal in E^{n+k+1} to the manifold M^{k+1} at the point $x \in M^{k+1}$. Since the map $g_* \phi_*^{-1}$ is proper at the point x, it is regular on N_x at the point x and so the system of vectors $e_1, \cdots, e_n$ corresponds in N_x to the system of vectors $U(x) = \{u_1(x), \cdots, u_n(x)\}$ (compare definition 4). With respect to the map g_* we make one further hypothesis.

(b) The manifold M^{k+1} is orthogonal, at points of its boundary, to the boundary of the strip E_*^{n+k+1} (see definition 2).

From hypothesis (b) it follows that $U(x)$ is a frame for the manifold M^{k+1} and it is easy to see that the framed manifold (M^{k+1}, U) realises the homology between the framed manifolds (M_0^k, U_0) and (M_1^k, U_1) (see definition 3). Thus to prove the theorem it is sufficient to construct a smooth homotopy g_t, connecting the maps g_0 and g_1, for which hypotheses (a) and (b) are satisfied. We do this.

Let h_t be an arbitrary smooth homotopy connecting the maps g_0 and g_1. We modify it to realise hypothesis (a). By assumption p is a proper point of the maps g_0 and g_1 distinct forms $g_0(q')$ and $g_1(q')$. Thus it follows that there exists a positive ϵ satisfying the following conditions. For all $p_* \in S^n$ such that $\rho(p, p_*) < \epsilon$ and for all t such that $t \leq \epsilon$ or $t \geq 1 - \epsilon$ the point p_* is a proper point of the map h_t and $\rho(h_t(q'), p) > \epsilon$. We fix the positive number ϵ to satisfy the given conditions and choose a point p_* which is a proper point of the map h_* not belonging to $h_*(q' \times I)$ and satisfying the conditions $\rho(p, p_*) < \epsilon$. Such a point p_* exists by Theorems 4 and 1. We will regard the sphere S^n as situated in the Euclidean vector space E^{n+1} and write E^{n-1} for the linear subspace orthogonal to the vectors p and p_*. We denote by θ_α the rotation of S^n through an angle α round E^{n-1} and suppose that $\theta_\beta(p) = p_*$, $0 < \beta < \pi$. We suppose that $\chi(t)$ is a smooth real-valued function of the parameter t, defined on the interval $0 \leq t \leq 1$ and satisfying the conditions

$$0 \leq \chi(t) \leq 1, \quad \chi(0) = \chi(1) = 0;$$

$$\chi(t) = 1 \text{ for } \epsilon \leq t \leq 1 - \epsilon.$$

We put $\eta_t = \theta_{\beta\chi(t)}$. Then the given rotation η_t of the sphere S^n round E^{n-1}, depending on the parameter t, transforms the point p into the point p_* as t varies from 0 to ϵ and then transforms it back to its initial position as t varies from $1 - \epsilon$ to 1. We now define a family of maps l_t by setting

$$l_t = (\eta_t)^{-1} h_t.$$

The family is smooth and connects g_0 and g_1. It turns out that for $g_t = l_t$ hypothesis (a) is fulfilled. For $0 \leq t \leq \epsilon$ or $1 - \epsilon \leq t \leq 1$ we have $l_t(q') \neq p$. For $\epsilon \leq t \leq 1 - \epsilon$ the set $l_t^{-1}(p)$ coincides with the set $h_t^{-1}(p_*)$ and so $l_t(q') \neq p$ in this case also. We show that p is a proper point of the map l_*. Let $(a, t_0) \in l_*^{-1}(p)$ and let $x^1, \cdots, x^{n+k}$ be local coordinates in the neighbourhood of the point a. For the point (a, t_0) to be proper for the map l_* it is necessary and sufficient that there be n linearly independent vectors among the vectors

$$\frac{\partial \phi l_*(a, t_0)}{\partial x^1}, \cdots, \frac{\partial \phi l_*(a, t_0)}{\partial x^{n+k}}, \frac{\partial \phi l_*(a, t_0)}{\partial t}.$$

For $0 \leq t_0 \leq \epsilon$ or $1 - \epsilon \leq t_0 \leq 1$ there are indeed n linearly independent vectors among the first $(n + k)$ of these, in view of the choice of ϵ. For $\epsilon \leq t_0 \leq 1 - \epsilon$ there are n linearly independent vectors among these, in view of the choice of p_*. Thus for $g_t = l_t$ hypothesis (a) is fulfilled.

In order to realise hypothesis (b), we construct a smooth real function $s(t)$ of the parameter t, $0 \leq t \leq 1$, satisfying the conditions

$$s(t) = 0 \text{ for } 0 \leq t \leq \frac{1}{3},$$

$$s(t) = 1 \text{ for } \frac{2}{3} \leq t \leq 1,$$

$$\frac{ds}{dt} > 0 \text{ for } \frac{1}{3} < t < \frac{2}{3},$$

and we put

$$g_t = l_{s(t)}.$$

We show first that as before the hypothesis (a) is fulfilled for the homotopy g_t. Since $l_{s(t)}(q') \neq p$, $g_t(q') \neq p$. Now let (a, t_0) be an arbitrary point of the set $g_*^{-1}(p) \subset \Sigma^{n+k} \times I$, and let $x^1, \cdots, x^{n+k}$ be coordinates in a neighbourhood in Σ^{n+k} of the point a. Since $(a, t_0) \in g_*^{-1}(p)$, it follows that $(a, s(t_0)) \in l_*^{-1}(p)$. For g_* to be proper at the point (a, t_0) it is necessary and sufficient to have n linearly independent vectors among the vectors

$$\frac{\partial \phi g_*(a, t_0)}{\partial x^1}, \cdots, \frac{\partial \phi g_*(a, t_0)}{\partial x^{n+k}}, \frac{\partial \phi g_*(a, t_0)}{\partial t}.$$

For l_* to be proper at the point $(a, s(t_0))$ it is necessary and sufficient to have n linearly independent vectors among the vectors

$$\frac{\partial \phi l_*(a, s(t_0))}{\partial x^1}, \cdots, \frac{\partial \phi l_*(a, s(t_0))}{\partial x^{n+k}}, \frac{\partial \phi l_*(a, s(t_0))}{\partial s}.$$

For $\frac{1}{3} < t_0 < \frac{2}{3}$ we have $\frac{\partial \phi g_*(a, t_0)}{\partial t} = \frac{\partial \phi l_*(a, s(t_0))}{\partial s} \cdot \frac{ds(t_0)}{dt}$, where $\frac{ds(t_0)}{dt} > 0$, and so, since $(a, s(t_0))$ is proper for l_*, it follows that (a, t_0) is proper for g_*. For $0 \leq t_0 \leq \frac{1}{3}$ or $\frac{2}{3} \leq t_0 \leq 1$ the point a belongs either to $l_0^{-1}(p)$ or to $l_1^{-1}(p)$ and so, in fact, among the vectors

$$\frac{\partial \phi g_*(a, t_0)}{\partial x^1}, \cdots, \frac{\partial \phi g_*(a, t_0)}{\partial x^{n+k}}$$

there are n which are linearly independent. Thus hypothesis (a) is fulfilled for the homotopy g_t.

Since for $0 \leq t \leq \frac{1}{3}$ or for $\frac{2}{3} \leq t \leq 1$ we have $g_t = g_0$ or $g_t = g_1$ respectively, the orthogonality of the manifold M^{k+1} to the boundary of the strip E_*^{n+k+1} is evident.

Thus Theorem 9 is proved.

Theorem 9 is here proved only for the case $n \geq 2$; it can easily be proved also for $n = 1$. However for this case it is without interest since the homotopy classification of maps of Σ^{k+1} into S^1 is quite elementary (see Theorem 12 for the case $k > 0$).

Transition from framed manifolds to maps.

B) Let N^r be a vector space with a specified basis $u_1, \cdots, u_r$. We denote by K'_α the region of N^r consisting of the vectors $\xi = \xi^1 u_1 + \cdots + \xi^r u_r$ for which the inequality $(\xi^1)^2 + \cdots + (\xi^r)^2 < \alpha^2$ is satisfied. We define a map λ_α of N^r on to the sphere S^r by mapping the point $\xi \in K'_\alpha$ to the point of S^r with coordinates

$$x^i = \frac{\xi^i \cdot \alpha^{2m}}{(\alpha^2 - (\xi^1)^2 - \cdots - (\xi^r)^2)^m}$$

(see(A)) and sending the whole of $N^r - K'_\alpha$ to the point $q \in S^r$. From the relations (1) and (2) it readily follows that the map λ_α has smoothness class m. Further it is easily seen that the functional matrix of the map λ_α at the point $\xi = (0, \cdots, 0)$ is the identity. We now take for N^r the space T_p with basis $e_1, \cdots, e_r$ and we set $\omega_\alpha(x) = \lambda_\alpha \phi(x)$, $x \in S^r - q$, $\omega_\alpha(q) = q$. We thus obtain a smooth map ω_α of class m of S^r into itself and it is easily seen that ω_α is homotopic to the identity. It is a smooth homeomorphism of the spherical neighbourhood $K_\alpha = \phi^{-1}(K'_\alpha)$ on to $S^r - q$ and maps the set $S^r - K_\alpha$ to the point q.

Theorem 10. *Let Σ^{n+k} and S^n be two oriented spheres, let p' be a fixed point of Σ^{n+k} and let E^{n+k} be the tangent space to Σ^{n+k} at the point p'. Further let p_0 and p_1 be two points of S^n and let $e_{10}, \cdots, e_{n0}$; $e_{11}, \cdots, e_{n1}$ be orthonormal systems of vectors, tangent to S^n at the points p_0 and p_1 respectively. Let (M_0^k, U_0) and (M_1^k, U_1) be two homologous smoothly-framed manifolds in E^{n+k}. It turns out that there exists a map g_0 of Σ^{n+k} into S^n such that (see definition 4)*

$$(g_0;\ p_0, e_{10}, \cdots, e_{n0}) \rightarrow (M_0^k, U_0).$$

It further turns out that if f_0 and f_1 are two maps of Σ^{n+k} into S^n such that

$$(f_0;\ p_0, e_{10}, \cdots, e_{n0}) \rightarrow (M_0^k, U_0),$$

$$(f_1;\ p_1, e_{11}, \cdots, e_{n1}) \rightarrow (M_1^k, U_1),$$

the f_0 and f_1 are homotopic.

Proof. Since the systems of tangent vectors $e_{10}, \cdots, e_{n0}$ and $e_{11}, \cdots, e_{n1}$ give the same orientation to the sphere S^n there exists an isometric map θ of S^n on to itself, obtained from the identity by rotation, by which the system $p_0, e_{10}, \cdots, e_{n0}$ is transformed into $p_1, e_{11}, \cdots, e_{n1}$. The maps f_1 and $\theta^{-1} f_1$ are homotopic and

$$(\theta^{-1} f_1;\ p_0, e_{10}, \cdots, e_{n0}) \rightarrow (M_1^k, U_1).$$

Thus to prove the second part of the theorem it is sufficient to consider the case when

$$(p_0, e_{10}, \cdots, e_{n0}) = (p_1, e_{11}, \cdots, e_{n1}),$$

that is, to prove that from the relations

$$(f_0;\ p, e_1, \cdots, e_n) \rightarrow (M_0^k, U_0), \tag{4}$$

$$(f_1;\ p, e_1, \cdots, e_n) \rightarrow (M_1^k, U_1) \tag{5}$$

it follows that f_0 and f_1 are homotopic.

We prove first that if

$$(M_0^k, U_0) = (M_1^k, U_1) = (M^k, U) \tag{6}$$

then the maps f_0 and f_1 are homotopic.

Let N_a be the normal in the Euclidean space E^{n+k} to the manifold M^k at the point a, and let $\eta^1, \cdots, \eta^n$ be the components of the vector $\eta \in N_a$ relative to the basis $u_1(a), \cdots, u_n(a)$. In the neighbourhood $S^n - q$ of the point p in the sphere S^n we introduce coordinates $x^1, \cdots, x^n$, with origin at the north pole p, by choosing an orthonormal system $e_1, \cdots, e_n$ at p (see (A)). From relations (4)–(6) it follows that, near the point a, the maps f_0 and f_1 of the space N_a into S^n appear in coordinate form as

$$x^i = \eta^i + \cdots;\ i = 1, \cdots, n,$$

$$x^i = \eta^i + \cdots;\ i = 1, \cdots, n,$$

where we have written only terms of the first order and terms of higher order have been omitted. Thus the maps f_0 and f_1 of the space N_a into S^n coincide near to a up to terms of the second order. From this it follows that for $\eta \in W_\delta$, where δ is sufficiently small (see §5, A), the geodesic arc $(f_0\phi^{-1}(\eta), f_1\phi^{-1}(\eta))$ on the sphere S^n, connecting $f_0\phi^{-1}(\eta)$ and $f_1\phi^{-1}(\eta)$, does not pass through the point p. We put $W'_\delta = \phi^{-1}(W_\delta)$. Since the region W'_δ contains the set $f_0^{-1}(p) = f_1^{-1}(p) = \phi^{-1}(M^k)$, the closed sets $f_0(S^n - W'_\delta)$ and $f_1(S^n - W'_\delta)$ do not contain p. For $\xi \in W'_\delta$ we put $\sigma(\xi) = \rho(\phi(\xi), \pi\phi(\xi))$. We transport the point $f_0(\xi)$, $\xi \in W'_\delta$, by uniform motion along the geodesic segment $(f_0(\xi), f_1(\xi))$ so that it traverses the segment in unit time and denote by $h(\xi, t)$ its position at time t, $0 \leq t \leq 1$. Let $\chi(\sigma)$ be a real function of the variable σ, defined on the interval $0 \leq \sigma \leq \delta$ and having the following properties:

$$\chi(\sigma) = 1 \text{ for } 0 \leq \sigma \leq \tfrac{1}{2}\delta,\quad \chi(\delta) = 0,$$

$$0 \leq \chi(\sigma) \leq 1 \text{ for } 0 \leq \sigma \leq \delta.$$

We put

$$h_t(\xi) = h(\xi, t\chi(\sigma(\xi))) \text{ for } \xi \in W'_\delta,$$

$$h_t(\xi) = f_0(\xi) \text{ for } \xi \in S^n - W'_\delta.$$

The family of maps h_t, $0 \leq t \leq 1$, yields a continuous deformation of the map $f_0 = h_0$

into the map h_1. Here the map h_1 has the following properties. There exists a spherical neighbourhood K_a of the point p in the sphere S^n, so small that

$$h_1^{-1}(K_a) = f_1^{-1}(K_a) = V \subset W_\delta',$$

and for $\xi \in V$ we have

$$h_1(\xi) = f_1(\xi). \tag{7}$$

It is now easy to prove that the maps f_0 and f_1 are homotopic. From (7) it follows (see (B)) that the maps $\omega_a h_1$ and $\omega_a f_1$ coincide. Since ω_a is homotopic to the identity, the maps h_1 and f_1 are homotopic and so f_0 and f_1 are also homotopic.

Thus it is proved that under the condition (6) it follows from (4) and (5) that f_0 and f_1 are homotopic.

We now prove that if the framed manifolds (M_0^k, U_0) and (M_1^k, U_1), corresponding to the maps f_0 and f_1, are homologous then the maps are homotopic. Let (M^{k+1}, U) be a framed submanifold of the strip $E^{n+k} \times I \subset E^{n+k} \times E^1 = E^{n+k+1}$, realising the homology between (M_0^k, U_0) and (M_1^k, U_1), (see definition 3). We denote the normal in E^{n+k+1} to the manifold M^{k+1} at the point $a \in M^{k+1}$ by N_a and take W_δ to be the neighbourhood of the manifold M^{k+1} in the strip $E^{n+k} \times I$, constructed in proposition (A) of §5. In the vector space N_a we are given a basis $U(a)$. We choose a positive number α such that for arbitrary points $a \in M^{k+1}$ we have the inclusion $\overline{K}_\alpha \subset W_\delta$ (see (B)). We now define a map g_* of the manifold $\Sigma^{n+k} \times I$ into the sphere S^n by setting

$$g_*(\xi) = \lambda_\alpha(\phi_*(\xi)) \text{ for } \phi_*(\xi) \in H_\delta(a) \quad (\text{see §5, A}),$$

$$g_*(\xi) = q \text{ for } \phi_*(\xi) \notin W_\delta \quad (\text{see (3)}).$$

To the map g_* of $\Sigma^{n+k} \times I$ into S^n corresponds a deformation g_t of maps of Σ^{n+k} into S^n. From the properties of the map λ_α (see (B)) it follows immediately that the framed manifolds corresponding to the maps g_0 and g_1 coincide with the given manifolds (M_0^k, U_0) and (M_1^k, U_1). Since the maps f_0 and g_0 correspond to the same framed manifold (M_0^k, U_0), the maps f_0 and g_0 are homotopic in view of what was proved earlier. Similarly f_1 and g_1 are homotopic. Since g_0 and g_1 are connected by the homotopy g_t, they are also homotopic. In view of the transitivity property of the homotopy relation the maps f_0 and f_1 are themselves homotopic.

Thus the second part of the theorem is proved. The proof of the first part is contained in the last construction, as we now demonstrate [1]. We are given a framed manifold (M^k, U). We denote the normal at $a \in M^k$ by N_a; in the vector space N_a we have a basis $U(a)$. We define a positive number α such that for arbitrary $a \in M^k$ the inclusion $\overline{K}_\alpha \subset W_\delta$ holds (see §5, A). A map g of the sphere Σ^{n+k} into the sphere S^n is given by the rule

1) Translator's note: The rest of the argument may well appear to the reader to be superfluous.

$$g(\xi) = \lambda_\alpha(\phi(\xi)) \text{ for } \phi(\xi) \in H_\delta(a),$$

$$g(\xi) = q \text{ for } \phi(\xi) \notin W_\delta.$$

From the properties of the map λ_α it follows immediately that the framed manifold (M^k, U) corresponds to the map g. Thus the first part of the theorem is also proved.

Thus Theorem 10 is completely proved.

It is easy to prove that each framed submanifold (M^k, U) of Euclidean space E^{k+1} is homologous to zero. Thus Theorems 9 and 10 are without interest for $n = 1$.

§7. Homology groups of framed manifolds

In this section we first define the notion of a deformation of a framed manifold. If the manifold is deformed smoothly in Euclidean space without self-intersections and carries its frame continuously with it, then we say that we have a continuous deformation of the framed manifold. It is easily proved that two framed manifolds obtained from each other by deformation are homologous. Further an addition operation is introduced into the set of homology classes of framed manifolds in Euclidean space in such a way that the set becomes a commutative group. If π_1 and π_2 are two homology classes and $(M_1^k, U_1) \in \pi_1$, $(M_2^k, U_2) \in \pi_2$, the sum $\pi_1 + \pi_2$ is defined to be the class containing the union of the two framed manifolds. Here it is necessary that M_1^k and M_2^k be disjoint and that they are not 'entangled' with each other, as is possible when the dimension of the ambient Euclidean space is less than $(2k + 2)$. The absence of entangling signifies that the manifolds M_1^k and M_2^k may be pulled apart by means of a deformation of each. In order that both these conditions may be fulfilled, it is assumed that M_1^k and M_2^k lie on opposite sides of some hyperplane.

Homotopies of framed manifolds.

A) Let E^r be a Euclidean space, let X be a compact metric space and let $N_{x,t}^n$ be a linear subspace with fixed origin $O(x, t)$, depending continuously on the pair (x, t), $x \in X$, $0 \le t \le 1$. Further let $U(x) = \{u_1(x), \cdots, u_n(x)\}$ be a basis of the vector space $N_{x,0}^n$, depending continuously on $x \in X$. Then there exists a basis $U(x, t)$ of $N_{x,t}^n$, depending continuously on (x, t) and coinciding with $U(x)$ for $t = 0$. If moreover the vector space $N_{x,t}^n$ does not depend on t for $x \in Y \subset X$, then we have $U(x, t) = U(x)$ for $x \in Y$.

We prove proposition (A). Let ϵ be a positive number so small that for $|t - t'| \le \epsilon$, $x \in X$, orthogonal projection of $N_{x,t}^n$ on to $N_{x,t'}^n$ is non-degenerate. We put $U(x, t) = U(x)$. We suppose a basis $U(x, t)$ already constructed for $0 \le t \le p\epsilon < 1$, $x \in X$ (p a non-negative integer). For $p\epsilon \le t \le (p + 1)\epsilon$ we construct a basis $U(x, t)$ by parallel transfer of the basis $U(x, p\epsilon)$ to the point $O(x, t)$ and orthogonal projection on to $N_{x,t}^n$.

B) Let E^{n+k} be a Euclidean space with Cartesian coordinates $y^1, \cdots, y^{n+k}$

chosen in it; let E_*^{n+k} be the strip given by the conditions $c_0 \leq y^{n+k} \leq c_1$, bounded by the hyperplanes E_0 and E_1, and let M^k be a smooth submanifold of the strip E_*^{n+k}, orthogonal at its boundary points to the boundary $E_0 \cup E_1$ of the strip E_*^{n+k} (see §5, A). We will call a smooth family of maps e_t, $0 \leq t \leq 1$, of M^k into E_*^{n+k} a smooth deformation of the submanifold M^k of the strip E_*^{n+k} if e_0 is the identity map and e_t is a regular homeomorphism of M^k on to a submanifold $e_t(M^k)$ of E_*^{n+k} which is orthogonal at its boundary points to the boundary of E_*^{n+k}. If U is a frame for M^k and if a frame $e_t(U)$ is given for the manifold $e_t(M^k)$, depending continuously on t, for which $e_0(U) = U$, then we say that e_t is a deformation of the framed manifold (M^k, U). (In the case of a closed manifold M^k, we take $E_*^{n+k} = E^{n+k}$). If for arbitrary t the map e_t of M^k is the identity, then e_t gives a deformation of the frame U of a fixed manifold M^k, yielding a homotopy of the frames $e_0(U)$ and $e_1(U)$ of M^k. It turns out that if e_t is a smooth deformation of the submanifold M^k of the strip E_*^{n+k} and if a frame U is given for the manifold M^k, then there exists a deformation e_t of the framed manifold (M^k, U). If moreover the deformation e_t holds the boundary points of M^k fixed then the frame $e_t(U)$, $0 \leq t \leq 1$, of $e_t(M^k)$ may be constructed to coincide with the original frame U on the boundary points of $e_t(M^k)$.

We prove proposition (B). Let (M^k, U) be a framed submanifold of the strip E_*^{n+k} and let e_t be a given smooth deformation of M^k. We construct a frame $e_t(U)$ of the submanifold $e_t(M^k)$, depending continuously on t, such that $e_0(U) = U$. We denote the normal at the point $e_t(x)$ to the manifold $e_t(M^k)$ by N_{xt}. By taking the system of vectors $U(x)$ as the initial basis of the space N_{x0}, we obtain, according to (A), a basis $U(x, t)$ of the vector space N_{xt} with origin at $e_t(x)$. The systems of vectors $U(x, t)$, $x \in M^k$, provide for fixed t the desired frame $e_t(U)$. Thus proposition (B) is proved.

C) Let (M^k, U) be a smooth framed submanifold of the Euclidean space E^{n+k}, and let e_t be a deformation of it in E^{n+k}. It turns out that the framed manifolds $(e_0(M^k), e_0(U))$ and $(e_1(M^k), e_1(U))$ are homologous.

We prove this. We put $s(t) = 3t^2 - 2t^3$. It may readily be seen that the function $s(t)$ satisfies the conditions

$$s(0) = 0, \quad s(1) = 1, \quad s'(0) = 0, \quad s'(1) = 0,$$

$$s'(t) > 0 \text{ for } 0 < t < 1.$$

We introduce the deformation f_t of (M^k, U) by putting $f_t = e_{s(t)}$. Evidently we have

$$f_0 = e_0, \quad f_1 = e_1.$$

To prove that $(f_0(M^k), f_0(U))$ and $(f_1(M^k), f_1(U))$ are homologous in the strip $E_*^{n+k+1} = E^{n+k} \times I \subset E^{n+k+1}$, we define the submanifold M^{k+1} as the collection of all points of the form $(f_t(x), t)$, $x \in M^k$, $0 \leq t \leq 1$. Let N'_{xt} be the normal in E^{n+k}

to the manifold $f_t(M^k)$ at the point $f_t(x)$. We denote by N_{xt} the normal in E^{n+k+1} to the manifold M^{k+1} at the point $(f_t(x), t)$. It is easily seen that $N_{xt} = (N'_{xt}, t)$ for $t = 0, 1$; that is, M^{k+1} is orthogonal at its boundary points to the boundary of the strip E_*^{n+k+1}. At points $(f_t(x), t)$, $t \neq 0, 1$, the normals (N'_{xt}, t) and N_{xt} are distinct and so the system $f_t(U(x))$ does not lie in N_{xt}. To obtain, from the system $f_t(U(x))$, a system $U(x, t)$ lying in N_{xt}, we project the system $f_t(U(x))$ orthogonally on to the space N_{xt}. It is easy to see that the projection involves no degeneration. Thus the system $U(x, t)$ is linearly independent and so constitutes a frame for the manifold M^{k+1}. Since on the boundary sets $(f_0(M^k), 0)$ and $(f_1(M^k), 1)$ the frame $U(x, t)$ coincides with the given frames $(f_0(U), 0)$ and $(f_1(U), 1)$, the framed manifold (M^{k+1}, U) realises the homology between

$$(f_0(M^k), f_0(U)) \text{ and } (f_1(M^k), f_1(U)).$$

D) Every frame of a smooth manifold of class m is homotopic to a smooth frame of class $(m - 1)$ of the same manifold.

Let M^k be a smooth submanifold of class m of the strip E_*^{n+k} and let $U(x) = \{u_1(x), \cdots, u_n(x)\}$ be a framing of M^k. We denote the components of the vector $u_i(x)$ in E^{n+k} by $u_i^1(x), \cdots, u_i^{n+k}(x)$. Let ϵ be a positive number and let v_i^j be real-valued smooth functions of class m, defined on M^k, such that $|u_i^j(x) - v_i^j(x)| < \epsilon$ (see §5, B). We denote by $v_i(x)$ the vector in E^{n+k} with components $v_i^1(x), \cdots, v_i^{n+k}(x)$, and by $w_i(x)$ the orthogonal projection of the vector $v_i(x)$ on to N_x. We put

$$W_t(x) = \{(1 - t)u_i(x) + tw_i(x)\}, \qquad i = 1, \cdots, n.$$

The system $W_t(x)$ is non-degenerate for sufficiently small ϵ and achieves a deformation of the original frame $U(x) = W_0(x)$ into the frame $W_1(x)$ which is smooth of class $(m - 1)$.

In §6 it was shown that the homotopy classification of maps of the sphere Σ^{n+k} into the sphere S^n is equivalent to the homology classification of smoothly-framed k-dimensional submanifolds of Euclidean space E^{n+k} (see Theorems 9 and 10). In view of propositions (C) and (D), it is possible to remove the requirement that the frame be smooth and consider arbitrary continuous frames of smooth manifolds. Indeed each continuous frame of a smooth manifold is homotopic to a smooth frame (see (D)) and smooth frames of a given manifold which are homotopic (not necessarily in a smooth way) are homologous and so correspond to homotopic maps of spheres.

The homology group Π_n^k of framed manifolds.

E) Let (M^k, U) be a framed submanifold of the Euclidean space E^{n+k} and let f be a homothetic map of E^{n+k} on to itself. It is plain that $(f(M^k), f(U))$ is also a framed submanifold of the Euclidean space E^{n+k}. If the map f preserves the orientation of the space E^{n+k}, then, as is easily seen, there exists a smooth family f_t of homothetic maps of E^{n+k} on itself, depending smoothly on the parameter t, $0 \leq t \leq 1$, such that f_0 is the identity and $f_1 = f$. The family f_t, $0 \leq t \leq 1$, realises a smooth deformation of the framed manifold (M^k, U) into the framed manifold $(f(M^k), f(U))$. Thus these framed manifolds are homologous (see (C)). From this it follows that if a framed manifold is transported in space as a rigid body or is compressed homothetically then it does not go out of its homology class.

Definition 5. We divide the collection of all framed k-dimensional submanifolds of Euclidean space E^{n+k} into homology classes. We denote the set of classes so obtained by Π_n^k and we define an addition operation in it in the following way. Let π_1 and π_2 be two elements of Π_n^k. We choose at random a hyperplane E^{n+k-1} in E^{n+k} and from the classes π_1 and π_2 we choose representatives (M_1^k, U_1) and (M_2^k, U_2) in such a way that M_1^k and M_2^k lie on opposite sides of the hyperplane E^{n+k-1}. In view of proposition (E) this is always possible. The framed manifold $(M^k, U) = (M_1^k, U_1) \cup (M_2^k, U_2)$ we define to be the union of the manifolds M_1^k and M_2^k, taken with their given frames. It turns out that the homology class π of the framed manifold (M^k, U) does not depend on the particular choice of hyperplane E^{n+k-1} and of representatives (M_1^k, U_1), (M_2^k, U_2) of the homology classes π_1 and π_2. By definition we have $\pi = \pi_1 + \pi_2$. It turns out, further, that by means of this definition of addition the set Π_n^k becomes a commutative group. The homology class of the empty framed manifold serves as the zero of the group. The element $-\pi$ inverse to π can be described in the following way. Let E^{n+k-1} be an arbitrary hyperplane in E^{n+k}, let (M^k, U) be a framed manifold in the class π and let σ be the reflexion of the space E^{n+k} relative to the hyperplane E^{n+k-1}. The homology class $-\pi$ contains the framed manifold $(\sigma(M^k), \sigma(U))$.

We prove first of all that the definition of addition in the set Π_n^k is invariant. Suppose that, together with the hyperplane E^{n+k-1} and framed manifolds (M_1^k, U_1) and (M_2^k, U_2) in the space E^{n+k}, we have chosen a hyperplane $\hat{E}^{n+k-1}$ and framed manifolds $(\hat{M}_1^k, \hat{U}_1)$ and $(\hat{M}_2^k, \hat{U}_2)$ in the classes π_1 and π_2. We show that the framed manifolds $(M_1^k, U_1) \cup (M_2^k, U_2)$ and $(\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)$ belong to the same homology class. Then the invariance of addition will be proved. Clearly there exists an orientation-preserving isometry f of E^{n+k} on to itself, for which $f(\hat{E}^{n+k-1}) = E^{n+k-1}$ and the manifolds $f(\hat{M}_1^k)$ and M_1^k lie on the same side of the hyperplane E^{n+k-1}. In view of remark (E) we have

$$f(\hat{M}_i^k, \hat{U}_i) \sim (\hat{M}_i^k, \hat{U}_i), \qquad i = 1, 2;$$

$$f((\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)) \sim (\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2).$$

Thus the question reduces to the case when $\hat{E}^{n+k-1} = E^{n+k-1}$ and both representatives $(\hat{M}_1^k, \hat{U}_1)$ and (M_1^k, U_1) of the class π_1 lie on the same side of the hyperplane E^{n+k-1} in the half-space E_-^{n+k}, while both representatives $(\hat{M}_2^k, \hat{U}_2)$ and

(M_2^k, U_2) of the class π_2 lie on the other side of E^{n+k-1} in the half-space E_+^{n+k}. Let (M_1^{k+1}, U_1^*) be a framed manifold in the strip $E^{n+k} \times I$ realising the homology $(M_1^k, U_1) \sim (\hat{M}_1^k, \hat{U}_1)$ and let (M_2^{k+1}, U_2^*) be defined similarly. If the inclusions $M_1^{k+1} \subset E_-^{n+k} \times I$ and $M_2^{k+1} \subset E_+^{n+k} \times I$ hold, the framed manifolds (M_1^{k+1}, U_1^*) and (M_2^{k+1}, U_2^*) do not intersect and their union is a framed manifold realising the homology $(M_1^k, U_1) \cup (M_2^k, U_2) \sim (\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)$. Let e be a vector in the space E^{n+k}, orthogonal to E^{n+k-1} and directed towards the half-space E_+^{n+k}. We denote by g_t the map of E^{n+k} to itself sending the point x to the point $x + te$. We choose the vector e so long that the inclusions

$$g_{-1}(M_1^{k+1}) \subset E_-^{n+k} \times I, \; g_1(M_2^{k+1}) \subset E_*^{n+k} \times I$$

hold. We remark, finally, that the framed manifolds $g_{-t}(M_1^k, U_1) \cup g_t(M_2^k, U_2)$ achieve a deformation of $(M_1^k, U_1) \cup (M_2^k, U_2)$ into $g_{-1}(M_1^k, U_1) \cup g_1(M_2^k, U_2)$ so that, in view of (C), $g_{-1}(M_1^k, U_1) \cup g_1(M_2^k, U_2) \sim (M_1^k, U_1) \cup (M_2^k, U_2)$. In exactly the same way $g_{-1}(\hat{M}_1^k, \hat{U}_1) \cup g_1(\hat{M}_2^k, \hat{U}_2) \sim (\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)$. Thus $(M_1^k, U_1) \cup (M_2^k, U_2) \sim (\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)$.

Since addition is independent of the choice of representative it follows that the class containing the empty framed manifold is the zero of the group Π_n^k; that is, the zero is the class of nullhomologous framed manifolds. We prove that the element $-\pi$ inverse to π may be described in the way indicated in the definition.

We will regard the Euclidean space E^{n+k} as lying in E^{n+k+1} and defined by the equation $y^{n+k+1} = 0$. We will also suppose that the points of M^k are all at a distance less than 1 from E^{n+k-1} (see (C)). Let E_+^{n+k} and E_-^{n+k} be the half-spaces into which E^{n+k} is cut by the hyperplane E^{n+k-1}, and let $M^k \subset E_+^{n+k}$. We rotate E_+^{n+k} in the half-space $y^{n+k+1} \geq 0$ of the space E^{n+k+1} until it coincides with E_-^{n+k}; in the course of the motion the framed manifold (M^k, U) sweeps out a framed submanifold (M^{k+1}, U^*) of the half-space $y^{n+k+1} \geq 0$. The framed manifold (M^{k+1}, U^*) lies entirely in the strip $0 \leq y^{n+k+1} \leq 1$ of E^{n+k+1} and realises the nullhomology of the framed manifold $(M^k, U) \cup (\sigma(M^k), \sigma(U))$.

We divide the set of maps of Σ^{n+k} into S^n into homotopy classes and denote by[1] $\pi_{n+k}(S^n)$ the collection of these classes. Since there exists a well-defined (1, 1) correspondence between elements of the group Π_n^k and elements of the set $\pi_{n+k}(S^n)$ (see §6), the addition operation defined in Π_n^k induces an addition in $\pi_{n+k}(S^n)$. It is not difficult to show that the addition operation in $\pi_{n+k}(S^n)$ so defined coincides with the usual addition of elements of homotopy groups (see [10]). This fact, however, is neither proved nor used in the present work. For readers familiar with the elements of homotopy theory the proof of this fact presents no

1) Translator's note: Pontryagin uses '$\pi^{n+k}(S^n)$'; but we have preferred the standard homotopy group symbol to avoid confusion with cohomotopy.

problem.

Orthogonalization of frames. Proposition (G) below show that in the homology theory of framed manifolds it is possible to restrict attention to orthonormal frames. Proposition (H) gives an approach to the problem of the homotopy classification of orthonormally framed manifolds of Euclidean space.

F) Let $U = \{u_1, \cdots, u_n\}$ be a linearly independent system of vectors in a Euclidean vector space E^l. We will subject this system to a process of orthogonalization, that is, we will find an orthonormal system $\bar{U} = \{\bar{u}_1, \cdots, \bar{u}_n\}$, obtained from the system U by the formula

$$\bar{u}_j = \sum_{i=1}^{n} a^i_j u_i, \qquad j = 1, \cdots, n,$$

where the coefficients a^i_j satisfy the conditions

$$a^i_j = 0 \text{ for } i > j; \; a^i_j > 0 \text{ for } i = j.$$

The coefficients a^i_j are uniquely determined by these conditions – they are expressed by means of scalar products of vectors of the system U. If the system U is orthonormal then $\bar{U} = U$. We put

$$U^t = \{u^t_1, \cdots, u^t_n\},$$

where

$$u^t_i = u_i(1 - t) + \bar{u}_i t.$$

The systems U^t are linearly independent since the matrices $\|(1 - t)\delta^i_j + ta^i_j\|$ are non-singular. Thus the system $\bar{U} = U^1$ is obtained from $U = U^0$ by means of a continuous deformation uniquely determined by U.

G) Let $U(x)$ be a frame for the manifold M^k. The frame $U^t(x)$ (see (F)) realises a continuous deformation of the original frame $U(x)$ into an orthonormal frame $\bar{U}(x)$. If the original frame is smooth of class m, then so are all the frames $U^t(x)$. Finally if there is given a deformation $U_t(x)$, $0 \leq t \leq 1$, of an orthonormal frame $U_0(x)$ into another orthonormal frame $U_1(x)$, then there exists an orthonormal deformation $\bar{U}_t(x)$ of $U_0(x)$ into $U_1(x)$.

H) Let (M^k, V) be an orthomormally framed submanifold of oriented Euclidean space E^{n+k}. In view of the remarks in definition 2 the manifold M^k has a well-defined orientation and we say that V is a frame for the *oriented* manifold M^k. Let U be an arbitrary orthonormal frame for the oriented manifold M^k. We compare the frames V and U. In each normal N_x to the manifold M^k there are two orthonormal systems of vectors

$$V(x) = \{v_1(x), \cdots, v_n(x)\} \text{ and } U(x) = \{u_1(x), \cdots, u_n(x)\};$$

so we have

$$u_i(x) = \sum_{j=1}^{n} f_{ij}(x) v_j(x), \qquad i = 1, \cdots, n,$$

where $f(x) = \|f_{ij}(x)\|$ is an orthogonal matrix with positive determinant. Thus, to each orthonormal frame U there exists, for fixed V, a map f of M^k into the manifold H_n of all orthogonal matrices with positive determinant: $U \to f$. It is evident that conversely to each map f of M^k into H_n there corresponds a unique orthonormal frame $U: f \to U$. We suppose that, together with the fixed frame V, we are given two orthonormal frames U_0 and U_1 of the oriented manifold M^k; let $U_0 \to f_0$, $U_1 \to f_1$. It is easy to see that the frames U_0 and U_1 are homotopic if and only if the maps f_0 and f_1 are homotopic. Thus the homotopy classification of frames of the oriented submanifold M^k of oriented Euclidean space E^{n+k} is equivalent to the homotopy classification of maps M^k into the manifold H_n of all orthonormal matrices of order n with positive determinant.

§8. The suspension operation

In this section we will define and to a certain extent study the suspension operation for framed manifolds which plays an important role in the question of the homotopy classification of maps of spheres into spheres. Let (M^k, U) be a framed submanifold of the Euclidean space E^{n+k}, regarded as a subspace of E^{n+k+1}. At each point $x \in M^k$ we draw in E^{n+k+1} a unit vector $u_{n+1}(x)$, perpendicular to the hyperplane E^{n+k}, in such a way that all the vectors $u_{n+1}(x)$, $x \in M^k$, are directed in the same sense, and we put $EU(x) = \{u_1(x), \cdots, u_n(x), u_{n+1}(x)\}$. The framed manifold $E(M^k, U) = (M^k, EU)$ of the Euclidean space E^{n+k+1} is called the *suspension* of (M^k, U). It turns out that the suspensions of homologous framed manifolds are homologous and that the induced map E of the group Π^k_n into the group Π^k_{n+1} (see definition 5) is a homomorphism. In Theorem 11 it is proved that E is on to Π^k_{n+1} if $n \geq k + 1$ and an isomorphism if $n \geq k + 2$ so that the groups Π^k_{k+2}, Π^k_{k+3}, $\cdots$ are all naturally isomorphic to each other.

In terms of maps of spheres the suspension operation can be described as follows. Let p' and q' be the poles of the sphere Σ^{n+k+1} and let Σ^{n+k} be its equator, that is, the section by the hyperplane perpendicular to the segment $p'q'$ and passing through the midpoint of this segment. In the same way let p and q be poles of S^{n+1} and S^n its equator. With the map f of Σ^{n+k} into S^n we associate the map Ef of Σ^{n+k+1} into S^{n+1} which maps the meridan $p'xq'$, $x \in \Sigma^{n+k}$, of the sphere Σ^{n+k+1} isometrically on to the meridan $pf(x)q$ of S^{n+1}. The suspension Ef of the map f was introduced, in the form described here, by Freudenthal [11]. In this work this form will not be used. The fact that the suspension of maps and the suspension of framed manifolds correspond to each other in the sense of definition 4 may be easily proved, but the proof will not be given here.

Definition 6. Let (M^k, U), $U(x) = \{u_1(x), \cdots, u_n(x)\}$, be a framed submanifold of the oriented Euclidean space E^{n+k} and let E^{n+k+1} be an oriented Euclidean space containing E^{n+k}. Let $e_1, \cdots, e_{n+k}$ be a basis of the vector space E^{n+k}, determining its orientation, and let e_{n+k+1} be a unit vector of E^{n+k+1}, orthogonal to E^{n+k} and such that the basis $e_1, \cdots, e_{n+k}, e_{n+k+1}$ determines the orientation of E^{n+k+1}. We designate by $u_{n+1}(x)$ the vector emanating from the point $x \in M^k$ and obtained from e_{n+k+1} by parallel displacement. We set

$$EU(x) = \{u_1(x), \cdots, u_n(x), u_{n+1}(x)\}.$$

Then $E(M^k, U) = (M^k, EU)$ is a framed submanifold of E^{n+k+1}. The framed manifold $E(M^k, U)$ is called the *suspension* of (M^k, U). It turns out that if $(M_0^k, U_0) \sim (M_1^k, U_1)$, then $E(M_0^k, U_0) \sim E(M_1^k, U_1)$. Thus the correspondence $(M^k, U) \to E(M^k, U)$ induces a map of Π_n^k into Π_{n+1}^k. This map turns out to be a homomorphism; we will also denote it by E.

We show that if $(M_0^k, U_0) \sim (M_1^k, U_1)$, then $E(M_0^k, U_0) \sim E(M_1^k, U_1)$. Let (M^{k+1}, U^*) be a framed submanifold of the strip $E^{n+k} \times I$ realising the homology $(M_0^k, U_0) \sim (M_1^k, U_1)$. At the point $y \in M^{k+1}$ we choose a unit vector $u_{n+1}^*(y)$ in the strip $E^{n+k+1} \times I$, orthogonal to the strip $E^{n+k} \times I$ and in the same direction as e_{n+k+1}. We put $EU^*(y) = \{u_1^*(y), \cdots, u_n^*(y), u_{n+1}^*(y)\}$. It is plain that the framed submanifold $E(M^{k+1}, U^*) = (M^{k+1}, EU^*)$ of the strip $E^{n+k+1} \times I$ realises the homology $E(M_0^k, U_0) \sim E(M_1^k, U_1)$.

That E is homomorphic is seen still more simply. The homomorphism E turns out, in a range of cases, to be an epimorphism or even an isomorphism. We will study these cases. For this we first prove the following proposition.

A) Let E^{n+k+1} be an oriented Euclidean space and let E^{n+k} be an oriented hyperplane. Further let (M^{k+1}, V) be an orthonormally framed submanifold of the strip $E^{n+k+1} \times I$ such that M^{k+1} actually lies in $E^{n+k} \times I$. The case of a closed manifold M^{k+1} is not excluded. We will suppose the boundary of M^{k+1} to consist of manifolds $M_0^k \times 0$ and $M_1^k \times 1$ such that $M_0^k \subset E^{n+k}$, $M_1^k \subset E^{n+k}$. We suppose that V has the form of a suspension on the boundary sets $M_0^k \times 0$ and $M_1^k \times 1$, that is,

$$V(x, \tau) = EU_\tau(x) \times \tau, \qquad \tau = 0, 1,$$

where U_τ is a frame for the manifold M_τ^k, $\tau = 0, 1$, in E^{n+k}. At each point $x \in M^{k+1}$ we choose a unit vector $u_{n+1}(x)$ orthogonal to $E^{n+1} \times I$ and appropriately sensed. In the normal N_x to M^{k+1} at the point x in the space $E^{n+k} \times I$ we have a basis $v_1(x), \cdots, v_{n+1}(x)$. Therefore the vector $u_{n+1}(x)$, which also lies in N_x, satisfies

$$u_{n+1}(x) = \psi^1(x) v_1(x) + \cdots + \psi^{n+1}(x) v_{n+1}(x). \qquad (1)$$

Let N be a Euclidean space of dimension $(n+1)$ with a given Cartesian coordinate system and let $\mathfrak{S}^n$ be the unit sphere in N with centre at the origin. We denote the point $(0, \cdots, 0, 1)$ of $\mathfrak{S}^n$ by $\mathfrak{B}$. With each point $x \in M^{k+1}$ we associate the point $\psi(x)$ of $\mathfrak{S}^n$ with coordinates $\psi^1(x), \cdots, \psi^{n+1}(x)$. So defined, ψ is a map of M^{k+1} into $\mathfrak{S}^n$ sending the whole boundary to $\mathfrak{B}$. We suppose that there exists

a deformation ψ_t, $0 \le t \le 1$, of the map $\psi = \psi_0$ to a map ψ_1 sending M^{k+1} to $\mathfrak{B}$, the boundary of M^{k+1} staying at $\mathfrak{B}$ throughout the deformation. It then turns out that there exists a deformation of the frame V into a frame EU, where U is a frame for the manifold M^{k+1} in $E^{n+k} \times I$, such that the frame restricted to the boundary of M^{k+1} remains constant throughout the deformation. For a closed manifold M^{k+1} this means that the framed manifold (M^{k+1}, V) is homologous to a framed manifold $E(M^{k+1}, U)$. For a non-closed manifold M^{k+1} this permits us to deduce, from a homology $E(M_0^k, U_0) \sim E(M_1^k, U_1)$, realised by (M^{k+1}, V), a homology $(M_0^k, U_0) \sim (M_1^k, U_1)$.

We prove proposition (A). We introduce into N_x Cartesian coordinates corresponding to the basis $v_1(x), \cdots, v_{n+1}(x)$, and let λ_x be the coordinate map of N on to N_x. We set $\psi(x, t) = \lambda_x \psi_{1-t}(x)$. The vector $\psi(x, t)$ of $E^{n+k+1} \times I$ lies in N_x, depends continuously on the pair of variables x, t, and satisfies $\psi(x, 0) = v_{n+1}(x)$, $\psi(x, 1) = u_{n+1}(x)$. The subspace of N_x orthogonal to $\psi(x, t)$ we denote by P_{xt}. Since $\psi(x, 0) = v_{n+1}(x)$, the vectors $v_1(x), \cdots, v_n(x)$ constitute a basis for P_{x0}. By taking this initial basis and applying proposition (A) of §7 to the variable vector space P_{xt}, we obtain a basis $U(x, t)$ for P_{xt}. Together with the vector $\psi(x, t)$ this basis gives us the required deformation of the frame V. Thus proposition (A) is proved.

Theorem 11. *The homomorphism E of Π_n^k into Π_{n+1}^k is an epimorphism if $n \ge k+1$ and an isomorphism if $n \ge k+2$. Thus the groups $\Pi_{k+2}^k, \Pi_{k+3}^k, \cdots$ are in a natural way isomorphic to each other.*

Proof. Let $n \ge k+1$, $\hat{\pi} \in \Pi_{n+1}^k$, and $(\hat{M}^k, \hat{U})$ a framed submanifold of the Euclidean space E^{n+k+1} in the homology class $\hat{\pi}$. In view of proposition (D) of §2, there exists a one-dimensional direction E^1, projection along which maps $\hat{M}^k$ regularly without self-intersection on to a manifold M^k. We will regard the projection along E^1 as a map on to a hyperplane E^{n+k} orthogonal to E^1 and containing M^k. We move each point $x \in \hat{M}^k$ by rectilinear uniform motion along E^1 to coincide with its image in M^k under projection in such a way that it traverses the path in unit time. This gives a deformation of $\hat{M}^k$ into M^k. In view of propositions (B) and (G) of §7, there exists a deformation of the framed manifold $(\hat{M}^k, \hat{U})$ into an orthonormally framed manifold (M^k, V). Since $n \ge k+1$, the map ψ of the manifold M^k into the sphere $\mathfrak{S}^n$, constructed in proposition (A), is homotopic to the constant map at $\mathfrak{B}$, and so the frame V of M^k is homotopic to a frame EU for the same manifold. In view of proposition (C) of §7, we have $(\hat{M}^k, \hat{U}) \sim E(M^k, U)$. Let $\pi = \Pi_n^k$ be the homology class of (M^k, U); then $\hat{\pi} = E\pi$. Thus it is established that $\Pi_{n+1}^k = E\Pi_n^k$ for $n \ge k+1$.

We suppose now that $n \ge k+2$ and show that E is an isomorphism; thus we must show that for $\pi_0, \pi_1 \in \Pi_n^k$, the relation $E\pi_0 = E\pi_1$ implies that $\pi_0 = \pi_1$.

Let (M^k_0, U_0) and (M^k_1, U_1) be orthonormally framed manifolds in the classes π_0 and π_1, situated in the Euclidean space $E^{n+k} \subset E^{n+k+1}$. Further let $(\hat{M}^{k+1}, \hat{U})$ be a framed submanifold of the strip $E^{n+k+1} \times I$ realising the homology $E(M^k_0, U_0) \sim E(M^k_1, U_1)$. We denote by $\hat{E}^1$ the one-dimensional direction in the space $E^{n+k+1} \times I$ orthogonal to $E^{n+k} \times I$. In view of proposition (D) of §2 there exists in arbitrarily close proximity to $\hat{E}^1$ a direction E^1, projection along which maps the manifold $\hat{M}^{k+1}$ regularly and without self-intersection. We choose E^1 so close to $\hat{E}^1$ that the projection M^{k+1} of $\hat{M}^{k+1}$ in the direction E^1 lies in the strip $E^{n+k} \times I$. The deformation of $\hat{M}^{k+1}$ into M^{k+1} leaves the boundary fixed, and so there may be defined, in view of propositions (B) and (G) of §7, a deformation of $(\hat{M}^{k+1}, \hat{U})$ into an orthonormally framed manifold (M^{k+1}, V) which leaves the frame restricted to the boundary unaltered. Thus the homology $E(M^k_0, U_0) \sim E(M^k_1, U_1)$ may be achieved by the framed manifold (M^{k+1}, V) where $M^{k+1} \subset E^{n+k} \times I$; that is, the conditions of proposition (A) are fulfilled and so the framed manifolds (M^k_0, U_0) and (M^k_1, U_1) are homologous. Thus $\pi_0 = \pi_1$.

Theorem 11 is therefore proved.

CHAPTER III

The Hopf invariant

§9. The homotopy classification of maps of n-dimensional manifolds into the n-sphere

In this section we give the homotopy classification of maps of smooth closed oriented manifolds of dimension n into the n-sphere. The result is well-known even for arbitrary manifolds but it plays an important auxiliary role in this work. The proof we give is by arguments specific to smooth manifolds so that the application of the result in later sections is simplified. To begin with, the degree of a map is defined and the simplest of its properties are proved. Next, on the basis of the constructions of the earlier theory the classification of maps of Σ^n into S^n is given, providing an elementary illustration of the general results of preceding sections. Finally the classification of maps of n-dimensional manifolds into S^n is reduced to the classification of maps of Σ^n to S^n.

The degree of a map.

Definition 7. Let f be a smooth map of the r-dimensional oriented manifold P^r into the r-dimensional oriented manifold Q^r and let b be an interior point of Q^r, which is a proper point of the map f and such that $f^{-1}(b)$ is compact and does not meet the boundary of P^r. Under these assumptions $f^{-1}(b)$ consists of a finite number of points $a_1, \cdots, a_p$, at each of which the functional determinant of the map f is non-zero, and so has a definite sign (the manifolds P^r and Q^r are oriented). We denote by $\epsilon_i (= \pm 1)$ the sign of the functional determinant of f at the point a_i, $i = 1, \cdots, p$, and we will call ϵ_i the *degree* of f at a_i. The sum $\epsilon_i + \cdots + \epsilon_p$ we will call the *degree* of f at b. Now if both manifolds are closed, the set G of all points b fulfilling the requirements set out above is a region everywhere dense in Q^r (see Theorem 4). It will be shown below (see (B)) that if moreover the manifold Q^r is connected then the degree of f is the same for all points $b \in G$; it will then be called the *degree* of f. It will also be shown below that (see (B)) the degrees of homotopic maps coincide. Thus, if P^r is closed and Q^r is closed and connected, the degree is an invariant of homotopy class and so is defined for arbitrary continuous maps.

A) Let Q^r be a closed connected manifold, P^{r+1} a compact manifold with boundary P^r, f a smooth map of P^{r+1} into Q^r and $b \in Q^r$ a proper point of the map f of P^r into Q^r. It turns out that the map $f|P^r$ has degree 0 at b.

We prove this. Let V be a connected neighbourhood of b, consisting of proper points of $f|P^r$. It is easy to see that the degree is the same for all points of V. So without loss of generality we may suppose that b is a proper point of the map f of P^{r+1} (see Theorem 4). Then $f^{-1}(b)$ is a one-dimensional submanifold M^1 of P^{r+1} and so consists of finitely many components, some of which are homeomorphic to a circle, the rest to a line segment. All points of $f^{-1}(b)$ in P^r are end-points of components of the manifold M^1. Let L^1 be a component of M^1 homeomorphic to an interval; we denote its end-points by a_0 and a_1. In view of the results of §4 (see (2)), for a given system of coordinates $y^1, \cdots, y^r$ with origin at b, defined in some neighbourhood of b, it is possible to choose coordinates $x^1, \cdots, x^{r+1}$ in the neighbourhood of $a \in L^1$ such that the map f takes the form:

$$y^i = x^i, \quad i = 1, \cdots, r.$$

We will suppose that the coordinates $y^1, \cdots, y^r$ orient Q^r. In the coordinates $x^1, \cdots, x^{r+1}$ the curve L^1 is given by the equations $x^1 = 0, \cdots, x^r = 0$, that is, x^{r+1} is a parameter on the curve L^1. We will suppose that a point on the curve L^1 moves from a_0 to a_1 as the parameter x^{r+1} increases. Under these assumptions the coordinates $x^1, \cdots, x^{r+1}$ may not determine the given orientation of P^{r+1} and we denote by $\epsilon (= \pm 1)$ the sign which distinguishes the orientation given in P^{r+1} from the orientations determined by the coordinates $x^1, \cdots, x^{r+1}$. It is easy to verify that ϵ does not depend on the particular choice of system $x^1, \cdots, x^{r+1}$ and does not alter under variations of the point a along the curve L^1. From the definition of the orientation of the boundary (see §1, B) it follows that the degree of the map $f|P^r$ is equal to $-\epsilon(-1)^r$ at a_0 and $\epsilon(-1)^r$ at a_1. By applying these considerations to all components of the manifold M^1 which are homeomorphic to line segments, we conclude that the degree of f at b is zero.

B) Let f_0 and f_1 be two homotopic smooth maps of the closed manifold P^r into the connected closed oriented manifold Q^r and let G_t be the set of all proper points $b \in Q^r$ of the map f_t, $t = 0, 1$. It turns out that, if $b \in G_0 \cap G_1$, the degrees of f_0 and f_1 at b are equal. It further turns out that if b_0 and b_1 are two points of G_0, the degrees of f_0 at b_0 and b_1 are equal.

We prove proposition (B). Since f_0 and f_1 are homotopic there exists a smooth family f_t connecting them (see Theorem 8). To the family f_t there corresponds a map f_* of the product $P^r \times I$ (see §5, C). The boundary of the manifold $P^r \times I$ consists of the manifolds $P^r \times 0$ and $P^r \times 1$. We choose the orientation of $P^r \times I$ such that $P^r \times 0$ is contained in the boundary of the product $P^r \times I$ with positive sign; then $P^r \times 1$ is contained in the boundary of $P^r \times I$ with negative sign. From this and from proposition (A) it follows that the degrees of the maps f_0 and f_1 at the point b coincide.

We now prove that the degrees of the map f_0 coincide at all points $b \in G_0$. Let X be a coordinate system with origin at the point $c \in Q^r$; let V be a spherical neighbourhood of c in this coordinate system; and further let b_0 and b_1 be two points of $V \cap G_0$. It is easy to construct a smooth homeomorphism ϕ of Q^r onto itself, under which all points of the set $Q^r - V$ remain fixed and which maps b_0 to b_1. The map ϕ is evidently homotopic to the identity. It is also evident that the degree of ϕf_0 at b_1 is equal to the degree of f_0 at b_0; but, since ϕf_0 and f_0 are homotopic, the degrees of these maps at b_1 coincide. Thus the degrees of f_0 at all points $b \in V \cap G_0$ are equal. Since Q^r is connected and G_0 is everywhere dense in Q^r it follows that the degrees of f_0 at all points $b \in G_0$ are the same.

Maps of the n-sphere into the n-sphere.

C) Let (M^0, U) be a 0-dimensional framed submanifold of the oriented Euclidean space E^n. Since M^0 is compact it consists of finitely-many points $a_1, \cdots, a_r$. We give a_i the index $+1$ if the vectors $u_1(a_i), \cdots, u_n(a_i)$ give a positive orientation to E^n and the index -1 in the opposite case. We call the sum $I(M^0, U)$ of the indices of the points $a_1, \cdots, a_r$ the *index* of the framed manifold. It is clear that the index of (M^0, U) is equal to the degree of the associated map (see definition 4) of the oriented sphere Σ^n into the oriented sphere S^n.

Theorem 12. *If two maps f_0 and f_1 of the oriented sphere Σ^n into the oriented sphere S^n have the same degree then they are homotopic. Moreover every integer appears as the degree of some map.*

Proof. From proposition (C) and Theorem 10 it follows that it is sufficient to establish a homology between framed 0-dimensional manifolds of the same index and to prove the existence of framed manifolds of arbitrary index. It is easy to see that two framed manifolds (M_0^0, U_0) and (M_1^0, U_1), each of which consists of one point of index $+1$, are obtainable from each other by a deformation (see §7, B) and

so belong to the same homology class (see §7, C). Thus all 'one-point' framed manifolds of index $+1$ belong to one homology class ϵ. In exactly the same way all 'one-point' framed manifolds of index -1 belong to one homology class ϵ'. Since under reflexion in an arbitrary hyperplane the space E^n reverses orientation it follows that $\epsilon' = -\epsilon$ (see definition 5). Since, further, each 0-dimensional framed manifold (M^0, U) is the union of a finite number of 'one-point' framed manifolds, some of index $+1$, the other of index -1, it follows that ϵ is a generator of the group Π_n^0, such that (M^0, U) belongs to the class $I(M^0, U)\epsilon$. Thus two 0-dimensional framed manifolds of the same index are homologous. It is clear also that there exist 0-dimensional framed manifolds of arbitrary index.

Thereby Theorem 12 is proved.

From Theorem 12 and (C) it follows that the group Π_n^0 or, what is the same thing, the group $\pi_n(S^n)$ is free cyclic.

D) Let f be a smooth map of the oriented sphere Σ^{n+k} into the oriented sphere S^n and let g be a smooth map of Σ^{n+k} into itself of degree ν. We denote by π the element of Π_n^k corresponding to the map f and by π' the element of Π_n^k corresponding to fg. It then turns out that

$$\pi' = \nu\pi. \tag{1}$$

We prove (1). Let p' and q' be the north and south poles of Σ^{n+k}, let E^{n+k} be the tangent space to Σ^{n+k} at p' and let ϕ be the central projection from q' of the region $\Sigma^{n+k} - q'$ on to E^{n+k}. For $\nu = 1$ the map g is homotopic to the identity (see Theorem 12), so that in this case the relation (1) is true. We prove it for $\nu = -1$. Since all maps of Σ^{n+k} to itself of degree -1 are homotopic, it is sufficient to consider one particular map g of degree -1. Let E^{n+k-1} be an arbitrary hyperplane of E^{n+k}, passing through the point p', and let σ be the reflexion of E^{n+k} in this hyperplane. The map $g = \phi^{-1}\sigma\phi$ of the region $\Sigma^{n+k} - q'$ on to itself, together with the relation $g(q') = q'$ determines a map g of Σ^{n+k} on to itself of degree -1. For the map g so constructed the relation (1) is obvious.

Now let g be a smooth map of Σ^{n+k} into itself such that $g^{-1}(p')$ consists of proper points of g and does not contain q'; this may always be achieved by a small change in an arbitrary given map g. Let $\phi g^{-1}(p') = \{a_0, \cdots, a_r\}$; we denote by ϵ_i the sign of the functional determinant of g at the point $\phi^{-1}(a_i)$. Further let V_i be the ball of radius δ in E^{n+k} with centre a_i. We will take δ so small that a hyperplane E_1^{n+k-1} may be construced intersecting none of the V_i and having a preassigned part of the set $\{a_0, \cdots, a_r\}$ on one side of it, the rest of the set on the other. We choose a positive number α so small that, if K_α is the spherical neighbourhood of P' the set $g^{-1}(K_\alpha)$ consist of proper points of g and splits into regions $A_1, \cdots, A_r$, where $a_i \in A_i$, each region being mapped by

g smoothly and homeomorphically on to K_α. We suppose α so small, moreover, that $\phi(A_i) \subset V_i$. We now define a map h_i of Σ^{n+k} on to itself to coincide with $\omega_a g$ (see[1] (B)) on A_i and to map the set $\Sigma^{n+k} - A_i$ to q'. Since the map h_i has degree ϵ_i, the framed manifold (M_i^k, U_i) corresponding to the map fh_i belongs to the homology class $\epsilon_i\pi$. It is evident that $M_i^k \subset V_i$ and that the framed manifold $(M_1^k, U_1) \cup (M_2^k, U_2) \cup \cdots \cup (M_r^k, U_r)$ corresponds to the map $f\omega_a g$. Since the maps $\omega_a g$ and g are homotopic, then from what has been said and from the possibility of the indicated method of constructing such hyperplanes E_1^{n+k-1} the validity of the relation (1) follows.

Maps of n-dimensional manifolds into S^n. Theorem 13 below completely solves the problem of the homotopy classification of maps of closed orientable manifolds of dimension n into S^n. Theorem 13 is proved by the use of the information given by Theorem 12 on the classification of maps of Σ^n into S^n.

Theorem 13. *Two continuous maps f_0 and f_1 of the connected oriented smooth manifold M^n into the oriented sphere S^n are homotopic if and only if they have the same degree (see definition 7). If, in fact, the degree of a map is zero, then the map is nullhomotopic, that is, contractible to a point. Moreover there exist maps of arbitrary given degree.*

Proof. To prove the first part of the theorem it is enough to show that if f_0 and f_1 are smooth maps of the same degree then they are homotopic. To reduce the proof of this fact to Theorem 12, we show first that any finite collection Q of points of the manifold M^n is contained in a region B of M^n smoothly homeomorphic to an n-dimensional ball.

It is easy to construct a smooth simple closed curve K, lying without self-intersections in M^n, containing all the points of the set Q. We take M^n to be a submanifold of the Euclidean space E^{2n+1}, and we denote by N_x the normal in E^{2n+1} to the curve K at the point $x \in K$. Just as in proposition (A) of §5 we denote by $H_\delta(x)$ the ball in the Euclidean space N_x with centre x and radius δ. Then there exists a sufficiently small positive number δ such that the set $W_\delta = H_\delta(K)$ is a neighbourhood of the curve K in E^{2n+1} and such that by associating with each point $y \in W_\delta$ the point $x = \pi(y)$ for which $y \in H_\delta(x)$, we obtain a smooth map π of the manifold W_δ on to the curve K (see §5, A). We distinguish on the closed curve K a segment L containing all points of the set Q in its interior, and we introduce on L a smooth parameter t, $-1 \le t \le 1$. In this way, there corresponds to each value of the parameter t, $-1 \le t \le 1$, a point $x(t) \in K$. We denote by T_x the tangent to the manifold M^n at the point $x \in K$ and we set $N_t' = N_{x(t)} \cap T_{x(t)}$. In the vector space N_t' we introduce an orthonormal basis $e_1(t), \cdots, e_{n-1}(t)$.

1) Translator's note: The appropriate reference to ω_a and K_a seems to be §6, B.

Using proposition (A) of §7 and the orthogonalization process (see §7, G), it is not hard to see that this may be done in such a way that the basis $e_1(t), \cdots, e_{n-1}(t)$ depends smoothly on the parameter t. Let $W_\delta' = M^n \cap W_\delta$, and let π' be the restriction of π to W_δ'. We denote $\pi'^{-1}(x(t))$ by H_t'. Let ϵ be a positive number. We denote by H_t^* the ball of radius $\epsilon\sqrt{1-t^2}$, in the space N_t', with its centre at $x(t)$. By orthogonal projection χ_t of the manifold M^n on to the space $T_{x(t)}$ some neighbourhood of $x(t)$ in M^n is mapped by smooth regular homeomorphism on to some neighbourhood of $x(t)$ in $T_{x(t)}$. From this it follows that for sufficiently small δ the projection χ_t is a smooth regular homeomorphism of the manifold H_t' on to some neighbourhood of $x(t)$ in N_t', and so there exists ϵ so small that $H_t^* \subset \chi_t(H_t')$, $-1 \le t \le 1$. We denote the coordinates of the point $z \in H_t^*$ with respect to the basis $e_1(t), \cdots, e_{n-1}(t)$ by $\epsilon z^1, \cdots, \epsilon z^{n-1}$ and we take the numbers $z^1, \cdots, z^{n-1}, t$ as coordinates of the point $\chi_t^{-1}(z)$. The collection B of all points $\chi_t^{-1}(z)$, $-1 \le t \le 1$, $z \in H_t^*$, constitutes a region in M^n, in which smooth coordinates $z^1, \cdots, z^{n-1}, t$ have been introduced satisfying the condition $(z^1)^2 + \cdots + (z^{n-1})^2 + t^2 < 1$. Thus B is the smooth homeomorph of an open n-dimensional ball.

We now choose in S^n a point p such that the set $f_t^{-1}(p)$, $t = 0, 1$, consists of proper points of the map f_t (see Theorem 4); put $f_t^{-1}(p) = P_t$. Set $Q = P_0 \cup P_1$ and let B be a spherical region in M^n containing the finite set Q. We take the point p as north pole of S^n and denote the south pole by q. Let α be a positive number so small that the spherical nieghbourhood K_α of p (see §6, B) satisfies the conditions

$$\overline{A}_t \subset B, \text{ where } A_t = f_t^{-1}(K_\alpha),\ t + 0, 1, \tag{2}$$

and let ω_α be the map of S^n on to itself corresponding to the selected number α (see §6, B). Since ω_α is homotopic to the identity, the maps $\omega_\alpha f_t$ and f_t are homotopic, $t = 0, 1$. We will regard B as a ball of unit radius in some Euclidean space R^n. Then there exists a positive $\beta < 1$ such that the ball B_β of radius β concentric with B contains the sets $\overline{A}_t$, $t = 0, 1$. Let λ_β be the map of R^n on to S^n described in proposition (B) of §6. We define a map θ of M^n on to S^n to coincide with λ_β on the ball B and to map $M_n - B$ to q. Since the set $\overline{A}_t$, $t = 0, 1$, is contained in B_β, the map θ is a homeomorphism on $\overline{A}_t$. We now define maps g_t, $t = 0, 1$, of S^n on to itself in the following way. On the set $\theta(A_t)$ we define g_t by putting $g_t = \omega_a f_t \theta^{-1}$ and for $x \in S^n - \theta(A_t)$ we put $g_t(x) = q$. From this definition of g_t it follows that

$$g_t\theta = \omega_\alpha f_t, \quad t = 0, 1. \tag{3}$$

The maps f_t and $\omega_a f_t$ evidently have the same degree at p and from relation (3) it follows that the maps g_t and f_t also have the same degree at p. Since the maps f_0 and f_1 have the same degree by hypothesis, it follows that the maps g_0 and g_1 of S^n to itself have the same degree. Thus the maps g_0 and g_1 are homotopic (see Theorem 12). From this it follows that the maps $g_0\theta$ and $g_1\theta$ of M^n to S^n are homotopic and consequently (see (3)) the maps $\omega_\alpha f_0$ and $\omega_\alpha f_1$ are homotopic. Since the maps $\omega_\alpha f_0$ and $\omega_\alpha f_1$ are homotopic to f_0 and f_1 respectively, these last maps are also homotopic.

The construction of a map $M^n \longrightarrow S^n$ with a given degree proceeds without difficulty.

Thus Theorem 13 is proved.

§10. The Hopf invariant of maps of Σ^{2k+1} into S^{k+1}

In the homotopy classification of maps of spheres into spheres the Hopf invariant, which was first introduced to prove that there exist an infinity of classes of maps of the 3-sphere to the 2-sphere [12], plays an important role. The invariant was later defined by Hopf for maps of the $(2k+1)$-dimensional sphere to the $(k+1)$-dimensional sphere. However for even values of k the invariant is always zero. The Hopf invariant is defined as the linking coefficient of the counterimages in Σ^{2k+1} of two distinct points of S^{k+1}. In this section we first of all give the definition of the linking coefficient of two submanifolds of Euclidean space in the form first proposed by Brouwer [13], that is, by means of the degree of a map and not by means of the intersection index (as it is now defined). This form corresponds better to the character of the entire work. Then the Hopf invariant is defined and, finally, it is characterised in terms of the framed manifold associated with the map. Moreover we establish a series of connections between properties of framed manifolds and properties of the Hopf invariant. These connections play a decisive role in the classification of maps of the $(n+2)$-dimensional sphere into the sphere of dimension n.

The linking coefficient.

Definition 8. Let M^k and N^l be two closed smooth oriented manifolds of dimension k and l, and let f and g be maps of them into the oriented Euclidean space E^{k+l+1} of dimension $(k+l+1)$ such that the set $f(M^k)$ and $g(N^l)$ do not intersect. Further let S^{k+l} be a unit sphere in E^{k+l+1}, with centre at some arbitrary point O, oriented as the boundary of a ball, and let $M^k \times N^l$ be the oriented direct product (see §1, K) of M^k and N^l. To each point $(x, y) \in M^k \times N^l$, $x \in M^k$, $y \in N^l$, there corresponds the non-zero segment $(f(x), g(y))$ in E^{k+l+1}, passing from the point $f(x)$ to the point $g(y)$. We draw a ray from O, parallel to the segment $(f(x), g(y))$, and we denote by $\chi(x, y)$ the intersection of this ray with S^{k+l}. The degree of the map χ of the oriented manifold $M^k \times N^l$ into the oriented sphere S^{k+l} (see definition 7) is called the *linking coefficient*[1] of the manifold-maps (f, M^k) and (g, N^l) and denoted by $\mathfrak{v}((f, M^k), (g, N^l))$. It is evident that if the maps f and g vary continuously: $f = f_t$, $g = g_t$, in such a way that the sets $f_t(M^k)$ and $g_t(N^l)$ intersect for no value of t, then the map $\chi = \chi_t$ also varies continuously and so the linking coefficient does not vary. In the special case when M^k and N^l are submanifolds of the space E^{k+l+1} and the maps f and g are identity maps, the linking coefficient is defined and is denoted in this case by $\mathfrak{v}(M^k, N^l)$. It turns out that

$$\mathfrak{v}((g, N^l), (f, M^k)) = (-1)^{(k+1)(l+1)}\mathfrak{v}((f, M^k), (g, N^l)). \tag{1}$$

We prove formula (1). Let χ' be the map of $N^l \times M^k$ into S^{k+l} analogous to the map χ constructed above. We denote by λ the map of $N^l \times M^k$ on to $M^k \times N^l$ which transforms the point (y, x) into the point (x, y) and let μ be the map of S^{k+l} on to itself, mapping each point to the point diametrically opposite. It is evident that the degree of λ is $(-1)^{kl}$ and the degree of μ is $(-1)^{k+l+1}$. It is easy to see, moreover, that $\chi' = \mu\chi\lambda$. From this follows the validity of formula (1).

A) We suppose that, instead of one map (g, N^l) we are given two maps (g_0, N_0^l) and (g_1, N_1^l). We suppose further that there exists an oriented compact manifold N^{l+1} with boundary, such that the oriented boundary consists of the manifolds N_0^l and $-N_1^l$ and that there exists a map g of N^{l+1} into E^{k+l+1}, coinciding with g_0 on N_0^l and with g_1 on N_1^l, such that the sets $g(N^{l+1})$ and $f(M^k)$ do not intersect. It then turns out that

$$\mathfrak{v}((f, M^k), (g_0, N_0^l)) = \mathfrak{v}((f, M^k), (g_1, N_1^l)). \tag{2}$$

We prove this. The manifold $M^k \times N_0^l - M^k \times N_1^l$ serves as the boundary of $M^k \times N^{l+1}$. To each point $(x, y) \in M^k \times N^{l+1}$ there corresponds the segment $(f(x), g(y))$. We draw a ray from the point O, parallel to the segment $(f(x), g(y))$ and we denote by $\chi(x, y)$ its intersection with S^{k+l}. Thus we obtain a map χ of $M^k \times N^{l+1}$ into S^{k+l} and so the degree of the restriction of χ to the boundary of $M^k \times N^{l+1}$ is zero (see §9, A). From this formula (2) follows directly.

The Hopf invariant.

Definition 9. Let f be a smooth map of the oriented sphere Σ^{2k+1} into the oriented sphere S^{k+1}, $k \geq 1$. Further let p' and q' be the north and south poles of Σ^{2k+1}, let E^{2k+1} be the tangent space to Σ^{2k+1} at p', and let ϕ be the central projection from q' of the set $\Sigma^{2k+1} - q'$ on to E^{2k+1}. In S^{k+1} we choose distinct points a_0 and a_1, each different from $f(q')$ and proper points of the map f; then $M_0^k = \phi f^{-1}(a_0)$ and $M_1^k = \phi f^{-1}(a_1)$ are closed oriented submanifolds of the Euclidean space E^{2k+1} (see the introduction to §4, the orientation of the counterimage of a point). We put

$$\gamma(f) = \gamma(f, p', a_0, a_1) = \mathfrak{v}(M_0^k, M_1^k). \tag{3}$$

It turns out that $\gamma(f)$ is a homotopy invariant of the map f, and does not depend on the particular choice of points p', a_0 and a_1; and that for even values of k the invariant is always zero.

1) Translator's note: Or *looping coefficient*.

We prove the invariance of the number (f).

Let f_0 and f_1 be two smooth homotopic maps of Σ^{2k+1} into S^{k+1} and let f_t be a smooth deformation connecting them. To the deformation f_t corresponds a map f_* of the product $\Sigma^{2k+1} \times I$ into S^{k+1} (see §5, C). We note that for sufficiently small displacements of the points a_0 and a_1 the number $\gamma(f_t)$, $t = 0, 1$, does not change since the manifolds $\phi f_t^{-1}(a_i)$ suffer only small deformations. So we may suppose that the curve $f_t(q')$, $0 \le t \le 1$ does not pass through a_0 or a_1. Let r be a sufficiently large natural number so that, if $|t' - t| < \frac{1}{r}$, the sets $f_t^{-1}(a_0)$ and $f_{t'}^{-1}(a_1)$ do not intersect. We now displace the points a_0 and a_1 so that they are proper points of the map f_* and of the maps

$$f_t;\ t = 0, \frac{1}{r}, \cdots, \frac{r-1}{r}, 1.$$

We prove that

$$\gamma(f_1) = \gamma(f_0).$$

We denote by I_s the part of the interval I consisting of points t satisfying $\frac{s}{r} \le t \le \frac{s+1}{r}$, and let $M_{s,i}^{k+1}$ be the intersection of $f_*^{-1}(a_i)$ with $\Sigma^{2k+1} \times I_s$. In view of the conditions imposed on the points a_0 and a_1, the set $M_{s,i}^{k+1}$ is an oriented submanifold of $\Sigma^{2k+1} \times I_s$ with boundary $-f_{\frac{s}{r}}^{-1}(a_i) + f_{\frac{(s+1)}{r}}^{-1}(a_i)$. We denote by π the projection of $\Sigma^{2k+1} \times I$ along the axis I on to Σ^{2k+1}. The map $\phi\pi$ of $M_{s,i}^{k+1}$ determines a 'singular' manifold $(\phi\pi, M_{s,i}^{k+1})$ with boundary $-\phi f_{\frac{s}{r}}^{-1}(a_i) = \phi f_{\frac{(s+1)}{r}}^{-1}(a_i)$. Since the sets $\phi\pi(M_{s,0}^{k+1})$ and $\phi\pi(M_{s,1}^{k+1})$ do not intersect it follows from (A) that

$$\gamma(f_{\frac{(s+1)}{r}}) = \gamma(f_{\frac{s}{\pi}}),$$

whence $\gamma(f_1) = \gamma(f_0)$.

We prove now that $\gamma(f, p', a_0, a_1)$ does not depend on the choice of points a_0 and a_1. Let the points b_0 and b_1 be chosen in place of a_0 and a_1. There evidently exists a smooth homeomorphism λ of the sphere S^{k+1} on to itself, homotopic to the identity, such that $\lambda(a_i) = b_i$, $i = 0, 1$. It is clear that $\gamma(\lambda f, p', b_0, b_1) = \gamma(f, p', a_0, a_1)$ and since λf and f are homotopic, then in view of what was proved earlier we obtain $\gamma(f, p', b_0, b_1) = \gamma(f, p', a_0, a_1)$.

Similarly one proves that $\gamma(f, p', a_0, a_1)$ does not depend on the choice of p' since there is a rotation of Σ^{2k+1} transforming p' into any other point of Σ^{2k+1}.

Finally we show that $\gamma(f)$ is zero if k is even. Since $\gamma(f)$ does not depend on the choice of a_0 and a_1 we may interchange their roles so we have

$$\mathfrak{v}(M_0^k, M_1^k) = \mathfrak{v}(M_1^k, M_0^k).$$

Since however, in view of (1),

$$\mathfrak{v}(M_1^k, M_0^k) = (-1)^{(k+1)^2}\mathfrak{v}(M_0^k, M_1^k),$$

it follows that $\mathfrak{v}(M_0^k, M_1^k) = 0$ if k is even.

The Hopf invariant of a framed manifold. In as much as the homotopy classes of maps of Σ^{2k+1} into S^{k+1} stand in (1,1) correspondence with the homology classes of framed k-dimensional manifolds in $(2k+1)$-dimensional Euclidean space, the invariant $\gamma(f)$ can be expressed as an invariant of homology class of framed k-dimensional manifolds in $(2k+1)$-dimensional space. We give this expression for $\gamma(f)$.

B) Let (M^k, U), $U(x) = \{u_1(x), \cdots, u_{k+1}(x)\}$, be a framed submanifold of the oriented Euclidean space E^{2k+1} and let N_x be the normal to M^k at the point $x \in M^k$. We will regard N_x as a vector space with origin at x so that $U(x)$ is a basis of the space N_x. We choose an arbitrary vector $c = \{c^1, \cdots, c^{k+1}\}$ of some coordinate Euclidean space N and we associate with each point x of M^k the point $c(x) = c^1 u_1(x) + \cdots + c^{k+1} u_{k+1}(x)$ of N_x. If the vector c is sufficiently small the map c is a homeomorphism of M^k into E^{2k+1} (see §5, A). It is clear that if $c \neq 0$ the manifolds M^k and $c(M^k)$ do not intersect and that for two different non-zero vectors c and c' the manifolds $c(M^k)$ and $c'(M^k$ are homotopic in the space $E^{2k+1} - M^k$. Thus for sufficiently small non-zero c the linking coefficient $\mathfrak{v}(M^k, c(M^k))$ does not depend on c, and we set

$$\gamma(M^k, U) = \mathfrak{v}(M^k, c(M^k)).$$

It turns out that if $f \to (M^k, U)$ (see definition 4), then

$$\gamma(f) = \gamma(M^k, U). \tag{4}$$

Since $\gamma(f)$ is a homotopy invariant of the map f, $\gamma(M^k, U)$ is a homology invariant of the framed manifold (M^k, U).

We prove formula (4). Let f be a smooth map of Σ^{2k+1} into S^{k+1} and let $p \in S^{k+1}$ be a proper point of the map f distinct from $f(q')$. Then to construct the framed manifold (M^k, U) corresponding to the map f we may take the point p as north pole of S^{k+1} (see definition 4). Let $e_1, \cdots, e_{k+1}$ be an orthonormal basis for the tangent space at p to S^{k+1} and $x^1, \cdots, x^{k+1}$ the corresponding coordinates in the region $S^{k+1} - q$ (see §6, A). To obtain the invariant $\gamma(f)$ we take for the point a_0 the pole p and for the point a_1 the points with coordinates $c^1, \cdots, c^{k+1}$. With this choice of a_0 and a_1 the manifold M_0^k evidently coincides with M^k and the manifold M_1^k approximates to $c(M^k)$ to within the second order of magnitude of the vector c. We see from this that $\mathfrak{v}(M^k, c(M^k)) = \mathfrak{v}(M_0^k, M_1^k)$ and relation (4) is proved.

C) Let Π_{k+1}^k be the homology group of framed k-dimensional manifolds in oriented Euclidean space E^{2k+1}. With each element $\pi \in \Pi_{k+1}^k$ we associate the integer $\gamma(\pi) = \gamma(M^k, U)$, where (M^k, U) is a framed manifold in the class π. In view of what was proved earlier (see (B)), $\gamma(\pi)$ depends only on π and not on the particular choice of framed manifold (M^k, U). It turns out that γ is a homeomorphism of

Π^k_{k+1} into the additive group of integers. From this it follows that the set of all elements $\pi \in \Pi^k_{k+1}$ for which $\gamma(\pi) = 0$ is a subgroup of Π^k_{k+1}.

We prove proposition (C). Let π_1 and π_2 be two elements of Π^k_{k+1} and let (M^k_1, U_1), (M^k_2, U_2) be representatives of π_1, π_2 respectively lying on opposite sides of some hyperplane E^{2k} of the space E^{2k+1}. Further let S^{2k} be a unit sphere in E^{2k+1} with centre O belonging to E^{2k}. We choose an arbitrarily small vector c, defining a displacement of the manifold $M^k_1 \cup M^k_2$ (see (B)). We have:

$$\gamma(\pi_1 + \pi_2) = \mathfrak{v}(M^k_1 \cup M^k_2,\ c(M^k_1 \cup M^k_2)).$$

The linking coefficient on the right-hand side is defined as the degree of the map χ of the manifold $(M^k_1 \cup M^k_2) \times c(M^k_1 \cup M^k_2)$ on to the sphere S^{2k}, where the map χ is constructed by the method described in definition 8. We will determine the degree of χ at a point p situated very near to the hyperplane E^{2k}. Because of this choice of p, the segment $(x, c(y))$, where $x \in M^k_1$, $y \in c(M^k_2)$ cannot be parallel to (O, p). In the same way, the segment $(x, c(y))$, where $x \in M^k_2$, $y \in c(M^k_1)$ cannot be parallel to (O, p). From this it follows that

$$\mathfrak{v}(M^k_1 \cup M^k_2,\ c(M^k_1 \cup M^k_2)) = \mathfrak{v}(M^k_1,\ c(M^k_1)) + \mathfrak{v}(M^k_2,\ c(M^k)).$$

That is, $\gamma(\pi_1 + \pi_2) = \lambda(\pi_1) + \gamma(\pi_2)$.

Thus proposition (C) is proved.

D) Let f be a smooth map of the oriented sphere Σ^{2k+1} into the oriented sphere S^{k+1}; let g be a map of Σ^{2k+1} to itself of degree σ, and let h be a map of S^{k+1} to itself of degree τ. We set $f' = hfg$. It turns out that

$$\gamma(f') = \sigma\tau^2\gamma(f). \tag{5}$$

It is sufficient to prove proposition (D) separately for the case when h is the identity and the case when g is the identity. The validity of the relation (5) when h is the identity follows from proposition (C) of this section and proposition (D) of §9. We consider the case when g is the identity, that is, when $f' = hf$. Let a_0 and a_1 be distinct points of S^{k+1}, different from $f'(q')$ and proper points of h and hf. The $h^{-1}(a_t) = \{a_{t1}, \cdots, a_{tr_t}\}$; $t = 0, 1$, where the map f is proper at each point a_{ti}, $t = 0, 1$; $i = 1, 2, \cdots, r_t$. We denote by ϵ_{ti} the sign of the functional determinant of the map h at a_{ti}. We denote by E^{2k+1} the tangent space to Σ^{2k+1} at the north pole p' and by ϕ the central projection from q' of the set $\Sigma^{2k+1} - q'$ on to E^{2k+1}. We put $\phi f'^{-1}(a_t) = M^k_t$, $t = 0, 1$; $\phi f^{-1}(a_{ti}) = M^k_{ti}$. It is easy to see that

$$M^k_t = \epsilon_{t1}M^k_{t1} \cup \epsilon_{t2}M^k_{t2} \cup \cdots \cup \epsilon_{tr_t}M^k_{tr_t}, \tag{6}$$

where the signs ϵ_{ti} take account of the orientations of the manifolds. Since a_{0i} and a_{1j} are two distinct points of S^{k+1} and proper points of f, the invariant $\gamma(f)$ can be defined as $\mathfrak{v}(M^k_{0i}, M^k_{1j})$. From this and from relation (6) it follows that

$$\gamma(f) = \mathfrak{v}(\epsilon_{01}M^k_{01} \cup \cdots \cup \epsilon_{0r_0}M^k_{0r_0},\ \epsilon_{11}M^k_{11} \cup \cdots \cup \epsilon_{1r_1}M^k_{1r_1}) =$$

$$= \sum_{i=1}^{r_0}\sum_{j=1}^{r_1} \epsilon_{0i}\epsilon_{1j}\gamma(f) = \gamma(f)\left(\sum_{i=1}^{r_0}\epsilon_{0i}\right)\left(\sum_{j=1}^{r_1}\epsilon_{1j}\right) = \tau^2\gamma(f).$$

Thus proposition (D) is proved.

E) Let (M^k, V), $V(x) = \{v_1(x), \cdots, v_{k+1}(x)\}$ be an orthonormally and smoothly framed submanifold of the oriented Euclidean space E^{2k+1} such that M^k is itself situated in the hyperplane E^{2k} of E^{2k+1}. We denote by $u(x)$ the unit vector, issuing from the point $x \in M^k$ and perpendicular to the hyperplane E^{2k}. Then we have

$$u(x) = \psi^1(x)v_1(x) + \cdots + \psi^{k+1}(x)v_{k+1}(x). \tag{7}$$

Here $\psi(x) = \{\psi^1(x), \cdots, \psi^{k+1}(x)\}$ is a unit vector in a coordinate Euclidean space N, so that ψ is a map of the manifold M^k on to the unit sphere $\mathfrak{S}^k$ in N (the map ψ was considered in (A) of §8). It turns out that the degree of ψ is equal to $\epsilon\gamma(M^k, V)$, where $\epsilon = \pm 1$ and depends only on k.

We prove proposition (E). We will suppose that the point $\mathfrak{P} = (0, \cdots, 0, 1) \in \mathfrak{S}^k$ is a proper point of the map ψ. If this were not so, then it would be possible to achieve it by a single orthogonal transformation applied to all the systems $V(x)$, $x \in M^k$. To compute $\gamma(M^k, V)$ we choose in E^{2k+1} a unit sphere S^{2k} with centre at some point O and take as vector c the vector $\{0, \cdots, 0, \delta\}$. If the vector $u(x)$ undergoes parallel displacement to the point O its end-point lands up at a point of S^{2k} which we designate as u. We draw a ray from O parallel to the segment $(x, c(y))$; $x, y \in M^k$, and we denote by $\chi(x, y)$ the intersection of this ray with S^{2k}. By definition $\gamma(M^k, V)$ is the degree of the map χ of the manifold $M^k \times M^k$ into the sphere S^{2k}. We will calculate the degree of this map at the point u. In the course of the calculation it will be shown that u is a proper point of the map χ. Let $\chi(a, b) = u$, then the segment $(a, c(b))$ is orthogonal to the hyperplane E^{2k} and goes in the direction of the vector u, so that $c(b) \in \overline{H_\delta(a)}$ (see §5, A). Since moreover $c(b) \in \overline{H_\delta(b)}$, it therefore follows that for sufficiently small δ we have $b = a$ (see §5, A). Thus for $\chi(a, b) = u$ we have $b = a$ and $\psi(a) = \mathfrak{P}$. Conversely if $\psi(a) = \mathfrak{P}$, then $\chi(a, a) = u$. We take the point a as origin of coordinates O in the space E^{2k+1} and as basis we take the vectors $u_1 = v_1(a), \cdots, u_{k+1} = v_{k+1}(a)$, $u_{k+2}, \cdots, u_{2k+1}$, where $u_{k+2}, \cdots, u_{2k+1}$ is an orthonormal system of vectors tangent to the manifold M^k at the point a. The coordinates of the point $x \in M^k$ in this basis we will denote by $z^1(x), \cdots, z^{2k+1}(x)$. In the neighbourhood of the point a in M^k it is easy to introduce coordinates $x^1, \cdots, x^k$ for the point x so that the equations of the manifold M^k takes the form

$$z^1 = z^1(x), \cdots, z^{k+1} = z^{k+1}(x),\ z^{k+2} = z^{k+2}(x) = x^1, \cdots,$$

$$z^{2k+1} = z^{2k+1}(x) = x^k, \tag{8}$$

where $z^i(x)$, $i = 1, \cdots, k+1$ is small of the second order with respect to $\rho(a, x)$. Transforming the system $V(y)$ by parallel displacement to the point $O = a$, we express its vectors in terms of the basis $u_1, \cdots, u_{2k+1}$ by

$$v_j(y) = \sum_{a=1}^{k+1} a_{ja}(y) u_a + \sum_{\beta=k+2}^{2k+1} b_{j\beta}(y) u_\beta. \tag{9}$$

Here $b_{j\beta}$ is small of the second order relative to $\rho(a, y)$, and a_{ja}, $\alpha \neq j$, is small of the first order relative to $\rho(a, y)$. From this, in view of the orthonormality of the system $V(y)$, it follows directly that, with an accuracy up to the second order relative to $\rho(a, y)$, the equations

$$a_{jj}(y) = 1; \; a_{ji}(y), \; i \neq j, \tag{10}$$

hold. Since $a_{ij}(y) = (u_i, v_j(y))$; $i, j = 1, \cdots, k+1$, then in view of relations (7), (9) and (10) we have, with second order accuracy relative to $\rho(a, y)$: $\psi^j(y) = -a_{k+1,j}(y)$, $j = 1, \cdots, k$; $\psi^{k+1}(y) = 1$. Thus, with second order accuracy relative to $\rho(a, y)$ the point $c(y)$ has, in the basis $u_1, \cdots, u_{2k+1}$, the coordinates $-\delta\psi^1(y), \cdots, -\delta\psi^k(y), \delta, y^1, \cdots, y^k$. In the same way and with second order accuracy the point x has coordinates (see (8))

$$0, \cdots, 0, x^1, \cdots, x^k.$$

Thus the components of the segment $(x, c(y))$ in the basis $u_1, \cdots, u_{2k+1}$ are

$$-\delta\psi^1(y), \cdots, -\delta\psi^k(y), \delta, y^1 - x^1, \cdots, y^k - x^k$$

with second order accuracy relative to $\rho(a, x) + \rho(a, y)$. From this it follows that at the point (a, a) the sign of the functional determinant of the map χ differs from the sign of the functional determinant of the map ψ at the point a by a factor $\epsilon = \pm 1$ depending only on the dimension k. Thus proposition (E) is proved.

§11. Framed manifolds with zero Hopf invariant

The main aim of this section is the proof of Theorem 16 which asserts that every framed manifold with Hopf invariant zero is homologous to a suspension. This theorem represents a development of Theorem 11. Since the Hopf invariant of an even-dimensional manifold is always zero, it follows from Theorem 16 that every even-dimensional framed submanifold (M^k, U) of Euclidean space E^{2k+1} is homologous to a suspension. This proposition will be applied in the present work only to the case $k = 2$, for the classification of maps of Σ^{n+2} into S^n. From it and Theorem 11 the conclusion follows that the number of homotopy classes of maps of Σ^{n+2} into S^n, $n \geq 2$, cannot exceed the number of classes of maps of Σ^4 into S^2.

To prove Theorem 16 and also in some other cases it is desirable to deal with connected framed manifolds. Theorem 14 ensures that every framed manifold is homologous to a connected one. For the proof of this theorem it is necessary to perform a 'reconstruction' of the manifold to convert it into a connected one. This

reconstruction is somewhat cumbrously described in proposition (A) below, but the geometrical idea is simple and consists in the following. The equation $x^2+y^2-z^2 = -t$ represents a two-sheeted hyperboloid if $t > 0$ and a one-sheeted hyperboloid if $t < 0$. In the strip of the space of the variables x, y, z, t given by the inequality $-1 \leq t \leq 1$, the equation determines a submanifold; the part of its boundary lying in the hyperplane $t = -1$ is not connected, but the part of its boundary lying in the hyperplane $t = 1$ is connected. In proposition (A) the reconstruction described here is attached in a smooth way to a pair of parallel planes. In these planes 'dents' are formed which, converging towards each other in a similar way to the sheets of the two-sheeted hyperboloid, form tubes connecting spherical holes in the planes. To apply the reconstruction described to arbitrary manifolds the almost obvious proposition (C) is proved, asserting that a manifold may be flattened by deformation near any of its points. By flattening the manifold at two points of its points, belonging to different components, we make it possible to apply the reconstruction (A), uniting the two components of the manifold into one. Since it is necessary to reconstruct framed manifolds we need also to concern ourselves with the reconstruction of frames. Propositions (B) and (D) are devoted to such constructions. The reconstruction (A) is applied not only to obtain connected manifolds but also so that it should be possible to embed a k-dimensional manifold in $2k$-dimensional Euclidean space.

The reconstruction of a manifold.

A) Let E^{k+2} be Euclidean space with coordinates $\xi^1, \cdots, \xi^k, \eta, \tau$, and let E_*^{k+2} be the inequalities $-1 \leq \tau \leq 1$, whose boundary consists of the hyperplanes E_{-1}^{k+1} and E_{+1}^{k+1} with equations $\tau = -1$ and $\tau = +1$. Finally let H^{k+2} be the part of the strip E_*^{k+2} defined by the inequalities

$$(\xi^1)^2 + \cdots + (\xi^k)^2 \leq 1; \; -1 \leq \eta \leq 1. \tag{1}$$

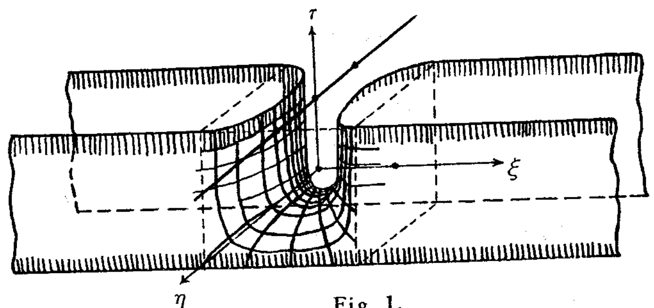

Fig. 1.

It turns out that there exists a smooth submanifold P^{k+1} of E_*^{k+2}, orthogonal at its boundary points to the boundary of the strip E_*^{k+2} and having the following properties (see Fig. 1):

a) Outside H^{k+2} the manifold P^{k+1} consists of all points satisfying the condition $|\eta| = 1$.

b) The manifold $P^k_{-1} = P^{k+1} \cap E^{k+1}_{-1}$ consists of all points of the hyperplane E^{k+1}_{-1}, satisfying the condition $|\eta| = 1$.

c) The intersection of $P^k_1 = P^{k+1} \cap E^{k+1}_1$ with the hyperplane given by the equation $\eta = \alpha$, for $|\alpha| < 1$, is a sphere of radius $\rho(\alpha) < 1$, given in the plane $\eta = \alpha$, $\tau = 1$ by the equation $(\xi^1)^2 + \cdots + (\xi^k)^2 = \rho^2(\alpha)$, such that $\rho(\alpha)$ tends to 1 as $|\alpha|$ tends to 1. Thus the set $P^k_1 \cap H^{k+2}$ does not intersect the line $\xi^1 = 0, \cdots, \xi^k = 0$, $\tau = 1$; moreover this set is connected if $k > 1$ and consists of two simple arcs if $k = 1$.

To construct the manifold P^{k+1} we consider first, for conceptual simplicity, the case $k = 1$. The coordinates ξ^1, η, τ we now designate as x, y, t. Let

$$\phi(x, y, t) = y^2 - (1+t)x^2 + t.$$

We consider the surface Q^2 with equation $\phi(x, y, t) = 0$. It may be verified directly that this surface has no singular points, that is, that the equations

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial t} = 0, \quad \phi = 0$$

are inconsistent. We consider the section C_β of the surface Q^2 by the plane $t = \beta$ ($|\beta| \leq 1$). The curve C_{-1} splits into a pair of parallel lines $y = \pm 1$. For $-1 < \beta < 0$ the curve C_β is a hyperbola whose real axis is the line $t = \beta$, $x = 0$. The curve C_0 splits into a pair of intersecting lines $y = \pm x$. Finally for $0 < \beta \leq 1$, the curve C_β is a hyperbola whose real axis is the line $t = \beta$, $y = 0$. For all values of β the curve C_β passes through the points $(\pm 1, \pm 1, \beta)$ and is symmetric with respect to the coordinate planes $x = 0$ and $y = 0$. In our case the set H^3 is a cube, given by the inequalities $|x| \leq 1$, $|y| \leq 1$, $|t| \leq 1$. The part Q^2_* of Q^2 lying in H^3 is supplemented by the points satisfying the conditions $|y| = 1$, $|x| \geq 1$, $|t| \leq 1$. The surface we obtain we denote by $\hat{P}^2$. Then $\hat{P}^2$ satisfies conditions (a) – (c), but it is not smooth and is not orthogonal, on its boundary, to the boundary to the boundary of the strip E^3_* ($|t| \leq 1$).

We now turn to the case of arbitrary k. We introduce a function

$$\phi(x^1, \cdots, x^k, y, t),$$

setting

$$\phi(x^1, \cdots, x^k, y, t) = y^2 - (1+t)((x^1)^2 + \cdots + (x^k)^2) + t.$$

It may be verified directly that the hypersurface given by

$$\phi(x^1, \cdots, x^k, y, t) = 0$$

has no singular points, that is, that the equations

$$\frac{\partial\phi}{\partial x^1} = 0, \cdots, \frac{\partial\phi}{\partial x^k} = 0, \quad \frac{\partial\phi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial t} = 0, \quad \phi = 0$$

are inconsistent. The hypersurface Q^{k+1} may be visualized with the aid of the remark that sections of it by arbitrary three-dimensional spaces containing the coordinate plane (y, t) are surfaces Q^2 of the sort described above. We put $Q^{k+1}_* = Q^{k+1} \cap H^{k+2}$. We supplement the set Q^{k+1}_* by the points satisfying the conditions $|y| = 1$, $(x^1)^2 + \cdots + (x^k)^2 > 1$, $|t| \leq 1$. Then the set $\hat{P}^{k+1}$ we obtain is a manifold satisfying the conditions (a) – (c), but not smooth at points of intersection with the boundary of H^{k+2}. Moreover at boundary points it is not orthogonal to the boundary of the strip E^{k+2}_*. We set about 'rectifying' the manifold $\hat{P}^{k+1}$.

Let $\chi(s)$ be a smooth function of class $m \geq 1$, which is odd, monotone increasing in its variable s, defined in the interval $-1 \leq s \leq 1$, and having the following properties:

$$\chi(-1) = -1, \quad \chi(1) = 1,$$

$$\chi'(-1) = \chi''(-1) = \cdots = \chi^{(m)}(-1) = \chi'(1) = \chi''(1) = \cdots = \chi^{(m)}(1) = 0,$$

$$\chi'(s) > 0 \text{ for } |s| < 1.$$

Plainly such a function exists. We now define a map σ of H^{k+2} onto itself by the rule

$$\sigma(x^1, \cdots, x^k, y, t) = (\xi^1, \cdots, \xi^k, \eta, t)$$

where $\xi^1 = x^1, \cdots, \xi^k = x^k$, $\eta = \chi(y)$, $\tau = \chi^{-1}(t)$, χ^{-1} being the function inverse to χ. It is evident that σ is a homeomorphism of H^{k+2} on to itself, and that σ and σ^{-1} are smooth at all interior points of H^{k+2}. It is plain that σ fails to be smooth only for $t = \pm 1$, and σ^{-1} only for $\eta = \pm 1$. It is easy to verify that, replacing the part Q^{k+1}_* of the manifold $\hat{P}^{k+1}$ by the set $\sigma(Q^{k+1}_*)$, we obtain a manifold P^{k+1} satisfying all the conditions of proposition (A).

B) Let W^{k+2} be the ϵ-neighbourhood of the set H^{k+2} in the Euclidean space E^{k+2} (see (A)), and let θ be a smooth homeomorphism of W^{k+2} into Euclidean space E^{n+k+1}. Then there exists a frame $V(\zeta) = \{v_1(\zeta), \cdots, v_n(\zeta)\}$ of the manifold $\theta(P^{k+1} \cap W^{k+2})$ in E^{n+k+1}, including a given orientation in the manifold.

We prove proposition (B). Let O be the centre of H^{k+2} and let X be the boundary of the convex set W^{k+2}. Further let ζ be an arbitrary point of W^{k+2} and (O, x) the segment passing through ζ and joining O to a boundary point $x \in X$. We denote by t the ratio of the segments (O, ζ) and (O, x) and we put $\zeta = (x, t)$. In this way we introduce polar coordinates in the region W^{k+2}, with $(x, 0) = O$. We denote by N_{xt} the normal at $\theta(\zeta)$ to the manifold $\theta(W^{k+2})$ in E^{n+k+1}. We choose an arbitrary basis $v_1, \cdots, v_{n-1}$ in N_{x0}. In view of proposition (A) of §7 a basis $v_1(x, t), \cdots, v_{n-1}(x, t)$ can be chosen in N_{xt} so that it depends continuously on the pair x, t and coincides for $t = 0$ with the basis $v_1, \cdots, v_{n-1}$. We put $v_i(\zeta) = v_i(x, t)$, $i = 1, \cdots, n-1$. We choose the vector $v_n(\zeta)$ at $\theta(\zeta)$, where $\zeta \in P^{k+1} \cap W^{k+2}$, to be a unit vector normal to the manifold $\theta(P^{k+1})$ at $\theta(\zeta)$ and tangent to the manifold $\theta(W^{k+2})$. The vector $v_n(\zeta)$ satisfying these conditions is uniquely determined up to sign. Since $P^{k+1} \cap W^{k+2}$ is connected, the entire field $v_n(\zeta)$ is uniquely determined up to sign, and, by choosing the direction of $v_n(\zeta)$ properly, we can arrange that the constructed frame $V(\zeta)$, $\zeta \in P^{k+1} \cap W^{k+2}$, induces on $\theta(P^{k+1} \cap W^{k+2})$ a given orientation.

C) Let M^k be a smooth closed submanifold of Euclidean space E^{n+k}, let $a \in M^k$, let T^k be the tangent space to M^k at a and let δ be a positive number. It turns out that there exists a smooth deformation τ_t, $0 \leq t \leq 1$, of the manifold M^k fulfilling the following conditions. Let $x \in M^k$; then: a) for $\rho(a, x) \geq \delta$ we have $\tau_t(x) = x$; b) for $\rho(a, x) \leq \delta$, $\rho(x, \tau_t(x))$ is of the second order of smallness relative to $\rho(x, a)$, that is, $\rho(x, \tau_t(x)) < c\rho^2(x, a)$, where c is a constant independent of δ; c) for $\rho(a, x) < \frac{\delta}{2}$, we have $\tau_1(x) \in T^k$.

We prove proposition (C). We will suppose δ so small that the δ-neighbourhood of a in M^k is mapped smoothly, regularly and homeomorphically into T^k by the orthogonal projection π. Further let $\mu(s)$ be a smooth even function of the parameter s, $-\infty < s < \infty$, taking the value 0 for $0 < s < \frac{\delta}{2}$, monotone increasing for $\frac{\delta}{2} \leq s \leq \delta$, and taking the value 1 for $s > \delta$. Then the required deformation τ_t is defined by the formula:

$$\tau_t(x) = x\lambda t + \pi(x)\cdot(1-\lambda)t + x(1-t),$$

where $\lambda = \mu(\rho(a, x))$.

D) Let (M^k, U) be a framed submanifold of the strip E_*^{n+k} in the Euclidean space E^{n+k} and let K' be a neighbourhood of an interior point $a \in M^k$ such that its closure $\overline{K}'$ is homeomorphic to a k-dimensional ball. We will take $\overline{K}'$ to be a ball of smaller radius concentric with K'. If there is given a frame V for the part $\overline{K}'$ of M^k inducing the same orientation as that induced by U, then there exists a frame U' for M^k, homotopic to U, and coinciding with U on $M^k - K'$ and with V on K.

We prove proposition (D). Let

$$U(x) = \{u_1(x), \cdots, u_n(x)\}, \quad V(x) = \{v_1(x), \cdots, v_n(x)\};$$

then we have

$$u_i(x) = \sum_{j=1}^{n} \lambda_{ij}(x) v_j(x), \qquad x \in \overline{K}',$$

where $\lambda(x) = \|\lambda_{ij}(x)\|$ is a matrix with positive determinant, depending continuously on $x \in \overline{K}'$, so that λ is a continuous map of $\overline{K}'$ into the manifold L_n of all matrices of order n with positive determinant. We regard $\overline{K}'$ as a ball in Euclidean space E^k, taken as a hyperplane of E^{k+1}, and let L be a rectilinear segment in E^{k+1} perpendicular to E^k and having one of its end-points at the centre a of $\overline{K}'$. We denote the other end of the segment L by b. One may easily construct a deformation ψ_t of maps of $\overline{K}'$ into $\overline{K}' \cup L$ under which all points on the boundary of $\overline{K}'$ remain fixed and such that $\psi_1(K) = b$. Since the manifold L_n is connected

the map λ can be extended to a map λ of $\overline{K}' \cup L$ into L_n which transforms b to the unit matrix. Then $\|\mu_{ij}(x)\| = \mu(x) = \lambda\psi_1(x)$ is a matrix with positive determinant depending continuously on $x \in \overline{K}'$. We define the frame U' on $\overline{K}'$ by setting

$$u_i'(x) = \sum_{j=1}^{n} \mu_{ij}(x) v_j(x).$$

On the set $M^k - K'$ the frame U' fulfills the requirements.

Manifolds with zero Hopf invariant.

Theorem 14. *Each framed manifold in a Euclidean space is homologous to a connected framed manifold in the same space.*

Proof. Let (M_{-1}^k, U) be an oriented framed submanifold of oriented Euclidean space E^{n+k}, where $n \geq 2$. The case $n = 1$ is trivial, since in this case every framed manifold is nullhomologous (see the end of §6). We suppose that M_{-1}^k is not connected and show that there exists a framed manifold (M_{-1}^k, U_*), homologous to (M_{-1}^k, U), such that M_1^k has one component less than M_{-1}^k. The theorem will then be proved. Let a_{-1} and a_1 be two points of M_{-1}^k belonging to different components. In view of proposition (C) we may suppose that M_{-1}^k is flat in the neighbourhoods of both a_{-1} and a_1. Since $n \geq 2$, M_{-1}^k does not separate E^{n+k}. From this it easily follows that in E^{n+k} there lies a smooth simple curve L given parametrically by

$$y = y(\eta), \qquad -2 \leq \eta \leq 2; \qquad y(-2) = y(2),$$

intersecting M_{-1}^k only in the points a_{-1} and a_1, given by $\eta = -1$ and $\eta = 1$ respectively. We suppose also that the curve L is orthogonal to M_{-1}^k at the points a_{-1} and a_1. Using proposition (A) of §7 and the orthogonalization process, it is possible to provide the segment $-1{,}5 \leq \eta \leq 1{,}5$ on the curve L with an orthonormal frame, that is, to construct at each point $y(\eta)$ of the segment an orthonormal system of vectors, $e_1(\eta), \cdots, e_{n+k-1}(\eta)$, orthogonal to L at the point $y(\eta)$ and smoothly dependent on the parameter η. We will suppose that the vectors $e_1(-1)$, $\cdots$, $e_k(-1)$, are tangent to M_{-1}^k and determine its orientation, and that the vectors $e_1(1), \cdots, e_k(1)$ are tangent to M_{-1}^k and determine the opposite orientation. It is possible to achieve this by subjecting the vectors $e_1(\eta), \cdots, e_{n+k-1}(\eta)$ to an orthogonal transformation depending continuously on η. Let $E_*^{n+k+1} = E^{n+k} \times I$, where I is the interval $-1 \leq t \leq 1$; we will regard the product space E_*^{n+k+1} as a strip in the Euclidean space E^{n+k+1}. We now construct a map θ of the subset H^{k+2} (see (A)) of E^{k+2} into E^{n+k+1}, depending on a positive number ρ, mapping the point $(\xi^1, \cdots, \xi^k, \eta, \tau)$ of H^{k+2} into the point (z, t) of E_*^{n+k+1}, where

$$z = y(\eta) + \sum_{i=1}^{k} \rho\xi^i e_i(\eta), \qquad (2)$$
$$t = \tau.$$

Here z and $y(\eta)$ are vectors of E^{n+k}. The given relations define a map θ not only on H^{k+2} but also on some ϵ-neighbourhood W^{k+2} of it in E^{k+2}. It is clear that if ρ is small enough the map θ is a smooth regular homeomorphism of the manifold W^{k+2}. Also, for sufficiently small ρ, the intersection of $\theta(W^{k+2})$ with $M^k_{-1} \times (-1)$ is contained in neighbourhoods of the points $a_{-1} \times (-1)$ and $a_1 \times (-1)$. We will assume ρ so small that this intersection is contained in flat neighbourhoods of these points in $M^k_{-1} \times (-1)$. Now $M^k_{-1} \times I$ is contained in the strip $E^{n+k} \times I$ as a submanifold. In this submanifold we replace the part lying in $\theta(H^{k+2})$ by $\theta(P^{k+1} \cap H^{k+2})$ (see (A)), namely we set:

$$M^{k+1} = (M^k_{-1} \times I - \theta(H^{k+2})) \cup \theta(P^{k+1} \cap H^{k+2}).$$

It may be seen directly that M^{k+1} is a smooth submanifold of E^{n+k+1}_*, orthogonal at its boundary points to the boundary of the strip E^{n+k+1}_*; moreover the part of its boundary lying in the hyperplane $E^{n+k} \times (-1)$ coincides with $M^k_{-1} \times (-1)$, and the part $M^k_1 \times 1$ lying in $E^{n+k} \times 1$ has one component fewer than M^k_{-1}.

We now concern ourselves with the construction of a frame V for the manifold M^{k+1} to realise a homology $(M^k_{-1}, U) \sim (M^k_1, U_*)$. We choose a frame V on the manifold $\theta(P^{k+1} \cap W^{k+2})$, as described in (B), so that at the point $a_{-1} \times (-1)$ the vectors $v_1, \cdots, v_n$ and $u_1 \times (-1), \cdots, u_n \times (-1)$ are obtained from each other by a transformation with positive determinant. In the light of proposition (D) we may suppose that the vectors $u_1 \times (-1), \cdots, u_n \times (-1)$ coincide with our constructed vectors $v_1, \cdots, v_n$ in the intersection $(M^k_{-1} \times (-1)) \cap \theta(H^{k+2})$. Thus the frame V is constructed on the part $\theta(P^{k+1} \cap H^{k+2})$ of the manifold. On the part $M^{k+1} - \theta(P^{k+1} \cap H^{k+2})$ we define the vectors $v_1, \cdots, v_n$ at the point (x, t), $x \in M^k_{-1}$, $t \in I$, to be parallel to the vectors $u_1(x) \times (-1), \cdots, u_n(x) \times (-1)$. Thus the framed manifold (M^{k+1}, V) is constructed.

So, Theorem 14 is proved.

Theorem 15. *Let (M^k_{-1}, U) be a framed submanifold of Euclidean space E^{n+k}, $n \geq k+1$. Then there exists a framed submanifold (M^k, W) of E^{n+k}, homologous to (M^k_{-1}, U) such that M^k is connected and lies in a $2k$-dimensional linear subspace E^{2k} of E^{n+k}.*

Proof. In view of Theorems 11 and 14 it is sufficient to prove Theorem 15 when $n = k+1$ and the manifold M^k_{-1} is connected. In view of proposition (B) of §4 there exists a hyperplane E^{2k} in E^{2k+1}, such that the orthogonal projection π of M^k_{-1} on to E^{2k} is typical. Let a_{-1} and a_1 be two distinct points of M^k_{-1} satisfying the condition $\pi(a_{-1}) = \pi(a_1)$. There are only finitely many such pairs in M^k_{-1} (see §4, A). We 'reconstruct' the manifold M^k_{-1} in the neighbourhood of the segment (a_{-1}, a_1). Such a reconstruction may be effected for each pair of coincidences of the map π of the manifold M^k_{-1}.

In view of (C) we may suppose that the manifold M^k_{-1} is flat near the points a_{-1} and a_1. Let $e_1, \cdots, e_k$ be a system of linearly independent vectors, tangent

to M^k_{-1} at the point a_{-1} and orienting the manifold M^k_{-1} and let $e_{k+1}, \cdots, e_{2k}$ be a system of linearly independent vectors, tangent to M^k_{-1} at a_1 and giving the opposite orientation to M^k_{-1}. We denote by e_{2k+1} the vector with origin at the midpoint O of the segment (a_{-1}, a_1) and ending at the point a_1. Taking the point O as origin of coordinates and transferring all the vectors constructed to it, we obtain a basis $e_1, \cdots, e_{2k+1}$ of the vector space E^{2k+1}. Let $E^{2k+2}_* = E^{2k+1} \times I$, where I is the interval $-1 \leq t \leq 1$; we consider E^{2k+2}_* as a strip in E^{2k+2}. We construct a map θ of the subset H^{k+2} (see (A)) of E^{k+2} into E^{2k+2}, depending on a positive number ρ, to be taken small enough to render the further constructions possible; namely, θ transforms the point $(\xi^1, \cdots, \xi^k, \eta, \tau) \in H^{k+2}$ into the point $(z, t) \in E^{2k+2}_*$, where

$$z = \eta e_{2k+1} + \rho \sum_{i=1}^{k} \xi^i \left(\cos\left(\frac{\pi}{4}\eta + \frac{\pi}{4}\right) e_i + \sin\left(\frac{\pi}{4}\eta + \frac{\pi}{4}\right) e_{i+k}\right),$$

$$t = \tau.$$

This formula defines a map θ not only of H^{k+2} but also of some ϵ-neighbourhood W^{k+2} of H^{k+2} in E^{k+2}. Here z is, in effect, a map of the set $H^{k+1}_{\tau_0}$ of points $(\xi^1, \cdots, \xi^k, \eta, \tau_0)$, satisfying condition (1), into the vector space E^{2k+1}. We remark that the map πz of the set H^{k+1}_τ is regular and homeomorphic everywhere except at points of the segment $\xi^i = 0$, $|\eta| \leq 1$, so that the map πz of the manifold $P^k_1 \subset H^{k+1}_1$ into the space E^{2k} is regular and homeomorphic. We replace the part of the submanifold $M^k_{-1} \times I$ of the strip $E^{2k+1} \times I$ lying in $\theta(H^{k+2})$ by $\theta(P^{k+1} \cap H^{k+2})$ (see (A)), namely we set

$$M^{k+1} = (M^k_{-1} \times I - \theta(H^{k+2})) \cup \theta(P^{k+1} \cap H^{k+2}).$$

It may be seen directly that M^{k+1} is a smooth submanifold of E^{2k+2}_*, orthogonal at its boundary points to the boundary of the strip E^{2k+2}_*; moreover the part of its boundary lying in the hyperplane $E^{2k+1} \times (-1)$ coincides with $M^k_{-1} \times (-1)$, and the part $M^k_1 \times 1$ lying in $E^{2k+1} \times 1$ is such that M^k_1 has one fewer pairs of coincidences of the map π than has M^k_{-1}. If $k > 1$ the connectedness of M^k_{-1} implies the connectedness of M^k_1. If $k = 1$ Theorem 15 follows immediately from proposition (D) of §13; there is no point in going through this proof if $k = 1$ since the manifold M^1_1 we construct may turn out not to be connected.

We now concern ourselves with the construction of a frame V for the manifold M^{k+1} to realise a homology $(M^k_{-1}, U) \sim (M^k_1, U_*)$. This construction[1] follows precisely the lines of the construction of the frame V in Theorem 14 (with $n = k+1$).

We will suppose that the reconstruction of M^k_{-1} which we have described has been carried out simultaneously on all pairs of coincidences of the map π. Then the manifold M^k_1 we obtain is mapped by the projection π regularly and homeomor-

Translator's note: The Russian text reproduces the construction of Theorem 14 word for word.

phically on to a submanifold $M^k = \pi(M_1^k)$ of the space E^{2k}. This projection can be realised by means of a deformation of the smooth submanifold M_1^k into the smooth submanifold M^k. In view of proposition (B) of §7, this deformation can be extended to a deformation of framed manifolds. Thus we obtain the required framed submanifold (M^k, W) of E^{2k+1}.

Thus, Theorem 15 is proved.

Theorem 16. *Let* (M_0^k, U_0) *be a framed submanifold of the Euclidean space* E^{2k+1} *with* $\gamma(M_0^k, U_0) = 0$ (which is always true if k is even, see definition 9), (see §10, B). *Then there exists a framed submanifold* (M_1^k, V_1) *of a hyperplane* E^{2k} *of* E^{2k+1} *such that* $(M_0^k, U_0) \sim E(M_1^k, V_1)$, (see definition 6).

Proof. In view of Theorems 14 and 15 there exists a closed connected framed submanifold (M_1^k, U_1) of E^{2k+1} such that $M_1^k \subset E^{2k}$ and $(M_1^k, U_1) \sim (M_0^k, U_0)$. In view of proposition (B) of § 10 we have $\gamma(M_1^k, U_1) = 0$. Thus in view of proposition (E) of §10 the degree of the map ψ of the manifold M_1^k on to the sphere $\mathfrak{S}^k$ is zero and so ψ is nullhomotopic (see Theorem 13). In view of proposition (A) of §8 the framed manifold (M_1^k, U_1) is homologous to a framed manifold $E(M_1^k, V)$, where (M_1^k, V) is a framed submanifold of the space E^{2k}.

Thus Theorem 16 is proved.

CHAPTER IV.

Classification of maps of $(n+1)$-dimensional and $(n+2)$-dimensional spheres into the n-dimensional sphere

§12. The rotation group of Euclidean space

The main aim of this section is to establish the most elementary topological properties of the group H_n of all rotations of the n-dimensional Euclidean vector space E^n, properties which will be used in the classification of maps of Σ^{n+k} into S^n for $k = 1, 2$. It turns out (see Theorem 17) that the manifold H_n is connected and that for $n \geq 3$ there exist precisely two homotopy classes of maps of the circle into H_n. As a means of establishing the topological properties of H_n we use the well-known covering homotopy lemma, which is of great significance in its own right, and also the description of the group H_3 by means of quaternions, which also have considerable independent interest and are used in the sequel.

Quaternions. We recall the notion of a *quaternion*, which will be used both in this and in the following sections.

A) Let k be a 4-dimensional Euclidean vector space with fixed Cartesian coordinates. We write an arbitrary vector $x = (x^1, x^2, x^3, x^4) \in K$ in the form: $x = x^1 + ix^2 + jx^3 + kx^4$, where i, j, k are the *quaternionic units*. We define a multiplication in K by demanding that it be distributive, that real numbers commute with the quaternionic units and that the units themselves multiply according to the rules

$$ij = -ji = k;\ jk = -kj = i;\ ki = -ik = j;\ ii = jj = kk = -1. \tag{1}$$

It is easy to verify that the multiplication in K so defined is associative. The quaternion $\bar{x}$, *conjugate* to x, is defined by putting $\bar{x} = x^1 - ix^2 - jx^3 - kx^4$. It is readily seen that

$$\overline{xy} = \bar{y}\bar{x}. \tag{2}$$

The *modolus* of the quaternion x is defined as the non-negative real number $|x| = \sqrt{x\bar{x}} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}$ we have $|xy|^2 = xy\overline{xy} = xy\bar{y}\bar{x} = x|y|^2\bar{x} = |y|^2 x\bar{x} = |x|^2|y|^2$. Thus

$$|xy| = |x|\,|y|. \tag{3}$$

If $x \neq 0$, then $|x| \neq 0$ and there exists a quaternion x^{-1}, *inverse* to x, namely $x^{-1} = \frac{\bar{x}}{|x|^2}$. Thus the collection K of all quaternions forms an algebraic field. The field K of quaternions contains the field D of real numbers, consisting of all quaternions of the form $x = x^1 + 0i + 0j + 0k$. The collection G of all quaternions x satisfying the condition $|x| = 1$ forms, in view of (3), a multiplicative group. The set G is the 3-sphere in the space K. A quaternion of the form $x^2 i + x^3 j + x^4 k$ is called *pure imaginary*. The collection J of such quaternions forms a 3-dimensional vector space, orthogonal to the line D in K.

B) Let K be the field of quaternions, D the subfield of real numbers, J the collection of all pure imaginary quaternions, and G the group of quaternions of modulus unity (see (A)). With each quaternion $g \in G$ we associate the map ψ_g of K into itself given by

$$\psi_g(x) = gxg^{-1}. \tag{4}$$

Since in view of (3) $|gxg^{-1}| = |x|$, the transformation ψ_g, being linear, is a rotation of the Euclidean space K. Since $\psi_g(D) = D$, the vector subspace J orthogonal to D is mapped by ψ_g into itself; that is, J is itself rotated. It turns out that by associating with each quaternion $g \in G$ the rotation $v(g) = \psi_g$ of J, we obtain a homomorphism v of the group G on to the group H_3 of all rotations of the Euclidean 3-space J. The kernel of the homomorphism v consists of the two elements 1 and -1. It further turns out that the subgroup S^1 of all quaternions $g \in G$ for which $\psi_g(i) = i$ consists of all quaternions of the form $\cos\alpha + i\sin\alpha$.

We prove proposition (B). First of all we have

$$\psi_{gh}(x) = ghxh^{-1}g^{-1} = \psi_g(hxh_{-1}) = \psi_g(\psi_h(x)),$$

so that v is a homomorphism of G into H_3. We show that $v(G) = H_3$. Let $l = aj + bk$ where $a^2 + b^2 = 1$. It is easy to see that

$$l^2 = -1, \qquad li = -il. \tag{5}$$

Now let $g = \cos\beta + l\sin\beta$. It follows from (5) that

$$\psi_g(i) = (\cos\beta + l\sin\beta)\,i\,(\cos\beta - l\sin\beta) = (\cos\beta + l\sin\beta)^2 i =$$
$$= (\cos 2\beta + l\sin 2\beta)\,i = i\cos 2\beta + (bj - ak)\sin 2\beta, \tag{6}$$

and from this it follows that by selecting the numbers a, b, β in a suitable way we can transform the quaternion i into any quaternion of the set $S^2 = J \cap G$. Further, by putting $a = 0$, $b = 1$ we obtain from (6)

$$\psi_g(i) = i\cos 2\beta + j\sin 2\beta, \qquad (7)$$

and since in this case g commutes with k it follows that it is possible by a transformation of the form ψ_g to realise an arbitrary rotation of J round the axis k. Since G is a group it follows from what has been proved that it is possible to realize an arbitrary rotation of J by a transformation of the form ψ_g. We remark further that it follows from the laws of multiplication (1) that the only quaternions of G commuting with i are those of the form $\cos\alpha + i\sin\alpha$ so that the group S^1 consists of the quaternions of the stated form. In the same way the quaternions of G commuting with j are of the form $\cos\alpha + j\sin\alpha$. Thus the kernel of the homomorphism v consists of the two elements $+1$ and -1.

Thus proposition (B) is proved.

Covering homotopies.

Lemma 1. *Let ϕ be a smooth map of the closed manifold P^p into the closed manifold Q^q, $p \geq q$, which is proper at every point. Further let f be a map of a compact metric space R into P^p and let g_t, $0 \leq t \leq 1$, be a deformation of maps of R into Q^q such that $g_0 = \phi f$. Then there exists a deformation f_t of maps of R into P^p such that $f_0 = f$ and $\phi f_t = g_t$. The deformation f_t is said to cover the deformation g_t. If, for some point $x \in R$ we have $g_t(x) = g_0(x)$, all t, then also $f_t(x) = f_0(x)$. If further R is a smooth manifold, f a smooth map and g_t a smooth deformation, then f_t is also smooth.*

Proof. We denote the counterimage of $y \in Q^q$ under ϕ by M_y: $M_y = \phi^{-1}(y)$. From formula (2) of §4 it follows that M_y is a $(p-q)$-dimensional submanifold of P^p. In view of Theorem 2 we may suppose that P^p is a smooth submanifold of a Euclidean vector space A of sufficiently high dimension. We denote the normal in A at x_0 to M_{y_0} by N_{x_0}. We show now that if the point y is close enough to y_0 there exists only one point of intersection $\gamma(x_0, y)$ of N_{x_0} with M_y close to x_0. To prove this we introduce such local coordinates into neighbourhoods of the points x_0 and y_0 of the manifolds P^p and Q^q, namely $x^1, \cdots, x^p$ and $y^1, \cdots, y^q$, with origins at x_0 and y_0 respectively, that ϕ takes the form

$$y^1 = x^1, \cdots, y^q = x^q \qquad (8)$$

(see §4, formula (2)). Let $x = \theta(x^1, \cdots, x^p)$ be the parametric equation of the manifold P^p near x_0. The normal N_{x_0} in the Euclidean vector space A is defined by the system oe equations

$$\left(x - x_0, \frac{\partial\theta(0,\cdots,0)}{\partial x^i}\right) = 0, \qquad i = q+1, \cdots, p, \qquad (9)$$

where x is a radius-vector describing the linear space N_{x_0}. As parametric equation of the manifold M_y we may take

$$x = \theta(y^1, \cdots, y^q, x^{q+1}, \cdots, x^p), \qquad (10)$$

where $y^1, \cdots, y^q$ are coordinates of the point y and $x^{q+1}, \cdots, x^p$ are local coordinates in the manifold M_y. Therefore to find the point $\gamma(x_0, y)$ we must substitute the value of x from equation (10) in the equations (9) and then solve the system of equations we obtain for the unknowns $x^{q+1}, \cdots, x^p$. By substitution we obtain

$$\left(\theta(y^1, \cdots, y^q, x^{q+1}, \cdots, x^p) - \theta(y_0^1, \cdots, y_0^q, x_0^{q+1}, \cdots, x_0^p), \frac{\partial\theta(0,\cdots,0)}{\partial x^i}\right) = 0, \qquad (11)$$

$$i = q+1, \cdots, p.$$

Here we have a system of $(p-q)$ equations in the $(p-q)$ unknowns $x^{q+1}, \cdots, x^p$. With the initial conditions $y^1 = 0, \cdots, y^q = 0$ the system (11) has the unique solution $x^{q+1} = 0, \cdots, x^p = 0$. The functional determinant of the system (10) with these initial conditions has the form

$$\left[\left(\frac{\partial\theta(0,\cdots,0)}{\partial x^j}, \frac{\partial\theta(0,\cdots,0)}{\partial x^i}\right)\right]; \quad i, j = q+1, \cdots, p,$$

which is non-zero since the vectors $\frac{\partial\theta(0,\cdots,0)}{\partial x^i}$, $i = q+1, \cdots, p$ are linearly independent. Thus if y is sufficiently close to y_0 there exists only one point x close to x_0 and satisfying the condition

$$x = \gamma(x_0, y) \in N_{x_0} \cap M_y.$$

Since P^p is compact it follows that there exists a positive number δ so small that if $\rho(y, \phi(x_0)) < \delta$ the function $\gamma(x_0, y)$ is defined and is a continuous function of its arguments $x_0 \in P^p$ and $y \in Q^q$. This function possesses the following two properties

$$\gamma(x_0, \phi(x_0)) = x_0, \qquad (12)$$

$$\phi(\gamma(x_0, y)) = y. \qquad (13)$$

Only these properties of γ will be used in the sequel.

We pass now to the construction of the deformation f_t, using the function γ. We put $f_0 = f$. Let ϵ be so small that if $|t - t'| \leq \epsilon$, $u \in R$, then $\rho(g_t(u), g_{t'}(u)) < \delta$. We suppose f_t defined for all values of t satisfying $0 \leq t \leq n\epsilon < 1$ where n is a non-negative integer. We define f_t for values of t satisfying $n\epsilon \leq t \leq (n+1)\epsilon$ by putting

$$f_t(u) = \gamma(f_{n\epsilon}(u), g_t(u)). \qquad (14)$$

From relations (12) and (13) it follows that f_t, so defined, is a continuous deformation satisfying the condition $g_t = \phi f_t$.

Thus Lemma 1 is proved.

The rotation group of Euclidean space.

C) Let E^n be a Euclidean vector space, let S^{n-1} be the sphere in it given by $(x, x) = 1$, let H_n be the rotation group of E^n and let a be a fixed point of S^{n-1}. It turns out that H_n is a smooth manifold of dimension $\frac{n(n-1)}{2}$ and that, by associating with each element h the point $\chi(h) = h(a)$ we obtain a smooth map χ of H_n on to S^{n-1} which is proper at each of its points.

We prove proposition (C). Let $e_1, \cdots, e_n$ be an orthonormal basis for the space E^n. If $h \in H_n$ then

$$h(e_j) = \sum_i h_{ij} e_i. \tag{15}$$

In this way, to each rotation h there corresponds an orthogonal matrix $\|h_{ij}\|$ with positive determinant: $h \to \|h_{ij}\|$, and, conversely, to each orthogonal matrix $\|h_{ij}\|$ with positive determinant there corresponds in view of (15) a definite rotation of E^n. In the light of the correspondence $h \to \|h_{ij}\|$ we identify the group H_n with the group of all orthogonal matrices of order n with positive determinant. As is known the orthogonality condition for matrices takes the form

$$F_{ij} = \delta_{ij}, \text{ where } F_{ij} = \sum_\alpha h_{i\alpha} h_{j\alpha}. \tag{16}$$

We show that the numbers h_{ij}, $i > j$, may be taken as local coordinates of the matrix $h \in H_n$ near the identity matrix $\|\delta_{ij}\|$. For this it is enough to show that, for the initial values $h_{ij} = \delta_{ij}$ the system of equations (16) is solvable for the variables h_{ij} where $i \leq j$. We remark that since $F_{ij} = F_{ji}$, we need only to consider the functions F_{ij} for $i \leq j$, so that the number of equations is equal to the number of unknowns. We have

$$\frac{\partial F_{ij}}{\partial h_{kl}} = \sum_\alpha (\delta_{ik}\delta_{\alpha l} h_{j\alpha} + h_{i\alpha}\delta_{jk}\delta_{\alpha l});$$

for $h_{ij} = \delta_{ij}$ this gives $\frac{\partial F_{ij}}{\partial h_{kl}} = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}$. If at least one of the inequalities $i \leq j$, $k \leq l$ is strict then the equations $j = k$, $i = l$ are inconsistent and the second summand is zero. Thus the functional matrix of the system of functions F_{ij}, $i \leq j$, of the variables h_{kl}, $k \leq l$, is diagonal and the diagonal elements are all equal to 1 or 2, so that the solubility of the system (2) is proved. Let U be a neighbourhood of the identity in which the equations (16) are solvable and in which, consequently, the numbers h_{ij}, $i > j$, may be taken as coordinates. Let $h_0 \in H_n$; then Uh_0 is a neighbourhood of the matrix h_0 and we take as coordinates of the element $hh_0 \in Uh_0$ the coordinates of the element h in U. Let Uh_0 and Uh_1 be two overlapping neighbourhoods. It is easy to see that the transformation from the coordinate system for Uh_0 to the coordinate system for Uh_1 is achieved by smooth functions. Thus H_n is a smooth manifold.

Since H_n is a group, some element of which transforms the point a into an arbitrary point of the sphere, it follows that $\chi(H_n) = S^{n-1}$ and that it is sufficient to demonstrate that χ is proper at one point of the manifold H_n, for instance, the point $\|\delta_{ij}\|$. For $a = e_1$ there corresponds to the matrix $\|h_{ij}\|$ under the map χ the point of S^{n-1} with coordinates h_{i1}, $i = 1, \cdots, n$. Since $h_{21}, h_{31}, \cdots, h_{n1}$ are coordinates of the element $\|h_{ij}\|$ in U, and the numbers $h_{21}, \cdots, h_{n1}$ can be taken as coordinates of the point $\chi(h)$, it is evident that χ is proper at the point $\|\delta_{ij}\|$.

Thus proposition (C) is proved.

Theorem 17. *Let H_n be the rotation group of the Euclidean vector space E^n, $n \geq 3$. It turns out that H_n is a connected manifold and that there exist precisely two homotopy classes of maps of the circle S^1 into H_n of which one consists of all nullhomotopic maps and the other of all maps which are not nullhomotopic. The maps in the latter class can be described in the following way. Let E^2 be an arbitrary two-dimensional subspace of the vector space E^n and let E^{n-2} be the orthogonal complement of E^2. The rotation group H_2 of the plane E^2, which is homeomorphic to S^1, may be regarded in a natural way as a subgroup of H_n by extending each rotation of E^2 to a rotation of the whole of E^n by defining it to be the identity on E^{n-2}. It turns out that a map g of S^1 into H_2 is nullhomotopic in H_n if and only if its degree is even. It further turns out that every map h of S^1 into H_n is deformable to a map g of S^1 into H_2 in such a way that every point x for which $h(x) \in H_2$ remains stationary during the deformation.*

Proof. Let S^{n-1} be the unit sphere in E^n, $a \in S^{n-1}$, and χ the map of H_n into S^{n-1} defined in proposition (C). Clearly the set $\chi^{-1}(a)$ is the subgroup H_{n-1} of H_n consisting of all rotations of the space E^{n-1} orthogonal to the vector a.

Let f_0 be a smooth map of the compact manifold M^r, $r \leq n-2$, into H_n. We show that there exists a deformation f_t, $0 \leq t \leq 1$, of the map f_0 which does not move the images of those points of M^r mapped into H_{n-1} and such that f_1 maps the whole of M^r to H_{n-1}: $f_1(M^r) \subset H_{n-1}$. In view of Theorem 1 the set $\chi f_0(M^r)$ is nowhere dense in S^{n-1} and so there exists a smooth deformation g_t of the map $g_0 = \chi f_0$ which leaves at a the images of the points in $g_0^{-1}(a)$ and such that $g_1(M^r) = a$. The deformation f_t covering g_t has the required properties (see Lemma 1).

If we consider the case $M^r = S^1$ we see that an arbitrary map of the circle S^1 into the manifold H_n is homotopic to a map into H_{n-1}. If $n - 1 \geq 3$ then, by a second application of the same argument, we see that an arbitrary map of S^1 into H_n is homotopic to a map into H_{n-2}, where H_{n-2} is the rotation group of some subspace E^{n-2} of E^{n-1}. Repeating this process further we conclude that an arbitrary map of S^1 into H_n is homotopic to a map into $H_2 \subset H_n$.

We show that if the map g of S^1 into H_2 is nullhomotopic in H_n then it is a also nullhomotopic in H_3, where $H_2 \subset H_3 \subset H_n$. Let K^2 be a disk bounded by the circle S^1. Since g is nullhomotopic in H_n it may be extended to a map of the whole disk K^2 into H_n. Applying the process described above in the case $M^r = K^2$ we deduce that g is nullhomotopic in H_3. To prove the theorem it remains to establish that a map g of S^1 into H_2 is nullhomotopic of and only if the degree σ of the map g is even.

To prove this fact we make use of the homomorphism v of the group G on to H_3 (see (B)). The map v is smooth and proper at every point and transforms exactly two points of G into each point of H_3. We remark further that $\Sigma^1 = v^{-1}(H_2)$ is a circle which is mapped by v on to the circle H_2 with degree 2 (see (7)).

We suppose that $\sigma = 2\rho$ and let ν be a map of S^1 into Σ^1 of degree ρ. Then the map $v\nu$ of S^1 into H_2 has degree $2\rho = \sigma$, and so is homotopic to g. Since the map ν is nullhomotopic in G, the map $v\nu$ is nullhomotopic in H_3. Thus the map g is also nullhomotopic in H_3.

We suppose now that the map g of S^1 into H_2 is nullhomotopic in H_3, so that there exists a deformation g_t, $0 \leq t \leq 1$, of maps of S^1 into H_3 such that $g_1 = g$ and $g_0(S^1)$ is a single point of H_3. Let p be a point of G such that $v(p) = g_0(S^1)$ and let f_0 map S^1 to p. Then $vf_0 = g_0$ and in view of Lemma 1 there exists a covering deformation f_t of the deformation g_t; $vf_t = g_t$. Consequently f_1 is a map of S^1 into Σ^1 such that $vf_1 = g_1$, so that the degree of g_1 is even (since the degree of $v|\Sigma^1$ is 2).

The connectedness of the manifold H_n is easily proved directly. It also follows from the fact that there exists only one class of nullhomotopic maps of S^1 into H_n.

Thus Theorem 17 is proved.

D) With each map h of a one-dimensional manifold M^1 into the group H_n, $n \geq 2$, we associate $\beta(h)$, a residue class mod 2. If $n \geq 3$ and M^1 is connected the class $\beta(h)$ is zero of h is nullhomotopic and non-zero otherwise. If M^1 is not connected we define $\beta(h)$ to be the sum of the classes $\beta(h)$ for each component. If $n = 2$ we define $\beta(h)$ to be the degree of the map h, reduced mod 2. Given two maps f and g of S^1 into H_n we define their *group-product* $h = fg$ by setting

$$h(x) = f(x)\,g(x), \qquad x \in S^1,$$

where on the right-hand side we have the product in the group H_n of the elements $f(x)$ and $g(x)$. It turns out that

$$\beta(h) = \beta(f) + \beta(g). \tag{17}$$

We prove formula (17). Let $T^2 = S^1 \times S^1$ be the direct product of the circle S^1 with itself, that is, the collection of all pairs (x, y), $x \in S^1$, $y \in S^1$. We define a map ϕ of the torus T^2 into H_n by setting

$$\phi(x, y) = f(x)\,g(y).$$

Further let a be a fixed point on S^1. Without loss of generality we may suppose that $f(a) = g(a) = e \in H_n$. We define three maps f', g', h' of S^1 into T^2 by setting

$$f'(x) = (x, a), \qquad g'(x) = (a, x), \qquad h'(x) = (x, x).$$

Plainly

$$\phi f' = f, \qquad \phi g' = g, \qquad \phi h' = h.$$

It is known and easy to prove that the map h' of S^1 into T^2 is homotopic to a map $\hat{h}$ of S^1 into the lemniscate $(S^1 \times a) \cup (a \times S^1)$ such that S^1 is mapped with degree 1 on to each side of $S^1 \times a$, $a \times S^1$. Then the maps $\phi h'$ and $\phi\hat{h}$ are homotopic. Moreover for the map $\phi\hat{h}$ it may immediately be verified that $\beta(\phi\hat{h}) = \beta(\phi f') + \beta(\phi g')$. Thus formula (17) is proved.

§13. Classification of maps of the 3-sphere into the 2-sphere

In this section the homotopy classification of maps of Σ^3 into S^2 is given, namely, it is proved that the Hopf invariant γ (see §10) is in this case the *unique* homotopy invariant of maps and can take arbitrary integer values. An important role in the proof is played by the Hopf map ω of Σ^3 into S^2 whcih is conveniently described by means of quaternions. Let K be the quaternionic field, G the collection of quaternions of modulus unity, and let J be the collection of all pure imaginary quaternions (see §12, A). As the sphere Σ^3 we take G and as the sphere S^2 we take the intersection $G \cap J$. With each element $g \in G$ we associate the element $\omega(g)$, by putting $\omega(g) = gig^{-1}$, where i is the quaternionic unit. It turns out that the map ω so defined is proper at each point and has Hopf invariant 1. Only these properties of the map ω are used for the solution of the classification problem of maps of Σ^3 into S^2. The solution also requires the fact that every map of the sphere S^n, $n \geq 2$, into the circle S^1 is nullhomotopic. The proof of this quite elementary theorem is also given here.

Maps of spheres into circles.

Theorem 18. *Every map of the sphere S^n into the circle S^1, $n \geq 2$, is nullhomotopic.*

Proof. Let p and q be the north and south poles of S^n and let S^{n-1} be its equator, that is, the section by the hyperplane perpendicular to the segment pq and passing through its midpoint. For each point $x \in S^{n-1}$ there is a unique meridian pxq on the sphere S^n passing through x, that is, a great semicircle on S^n joining p and q and passing through x. On the meridian pxq we introduce an angular coordinate α, reckoned from the point p. We denote the point y on the meridian pxq with angular coordinates α by (x, α). We have $(x, 0) = p$, $(x, \pi) = q$, and for each point $y \in S^n - (p \cup q)$ there is a unique expression $y = (x, \alpha)$, where $0 < \alpha < \pi$.

Let f be an arbirtaty map of S^n into S^1. On the circle S^1 we introduce the we obtain from (4), ignoring second order terms,

$$\omega(h) = i + 2(x^3 j + x^4 k)(\cos x - i \sin x) i. \tag{5}$$

From this it follows that

$$\tilde{\omega}_x(\xi) = 2\xi(\cos x - i \sin x). \tag{6}$$

Thus if we express the quaternion ξ in polar coordinates ρ, β, that is, if we put

$$\xi = j\rho(\cos\beta - i \sin\beta),$$

then it is plain that $\tilde{\omega}_x$ is a rotation in the plane l through an angle x with simultaneous magnification by a factor of 2.

The normal N_x^2 at the point $\cos x + i \sin x$ to the circle S^1 is given by the parametric equations (2). As in proposition (A) we put $u_1(x) = \{1, 0\}$, $u_2(x) = \{0, 1\}$. To the map ϕ there corresponds a map ϕ_x of the tangent space P_x^3 to G at x on to the linear space E^3 (see §1, E). It may be verified directly that

$$\phi_x q_x(j) = u_2(x), \quad \phi_x q_x(k) = u_1(x). \tag{7}$$

For the construction of the frame $V = \{v_1(x), v_2(x)\}$ corresponding to the map ω, we must, according to definition 4, choose vectors e_1 and e_2 in R^2 and then find vectors $v_1(x)$, $v_2(x)$ in N_x^2 such that $e_1 = (\omega\phi^{-1})_x v_1(x)$, $e_2 = (\omega\phi^{-1})_x v_2(x)$. In the choice of e_1 and e_2 and the calculation of the vectors $v_1(x)$ and $v_2(x)$ it is helpful to note that

$$(\omega\phi^{-1})_x^{-1} = \phi_x \omega_x^{-1} = \phi_x q_x q_x^{-1} \omega_x^{-1} r r^{-1} = \phi_x q_x \tilde{\omega}_x^{-1} r^{-1}. \tag{8}$$

So, by taking $e_1 = r\left(\frac{k}{2}\right)$, $e_2 = r\left(\frac{j}{2}\right)$, we obtain, according to (6)–(8),

$$v_1(x) = (\phi_x q_x)\tilde{\omega}_x^{-1}\left(\frac{k}{2}\right) = \phi_x q_x(k(\cos x + i \sin x)) =$$
$$= \phi_x q_x(k \cos x + j \sin x) = u_1(x)\cos x + u_2(x) \sin x,$$
$$v_2(x) = (\phi_x q_x)\tilde{\omega}_x^{-1}\left(\frac{j}{2}\right) = \phi_x q_x(j(\cos x + i \sin x)) =$$
$$= \phi_x q_x(-k \sin x + j \cos x) = -u_1(x)\sin x + u_2(x)\cos x.$$

Thus, in view of (A), we obtain $\gamma(S^1, V) = 1$ and so $\gamma(\omega) = 1$.

Lemma 1 is thereby proved.

Classification of maps of the 3-sphere into the 2-sphere.

Lemma 2. *Let $\pi_n(S^r)$ be the set of homotopy classes of maps of S^n into S^r, $n \geq 3$, $r = 2, 3$ and let ω be the map of Σ^3 into S^2 described in Lemma 1. It is evident that if f_0 and f_1 are two homotopic maps of S^n into Σ^3 then the maps ωf_0 and ωf_1 of S^n into S^2 are also homotopic. Thus for $\alpha \in \pi_n(\Sigma^3)$ the collection of maps $\omega\alpha$ belongs to a single class $\hat{\omega}(\alpha) \in \pi_n(S^2)$. It turns out that $\hat{\omega}$ maps the set $\pi_n(\Sigma^3)$ on to the set $\pi_n(S^2)$ and that only the zero element of $\pi_n(\Sigma^3)$ is mapped by $\hat{\omega}$ to the zero element of $\pi_n(S^2)$.*

It follows immediately from the definition of addition in the group $\pi_n(S^r)$, which has not been described explicitly in the present work, that $\hat{\omega}$ is a homomorphism of the group $\pi_n(\Sigma^3)$ into the group $\pi_n(S^2)$. Thus we may conclude from Lemma 2 that $\hat{\omega}$ is an isomorphism of $\pi_n(\Sigma^3)$ on to $\pi_n(S^2)$. However this result is not used in what follows.

Proof. We show first that $\hat{\omega}^{-1}(0) = 0$. Let f be a map of S^n into Σ^3 such that ωf is a nullhomotopic map of S^n into S^2. Then there exists a continuous family g_t, $0 \leq t \leq 1$, of maps of S^n into S^2, such that $g_0 = \omega f$ and g_1 maps S^n to a point c of S^2. In view of Lemma 1 of §12 there exists a continuous family f_t of maps of S^n into Σ^3 such that $f_0 = f$ and $\omega f_t = g_t$. Since $g_1(S^n) = c$, $f_1(S^n) \subset \omega^{-1}(c)$ and in view of Lemma 1 the set $\omega^{-1}(c)$ is homeomorphic to a circle. Thus, f_1, and consequently also f_0, is nullhomotopic in Σ^3 by Theorem 18.

We show now that, given any $\beta \in \pi_n(S^2)$, we can find an element $\alpha \in \pi_n(\Sigma^3)$ such that $\hat{\omega}(\alpha) = \beta$. We will think of S^n as the unit sphere with centre at the origin of coordinates in the Euclidean space E^{n+1} with a fixed coordinate system $x^1, \cdots, x^{n+1}$. We denote the set of all points of S^n satisfying the condition $x^{n+1} \leq 0$ by E_-, the set of all points satisfying the condition $x^{n+1} \geq 0$ by E_+, and the set of all points satisfying $x^{n+1} = 0$ by S^{n-1}. As north pole of the sphere S^n we take the point $p = (0, 0, \cdots, 0, 1)$, and as south pole the point $q = (0, 0, \cdots, 0, -1)$. It is clear that there exists a map of S^n on to itself which is homotopic to the identity and which maps the hemisphere E_- to q. From this it follows that the class β contains a map g transforming E_- to a single point c fo S^2. Let P_x^2 be the half-plane in E^{n+1} containing the point $x \in S^{n-1}$ and bounded by the line passing through p and q. The intersection of the half-plane P_x^2 with the sphere S^n and the hyperplane $x^{n+1} = 1 - t$ is denoted by (x, t). Thus to the pair (x, t), $x \in S^{n-1}$, $0 \leq t \leq 1$, corresponds a well-defined point $(x, t) \in E_+$ and each point $y \in E_+$ can be expressed in the form $y = (x, t)$, where the expression is unique if $y \neq p$ while $p = (x, 0)$ where x is an arbitrary point of S^{n-1}. We put $g_t(x) = g(x, t)$, This defines a family g_t, $0 \leq t \leq 1$, of maps of S^{n-1} into S^2 with $g_1(S^{n-1}) = c$ and $g_0(S^{n-1}) = g(p) = b$. Let a be an arbitrary point of the circle $\omega^{-1}(b)$ and let f be the map of S^{n-1} into Σ^3 sending the whole of S^{n-1} to a. In view of Lemma 1 of §12 there exists a deformation f_t, $0 \leq t \leq 1$, of maps of S^{n-1} into Σ^3 such that $f_0 = f$ and $\omega f_t = g_t$. We put $f(x, t) = f_t(x)$. The map f so defined is a map of the hemisphere E_+ under which S^{n-1} is mapped into the circle $\omega^{-1}(c)$. Since a map of S^{n-1} into the circle $\omega^{-1}(c)$ is nullhomotopic (see Theorem 18), the map f of the hemishpere E_+ can be extended to a map f of the whole sphere S^n such that $f(E_-) \subset \omega^{-1}(c)$. The map f so constructed satisfies the condition $\omega f = g$.

Thus Lemma 2 is proved.

Theorem 19. *The homomorphism γ of the group Π_2^1 into the group of integers is an isomorphism* (see §10, C). *From this it follows that two maps f_0 and f_1 of Σ^3 into S^2 are homotopic if and only if $\gamma(f_0) = \gamma(f_1)$ and, moreover, that for each integer c there exists a map f of Σ^3 into S^2 with $\gamma(f) = c$.*

Proof. We show first that the kernel of γ consists of the zero element of Π_2^1

only. For this it is sufficient to show that a map g of Σ^3 into S^2 satisfying the condition $\gamma(g) = 0$ is nullhomotopic. In view of Lemma 2, there exists a map f of Σ^3 into itself such that ωf and g are homotopic, and, consequently, $\gamma(\omega f) = 0$. By proposition (D) of §10 the degree σ of the map f is given by the equation $\gamma(\omega f) = 1^2\sigma = \sigma$, so that $\sigma = 0$. Thus (see Theorem 12) the map f of Σ^3 into itself and consequently also the maps ωf and g are nullhomotopic.

We show that for any integer σ there is a map f of Σ^3 into S^2 such that $\gamma(g) = \sigma$; that is, we show that γ is an epimorphism. In fact, let f be a map of Σ^3 on to itself of degree σ. Then we have, by (D) of §10, $\gamma(\omega f) = \sigma \cdot 1 = \sigma$.

Thus Theorem 19 is proved.

B) Putting together proposition (A) and Theorem 19, we see that each 1-dimensional framed manifold in 3-dimensional Euclidean space is homologous to a framed manifold $(S^1, V_{(r)})$, as constructed in (A), where r is a suitably chosen integer.

§14. Classification of maps of the $(n+1)$-sphere into the n-sphere

In this section it is proved that if $n \geq 3$ there exist exactly two homotopy classes of maps of Σ^{n+1} into S^n. The proof is based on the construction of a homology invariant $\delta(M^1, U)$ of framed manifolds in Euclidean space E^{n+1}, $n \geq 2$, which is a residue class mod 2 and can take either of the values 0 or 1. Thus, just from the existence of the invariant δ it follows that there are at least two classes of maps of Σ^{n+1} into S^n for $n \geq 2$. The invariant δ may be described in the following way. Let $U(x) = \{u_1(x), \cdots, u_n(x)\}$ be an orthonormal frame for the manifold M^1, and let $u_{n+1}(x)$ be a unit vector tangent to M^1 at the point x. The system $U'(x) = \{u_1(x), \cdots, u_{n+1}(x)\}$ is obtained from some fixed orthonormal basis of E^{n+1} by a rotation $h(x)$. Thus there arises a map h of the manifold M^1 into the manifold H_{n+1} of all rotations of E^{n+1}. In the case of a connected curve M^1 the invariant δ is defined to be zero if h is *not* nullhomotopic, and equal to 1 otherwise. In the case of a manifold M^1 which is not connected the invariant δ is defined to be the mod 2 sum of the values of δ on the components.

As a preliminary to the proof of the invariance of the residue class δ a general Lemma 1 is proved in which a framed manifold (M^{k+1}, U) inducing a homology is subjected to 'improvement'. An improved manifold (M^{k+1}, U) has the property that its intersection with the hyperplane $E^{n+k} \times t$ is a framed manifold for all values of the parameter t with the exception of a finite number of critical values. Since for non-critical values of the parameter t the framed manifold (M^k_t, U_t) depends continuously on the parameter t, the invariance of δ remains to be proved only when the parameter t passes through a critical value. In passing through a critical value the manifold (M^k_t, U_t) undergoes a comparatively simple reconstruc-

tion, thanks to which fact the proof of the invariance of δ goes through.

For a 1-dimensional framed submanifold of 3-space the two invariants γ and δ have been defined; it turns out that δ is obtained from the integer γ by reducing mod 2. Since every 1-dimensional framed manifold may be obtained by suspending a 1-dimensional framed submanifold of 3-space (see Theorem 11), we may use in the classification of maps of Σ^{n+1} into S^n for $n \geq 3$ the known classification of maps of the 3-sphere into the 2-sphere. Precisely, it is proved that if $\gamma(M^1, U)$ is even then $E(M^1, U) \sim 0$. It is thereby established that there cannot exist more than two classes of maps of Σ^{n+1} into S^n for $n \geq 3$.

The 'improvement' of a framed manifold inducing a homology.

Lemma 1. *Let (M^k_0, U_0) and (M^k_1, U_1) be two homologous framed submanifolds of the Euclidean space E^{n+k} and let (M^{k+1}, U) be a framed submanifold of the strip $E^{n+k} \times I$ inducing the homology $(M^k_0, U_0) \sim (M^k_1, U_1)$. We will call the point (x_0, t_0) critical – and the value t_0 of the parameter t a* ***critical value*** *– if the tangent T^{k+1}_0 to M^{k+1} at (x_0, t_0) lies in the hyperplane $E^{n+k} \times t_0$. It turns out that it is always possible to choose the framed manifold (M^{k+1}, U) inducing a homology between the given framed manifolds in such a way that* a) *there exist only finitely-many critical points of the manifold M^{k+1} and the critical values of t at distinct critical points are distinct;* b) *at an arbitrary critical point (x_0, y_0) of the manifold M^{k+1} an orthonormal basis $e_1, \cdots, e_{n+k}$ may be chosen so that, in the coordinates $x^1, \cdots, x^{n+k}$ corresponding to this basis, with origin at x_0, the manifold M^{k+1} is given near (x_0, t_0) by the equations*

$$t = t_0 + \sum_{i=1}^{k+1} \sigma^i (x^i)^2, \quad \sigma^i = \pm 1; \quad x^{k+2} = \cdots = x^{n+k} = 0, \tag{1}$$

and the frame $U = \{u_1(x, t), \cdots, u_n(x, t)\}$ is given near (x_0, t_0) by the formulae

$$u_1(x, t) = \sigma\Big(e - \sum_{i=1}^{k+1} 2\sigma^i x^i e_i\Big), \ \sigma = \pm 1; \ u_2(x, t) = e_{k+2}, \cdots, u_n(x, t) = e_{n+k}, \tag{2}$$

where e is the unit vector in the strip $E^{n+k} \times I$ in the direction of the t axis.

Proof. Let (N^{k+1}_0, V_0) be a framed manifold inducing the homology $(M^k_0, U_0) \sim (M^k_1, U_1)$. With each point $y = (x, t)$ of N^{k+1}_0 we associate the real number $f(y) = f(x, t) = t$. In view of Theorem 5 there exists a real-valued function $g(y)$, defined on N^{k+1}_0, coinciding with f near the boundary of N^{k+1}_0, and in ϵ-proximity of the first order to f, all of whose critical points are non-degenerate and such that the critical values at critical points are distinct. We now associate with each point $y = (x, t)$ of N^{k+1}_0 the point $\phi_s(y) = (x, t + s(g(y) - f(y))$, where s is a fixed number such that $0 \leq s \leq 1$. For sufficiently small ϵ the map ϕ_s is regular and homeomorphic. Thus ϕ_s is a deformation of the smooth submanifold N^{k+1}_0 into a submanifold $N^{k+1}_1 = \phi_1(N^{k+1}_0)$.

It is evident that the critical points of the function $g(y)$ coincide with the critical points of the manifold N_1^{k+1}. Thus condition a) above holds for the manifold $M^{k+1} = N_1^{k+1}$.

We subject the manifold N_1^{k+1} now to a further 'correction', so that condition b) may also be fulfilled.

Let $y_0 = (x_0, t_0)$ be an arbitrary critical point of N_1^{k+1} and let T_0^{k+1} be the tangent to N_1^{k+1} at the point y_0. The space T_0^{k+1} lies in the hyperplane $E^{n+k} \times t_0$, so that $T_0^{k+1} = T^{k+1} \times t_0$; $T^{k+1} \subset E^{n+k}$. In the space E^{n+k} we choose a basis $e_1, \cdots, e_{n+k}$ such that the vectors $e_1, \cdots, e_{k+1}$ lie in T^{k+1}. In the neighbourhood of the point (x_0, t_0) the manifold N_1^{k+1} is described in the corresponding coordinates by the equations

$$t = t_0 + \phi(x^1, \cdots, x^{k+1}) + \psi(x^1, \cdots, x^{k+1}), \tag{3}$$

$$x^{k+j} = \psi^j(x^1, \cdots, x^{k+1}), \quad j = 2, \cdots, n, \tag{4}$$

where ϕ is a non-singular quadratic form in the variables $x^1, \cdots, x^{k+1}$; ψ is a function which is small of the third order relative to $\xi = \sqrt{(x^1)^2 + \cdots + (x^{k+1})^2}$, and ψ^j are functions which are small of the second order relative to ξ. By choosing the axes in T^{k+1} in a suitable way, we can bring ϕ into the form

$$\phi = \sum_{i=1}^{k+1} \lambda^i (x^i)^2, \tag{5}$$

where λ^i is a non-zero real number. We begin by 'correcting' the manifold N_1^{k+1} in the neighbourhood of the point (x_0, t_0). Let $\chi(\eta)$ be a smooth real-valued monotone function of the variable $\eta = 0$, satisfying the conditions

$$\chi(\eta) = 0 \text{ for } 0 \le \eta \le \frac{1}{2}; \; \chi(\eta) = 1 \text{ for } \eta \ge 1.$$

We put

$$\chi(\xi, s) = s\chi\left(\frac{\xi}{\alpha}\right) + 1 - s,$$

where α is a sufficiently small positive number. We define the manifold N_{1+s}^{k+1} by the equations

$$t = t_0 + \sum_{i=1}^{k+1} \lambda^i (x^i)^2 + \chi(\xi, s)\psi(x^1, \cdots, x^{k+1}),$$

$$x^{k+j} = \chi(\xi, s)\psi^j(x^1, \cdots, x^{k+1}), \quad j = 2, \cdots, n,$$

for $\xi \le \alpha$, $|t - t_0| \le \alpha$, and we take $N_{1+s}^{k+1} = N_1^{k+1}$ elsewhere. It is clear that the submanifold N_{1+s}^{k+1} realises a smooth deformation of the submanifold N_1^{k+1} into the submanifold N_2^{k+1}, where the latter is given in $\xi \le \alpha$, $|t - t_0| \le \alpha$ by the equations

$$t = t_0 + \sum_{i=1}^{k+1} \lambda^i (x^i)^2 + \chi\left(\frac{\xi}{\alpha}\right)\psi(x^1, \cdots, x^{k+1}), \tag{6}$$

$$x^{k+j} = \chi\left(\frac{\xi}{\alpha}\right)\psi^j(x^1, \cdots, x^{k+1}), \quad j = 2, \cdots, n. \tag{7}$$

It is evident that in a sufficiently small neighbourhood of (x_0, t_0), namely, for $\xi < \frac{\alpha}{2}$, $|t - t_0| \le \alpha$, the manifold N_2^{k+1} is given by the equations

$$t = t_0 + \sum_{i=1}^{k+1} \lambda^i (x^i)^2; \; x^{k+j} = 0, \; j = 2, \cdots, n. \tag{8}$$

We show that N_2^{k+1} has no critical points distinct from those of N_1^{k+1}. For this it is sufficient to study the points of N_2^{k+1} given by the equations (6), (7) and satisfying the condition $\xi \le \alpha$, and show that among them only the point $\xi = 0$ is critical.

We have

$$\frac{dt}{dx^i} = 2\lambda^i(x^i, \theta^i),$$

where

$$2\lambda^i\theta^i = \frac{\chi'\left(\frac{\xi}{\alpha}\right)}{\alpha} \cdot \frac{x^i}{\xi} \cdot \psi(x^1, \cdots, x^{k+1}) + \chi\left(\frac{\xi}{\alpha}\right) \frac{\partial\psi(x^1, \cdots, x^{k+1})}{\partial x^i}.$$

Thus

$$|\theta^i| \le \frac{c_1}{\alpha}\xi^3 + c_2\xi^2.$$

It is clear that for sufficiently small α we have

$$|\theta^i| \le \frac{1}{k+1}\xi, \; \xi \le \alpha.$$

Now if for $\xi \le \alpha$ we have $\frac{dt}{dx_i} = 0$, $i = 1, \cdots, k+1$, then

$$x^i = -\theta^i. \tag{9}$$

Hence by squaring each of the equations (9) and summing we obtain $\xi^2 = \Sigma(\theta^i)^2 \le \frac{k+1}{(k+1)^2}\xi^2$, so that $\xi^2 \le \frac{1}{k+1}\xi^2$ and this is only possible if $\xi = 0$.

We now subject the manifold N_2^{k+1} to a further correction so that the equations for it near the critical point (x_0, t_0) take the form (1). We express the numbers λ^i in the form $\lambda^i = \frac{\sigma^i}{a_i^2}$, where the a_i are positive numbers and $\sigma^i = \pm 1$. Let α' be a sufficiently small positive number such that if $|x^i| \le \alpha'$, $|t - t_0| \le \alpha'$ the manifold N_2^{k+1} is given by the equations (8). We now define a smooth function $\kappa_a(\eta)$ of the variable η, $|\eta| \le \alpha'$, depending on the positive number a and satisfying the conditions

$$\kappa_a'(\eta) > 0; \; \kappa_a(\eta) = a\eta \text{ for } |\eta| \le \beta; \; \kappa_a(\eta) = \eta \text{ for } |\eta| > \frac{\alpha'}{2}.$$

Here β is a positive number so small that a function $\kappa_a(\eta)$ satisfying the stated conditions exists. We define the manifold N_{2+s}^{k+1} for $|x^i| \le \alpha'$ by the equations

$$t = t_0 + \Sigma\lambda^i((1-s)x^i + s\kappa_{a_i}(x^i))^2; \; x^{k+j} = 0, \; j = 2, \cdots, n, \tag{10}$$

and we take N_{2+s}^{k+1} to coincide with N_2^{k+1} elsewhere. It may be verified directly that the manifold N_{2+s}^{k+1} realises a smooth deformation of N_2^{k+1} into a manifold N_3^{k+1} and that the critical points of N_2^{k+1} and N_3^{k+1} coincide. Moreover the equations of N_3^{k+1} near the point (x_0, t_0) have the form (1).

By carrying out the reconstruction indicated near each critical point of the manifold N_1^{k+1} we construct a manifold M^{k+1}, and, since it is obtained from the manifold N_0^{k+1} by a series of consecutive smooth deformations, there must exist a frame V for the manifold M^{k+1} such that the framed manifold (M^{k+1}, V) induces the homology $(M_0^k, U_0) \sim (M_1^k, U_1)$. By choosing the number $\sigma = \pm 1$ in formula (2) in a suitable way we may ensure that the frames U and V determine the same orientation in M^{k+1} and so, near the critical point (x_0, t_0), it is possible so to deform the frame V that it passes into the frame U (see §11, D). By carrying out this correction near each critical point of the manifold M^{k+1}, we obtain the required frame U.

Thus Lemma 1 is completely proved.

The invariant δ of maps of Σ^{n+1} into S^n.

Theorem 20. *Let (M^1, U) be a 1-dimensional orthonormally framed submanifold of the oriented Euclidean space E^{n+1}, $n \geq 2$, $U(x) = \{u_1(x), \cdots, u_n(x)\}$. At each point $x \in M^1$ we draw the unit vector $u_{n+1}(x)$ tangent to the*[1] *curve M^1, so directed that the sequence $u_1(x), \cdots, u_n(x), u_{n+1}(x)$ determines the positive orientation of the space E^{n+1}. Further let $e_1, \cdots, e_{n+1}$ be an orthonormal basis of E^{n+1}, determining its positive orientation. Then*

$$u_i(x) = \sum_{j=1}^{n+1} h_{ij}(x)\, e_j, \quad i = 1, \cdots, n+1, \tag{11}$$

where $h(x) = \|h_{ij}(x)\|$ is an orthogonal matrix with positive determinant, depending continuously on $x \in M^1$. Thus h is a continuous map of M^1 into the manifold H_{n+1} of all rotations of the Euclidean space E^{n+1}. We set

$$\delta(M^1, U) \equiv \beta(h) + r(M^1) \pmod 2,$$

where $r(M^1)$ is the number of components of M^1 and the residue class $\beta(h)$ is defined in proposition (D) *of* §12. *It turns out that the residue class $\delta(M^1, U)$ is an invariant of the homology class of the framed manifold (M^1, U) so that if the map f of Σ^{n+1} into S^n corresponds to the framed manifold (M^1, U), then, by putting*

$$\delta(f) = \delta(M^1, U)$$

we obtain an invariant $\delta(f)$ of the homotopy class of the map f. The residue class $\delta(M^1, U)$ does not depend on the orientation of E^{n+1} nor on the particular choice of basis $e_1, \cdots, e_{n+1}$.

Proof. We prove first the invariance of $\delta(M^1, U)$ under change of basis. Let $e_1', \cdots, e_{n+1}'$ be another orthonormal basis of E^{n+1}, also determining its positive orientation; then

$$e_j = \sum_{k=1}^{n+1} a_{jk} e_k', \quad j = 1, \cdots, n+1,$$

where $a = \|a_{jk}\|$ is an orthogonal matrix with positive determinant. Using the basis $e_1', \cdots, e_{n+1}'$, there corresponds to the framed manifold (M^1, U) not the matrix

[1] Translator's note: Pontryagin uses the word 'curve' synonymously with 'one-dimensional manifold' – he does not assume connectedness.

$h(x)$ but the matrix $h'(x) = h(x)a$. Since the manifold H_{n+1} is connected, there exists a matrix $a_t \in H_{n+1}$, depending continuously on the parameter t, $0 \leq t \leq 1$, such that $a_1 = a$ and a_0 is the unit matrix. The map $h_t = ha_t$ realises a continuous deformation of h into h' so that h is homotopic to h' and thus the residue class $\delta(M^1, U)$ does not depend on the choice of basis $e_1, \cdots, e_{n+1}$.

We now show that $\delta(M^1, U)$ is independent of the orientation of E^{n+1}. Under change of orientation of E^{n+1} the vector $u_{n+1}(x)$ is replaced by $-u_{n+1}(x)$ and, in the basis $e_1, \cdots, e_{n+1}$, the orientation may be changed by replacing e_{n+1} by $-e_{n+1}$. With these changes we obtain instead of $h(x)$ the matrix $h'(x)$, obtained from $h(x)$ by multiplying by -1 the last row and column. We associate with each matrix $l \in H_{n+1}$ the matrix l' obtained from l by multiplying by -1 the last row and column. If we take for E^2 the plane with basis e_1, e_2, then it is seen that under the map $l \to l'$ the curve H_2, which is not nullhomotopic in H_{n+1} (this is described in Theorem 17), is mapped identically on to itself. Thus $\delta(M^1, U)$ does not alter under change of orientation of E^{n+1}.

Finally we prove the principal property of the residue class $\delta(M^1, U)$ – its invariance with respect to the choice of the framed manifold (M^1, U) from its homology class.

Let (M_0^1, U_0) and (M_1^1, U_1) be two framed submanifolds of the space E^{n+1} and let (M^2, U) be a framed submanifold of the strip $E^{n+1} \times I$ inducing the homology $(M_0^1, U_0) \sim (M_1^1, U_1)$ and chosen to satisfy conditions (a) and (b) of Lemma 1. The intersection $M^2 \cap (E^{n+1} \times t)$ lies in $E^{n+1} \times t$ and so has the form $M_t^1 \subset E^{n+1}$. It is easy to see that, if (x, t) is not a critical point of the surface M^2, the set M_t^1 is, near the point x, a smooth curve, so that, if t is not a critical value of the parameter, M_t^1 is a smooth submanifold of E^{n+1}. We construct a frame V_t of the manifold M_t^1. Let (x, t) be a non-critical point of the manifold M^2, let $V(x, t) \times t$ be the orthogonal projection of the system of vectors $U(x, t)$ on to the hyperplane $E^{n+1} \times t$, and let $V_t(x)$ be the system obtained from $V(x, t)$ by means of the orthogonalization process (see §7, G). Since all the vectors of the system $U(x, t)$ are orthogonal to M^2 at the point (x, t), it follows that all the vectors of the system $V(x, t)$ are orthogonal to M_t^1 at the point x. Since (x, t) is not a critical point it follows that the vectors of the system $V(x, t)$ are linearly independent. Thus the system $V_t(x)$ constitutes a frame for the manifold M_t^1 for non-critical values of t. To the framed manifold (M_t^1, V_t) there corresponds a map h_t of M_t^1 into H_{n+1}. From general continuity considerations it follows that when the parameter t varies continuously without passing through a critical value, the residue class $\delta(M_t^1, V_t)$ reamins unchanged. We prove that it also remains unchanged when the parameter t passes through a critical value t_0. The invariance of $\delta(M^1, U)$ will follow from this in view of the relations $V_0 = U_0$, $V_1 = U_1$.

Let (x_0, t_0) be the unique critical point of the manifold M^2 at which the parameter t has the critical value t_0. Near the point (x_0, t_0) the manifold M^2 is given by the equations

$$t = t_0 + \sigma^1(x^1)^2 + \sigma^2(x)^2,\ \sigma^1 = \pm 1,\ \sigma^2 = \pm 1,$$
$$x^3 = \cdots = x^{n+1} = 0$$

(see (1)). It follows from this that if t is near to t_0 the equation of the manifold M_t^1 near x_0 has the form

$$\sigma^1(x^1)^2 + \sigma^2(x^2)^2 = t - t_0;\ x^3 = \cdots = x^{n+1} = 0. \tag{12}$$

Further it follows from (2) that the system $V_t(x)$, for sufficiently small $|t - t_0|$ and for x near x_0, is given by the formulae

$$(v_t)_1(x) = \sigma\left(\sigma^1 \frac{x^1}{\xi} e_1 + \sigma^2 \frac{x^2}{\xi} e_2\right);\ (v_t)_j = e_{j+1},\ j = 2, \cdots, n, \tag{13}$$

where $\xi = \sqrt{(x^1)^2 + (x^2)^2}$. To study the residue class $\delta(M_t^1, V_t)$ we take for E^2 (see Theorem 17) the plane with basis e_1, e_2. We consider the two distinct cases: 1) $\sigma^1 = \sigma^2$ and 2) $\sigma^1 = -\sigma^2$.

In case 1 we will assume for definiteness that $\sigma^1 = \sigma^2 = -1$. Under this hypothesis the manifold M_t^1, with $t < t_0$, contains a component determined by equations (12) which is a circle of small radius in the ordinary metric. We denote this component by S^1. It is easy to see that the map h_t maps the circle S^1 on to the circle H_2 with degree 1. For $t > t_0$ the component given by equations (12) becomes imaginary, that is, vanishes, while at the same time all the other components of M_t^1 together with their frames vary continuously. Thus in the first case as the parameter t passes through a critical value t_0, the residue class $\beta(h_t)$ changes by 1 and so does the number of components of M_t^1 so that the residue class $\delta(M_t^1, V_t)$ does not vary.

In case 2 the set $M_{t_0}^1$ near the point x_0 is described by the equation $(x^1)^2 - (x^2)^2 = 0$, that is, it is a cross K_{t_0}, the union of two segments intersecting in a point. From this we see that the component L_{t_0} of M_{t_0} containing the cross K_{t_0} is homeomorphic to a lemniscate. Since the surface M^2 is orientable, the neighbourhood of the lemniscate L_{t_0} on M^2 is homeomorphic to a doubly-connected plane region and hence it may be seen that the part L_t of the set M_t^1 lying near the lemniscate L_{t_0} consists of two components S_1^1 and S_2^1 if t lies on one side of t_0, and consists of one component $\hat{S}^1$ if t lies on the other side of t_0. If we denote the residue classes $\beta(h_t)$ corresponding to the components S_1^1, S_2^1 and $\hat{S}^1$ by β_1, β_2 and $\hat{\beta}$ then to prove the invariance of δ it is sufficient to prove that $\beta_1 + \beta_2 \equiv \hat{\beta} + 1 \pmod 2$. We prove this. We denote by K_t the part of the curve L_t lying near the cross K_{t_0}. This part is described by the equation $(x^1)^2 - (x^2)^2 = \sigma^1(t - t_0)$,

and so is a hyperbola. From formula (13) we see that $h_t(K_t) \subset H_2$ and moreover that for $t < t_0$ the set $h_t(K_t)$ covers two quarters of the circle H_2 and for $t > t_0$ it covers the other two quarters. In view of Theorem 17 the map h_t of L_t is homotopic to a map h_t' for which $h_t'(L_t) \subset H_2$ and such that h_t and h_t' agree on K_t. From what has been said it follows easily that the sum of the degrees of the maps h_t' of the curves S_1^1 and S_2^1 for $t < t_0$ differs by unity from the degree of the map h_t' of the curve $\hat{S}^1$ for $t > t_0$. Thus $\beta_1 + \beta_2 \equiv \hat{\beta} + 1 \pmod 2$ and the invariance of the residue class $\delta(M^1, U)$ is completely proved.

Theorem 20 is thereby proved.

We pick out some easily verified properties of the invariant $\delta(M^1, U)$.

A) Let Π_n^1 be the group of homology classes of framed 1-dimensional submanifolds of the Euclidean space E^{n+1}. Since $\delta(M^1, U)$ is an invariant of homology class, we may put $\delta(\pi) = \delta(M^1, U)$, where (M^1, U) is a framed manifold in the class $\pi \in \Pi_n^1$. It may be verified immediately that δ is a homomorphism of Π_n^1 into the group of residue class mod 2. It is, further, clear that if $E\pi$ is the suspension of π, that is, $E(M^1, U) \in E\pi$ (see §8), then $\delta(E\pi) = \delta(\pi)$.

The classification of maps of Σ^{n+1} into S^n.

Theorem 21. *For $n \geq 3$ the homomorphism δ of Π_n^1 into the group of residue classes mod 2 is an isomorphism (onto), so that Π_n^1 is cyclic of order 2. Thus there exist precisely two homotopy classes of maps of Σ^{n+1} into $S^n (n \geq 3)$. Further the homomorphism δ of Π_2^1 is epimorphic and since Π_2^1 is mapped isomorphically on to the group of integers by the isomorphism γ (see* Theorem 19), *it follows that the homomorphism $\delta\gamma^{-1}$ of the group of integers on to the group of integers mod 2 is just reduction mod 2.*

Proof. Let (S^1, U) be an orthonormally framed submanifold of Euclidean space E^{n+1}, S^1 being homeomorphic to a circle, $U(x) = \{u_1(x), \cdots, u_n(x)\}$. To calculate the invariant $\delta(S^1, U)$ we denote by $u_{n+1}(x)$ a suitably directed unit vector tangent to S^1 at the point x, and let $e_1, \cdots, e_{n+1}$ be a basis for E^{n+1}. We have

$$u_i(x) = \sum_{j=1}^{n+1} h_{ij}(x) e_j,\ i = 1, \cdots, n+1, \tag{14}$$

so that $h(x) = \|h_{ij}(x)\|$ is an orthogonal matrix with positive determinant, and h is a continuous map of the circle S^1 into H_{n+1}. In view of the definition of the invariant $\delta(M^1, U)$ (see Theorem 20), we have

$$\delta(S^1, U) \equiv \beta(h) + 1 \pmod 2. \tag{15}$$

Further let $g(x) = \|g_{ij}(x)\|$ be an orthogonal matrix of order n with positive determinant, such that g is a continuous map of S^1 into H_n. We set

$$v_i(x) = \sum_{j=1}^{n} g_{ij}(x) u_j(x),\ i = 1, \cdots, n,$$

and denote by $g[U]$ the frame $V(x) = \{v_1(x), \cdots, v_{n+1}(x)\}$. To compute the invariant $\delta(S^1, g[U])$ we put $v_{n+1}(x) = u_{n+1}(x)$ and denote by $g'(x)$ the matrix of order $(n+1)$ obtained from $g(x)$ by adding elements $g_{i,n+1}(x)$ and $g_{n+1,i}(x)$, where only $g_{n+1,n+1}(x)$ is non-zero, and is equal to 1. It is clear that for $n \geq 2$ we have

$$\beta(g') = \beta(g)$$

(see §12, D). Further we have

$$v_i(x) = \sum_{j,k=1}^{n+1} g'_{ij}(x) h_{jk}(x) e_k, \quad i = 1, \cdots, n+1.$$

Thus in view of proposition (D) of §12 we have

$$\delta(S^1, g[U]) = \beta(g'h) + 1 = \beta(g') + \beta(h) + 1 = \delta(S^1, U) + \beta(g). \qquad (17)$$

It immediately follows from Theorem 11 and proposition (B) of §13 that, for each framed manifold (M^1, W) of the Euclidean space E^{n+1} there is a homology

$$(M^1, W) \sim E^{n-2}(S^1, V_{(r)}), \qquad (18)$$

where $(S^1, V_{(r)})$ is the framed submanifold of Euclidean 3-space E^3 constructed in proposition (A) of §13, and E^{n-2} is the suspension operation iterated $(n-2)$ times.

We have

$$V_{(r)} = g_{(r)}[V_{(0)}], \qquad (19)$$

where

$$g_{(r)}(x) = \begin{Vmatrix} \cos rx & \sin rx \\ -\sin rx & \cos rx \end{Vmatrix}$$

(see §13, A). Thus

$$\beta(g_{(r)}) \equiv r \pmod 2. \qquad (20)$$

It may be immediately verified that $\delta(S^1, V_{(0)}) = 0$. Hence, in view of (17), (19) and (20) it follows that

$$\delta(S^1, V_{(r)}) \equiv r \pmod 2. \qquad (21)$$

Since $\gamma(S^1, V_{(r)}) = r$ (see §13, A) it follows from (21) that the homomorphism $\delta\gamma^{-1}$ of the group of integers into the group of residue classes mod 2 is just reduction mod 2. The second part of Theorem 21 is thus proved.

Further we have

$$EV_{(r)} = g'_{(r)}[EV_{(0)}],$$

where

$$g'_{(r)}(x) = \begin{Vmatrix} \cos rx & \sin rx & 0 \\ -\sin rx & \cos rx & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

so that $\beta(g'_{(r)}) \equiv r \pmod 2$. Since $\gamma(S^1, V_{(0)}) = 0$, $(S^1, V_{(0)}) \sim 0$ (see Theorem 19) and so $E(S^1, V_{(0)}) \sim 0$. Consequently $E(S^1, V_{(r)}) \sim 0$ if the map $g'_{(r)}$ of the circle S^1 into H_3 is nullhomotopic (see §7, H), which is true if r is even. So $E(S^1, V_{(r)}) \sim 0$ if $\delta(E(S^1, V_{(r)})) = 0$. Hence and from relation (18) it follows that, for $n \geq 3$,

$\delta(M^1, W) = 0$ implies that $(M^1, W) \sim 0$. Since $\delta(E^{n-2}(S^1, V_{(1)}) = 1$ the framed manifold $E^{n-2}(S^1, V_{(1)})$ is not nullhomologous. Thus it is established that the homomorphism δ maps the group Π^1_n isomorphically on to the group of residue classes mod 2.

Theorem 21 is thereby proved.

§15. Classification of maps of the $(n+2)$-sphere into the n-sphere.

It is proved in this section that for $n \geq 2$ there are exactly two homotopy classes of maps of Σ^{n+2} into S^n. The proof is based on the construction of a homology invariant $\delta(M^2, U)$ of the framed manifold (M^2, U) in Euclidean space E^{n+2}, which is a residue class mod 2 and can take either of the values 0 or 1. Thus, just from the existence of the invariant δ, it follows that there exist at least two classes of maps of Σ^{n+2} into S^n. The invariant δ may be described in the following way. Let $U(x) = \{u_1(x), \cdots, u_n(x)\}$ be an orthonormal frame for a manifold M^2 and let C be a smooth simple closed curve on M^2. The unit normal to C touching the surface M^2 at the point $x \in C$ we denote by $u_{n-1}(x)$ and we set $V(x) = \{u_1(x), \cdots, u_{n+1}(x)\}$. The invariant $\delta(C, V)$ (see §14) is defined for the 1-dimensional framed manifold (C, V); we denote it in the given case by $\delta(C)$. We suppose first that M^2 is a connected surface whose genus we designate by p. There exists on M^2 a system of smooth simple closed curves $A_1, \cdots, A_p, B_1, \cdots, B_p$ such that A_i and B_i intersect, but do not touch, at a single point, $i = 1, \cdots, p$, but no two other curves intersect at all. It turns out that the residue class

$$\delta(M^2, U) = \sum_{i=1}^{p} \delta(A_i)\delta(B_i)$$

does not depend on the choices made in the construction and is a homology invariant of the framed manifold (M^2, U). If the surface M^2 is not connected the invariant δ is defined to be the sum of its values on the components.

It follows from Theorems 11 and 16 that the number of classes of maps of Σ^{n+2} into S^n does not exceed the number of classes of maps of Σ^4 into S^2. The number of such classes of maps of Σ^4 into S^2 is, in view of Lemma 2 of §13, no greater than the number of classes of maps of Σ^4 into S^3 and this last is, in view of Theorem 21, equal to 2. Thus it is established that the number of classes of maps of Σ^{n+2} into S^n is no greater than two.

A) Let M^2 be an orientable surface, that is, a smooth closed orientable manifold of dimension 2, and let M^1 be a curve, that is, a smooth closed 1-dimensional manifold. Further let f be a smooth regular map of M^1 into M^2 such that no three distinct points of M^1 are mapped to the same point of M^2. We will further make the hypothesis about the map f that if the two distinct points a and b of M^1 are mapped by f to the same point $c = f(a) = f(b)$, then neighbourhoods of a and b on M^1 are transformed by f into curves having different tangents at c. Under these conditions the set $C = f(M^1)$ is called a *smooth curve* on the surface M^2. If M^1 is oriented the curve $C = f(M^1)$ is also to be regarded as *oriented*. A point of the form $c = f(a) = f(b)$, where $a \neq b$, is called a *double* point of the curve C. It is easy to see that a curve on a surface has only finitely-many double points. If $C = f_1(M_1^1) = f_2(M_2^1)$, that is, if the curve C may be obtained as the result of two distinct maps f_1 and f_2 of distinct curves M_1^1 and M_2^1, such that f_1 and f_2 satisfy the conditions listed above, then there exists a smooth homeomorphism ϕ of M_1^1 on to M_2^1 such that $f_2\phi = f_1$. In view of this it is possible to define the components of C as the images of the components of the curve M^1. We will not exclude the case of an empty curve. It is easy to see that if $C = f(M^1)$ is a curve on the surface then, provided the map f' is sufficiently close to the map f in the sense of proximity of class 1, the set $f'(M^1)$ is also a curve on the surface. We will say that the curve $C' = f'(M^1)$ is obtained from the curve C by a *small displacement*.

B) The curve C on the surface M^2 is said to be *nullhomologous* (more precisely, nullhomologous mod 2) if there exists on M^2 an open set G such that $C = \overline{G} - G$ and that in any neighbourhood of a point $x \in C$ there are points of M^2 not belonging to $\overline{G}$; in symbols, $C = \Delta G$; $C \sim 0$. It is evident that a small displacement transforms a nullhomologous curve into a nullhomologous curve. Let C_1 and C_2 be two curves on M^2 such that the double points of one do not belong to the other and that at each point of intersection of the two curves the tangents to them are distinct. In this case $C_1 \cup C_2$ is again a curve and we will say that the curves C_1 and C_2 *admit addition*, writing $C_1 + C_2$ for the curve $C_1 \cup C_2$. It is easy to see that if two arbitrary curves are given on M^2 then, by subjecting one of them to a suitably chosen small displacement we obtain two curves which admit addition. If two curves C_1 and C_2 are nullhomologous and admit addition, their sum is again nullhomologous. In fact, let $C_1 = \Delta G_1$, $C_2 = \Delta G_2$. We set $G = (G_1 \cup G_2) - (\overline{G}_1 \cap \overline{G}_2)$. It is easy to see that $C_1 + C_2 = \Delta G$. We will write the relation $C_1 + C_2 \sim 0$ instead as $C_1 \sim C_2$. Then the relation $C_1 \sim C_2$ only makes sense if C_1 and C_2 admit addition. If the curves C_1 and C_2 do not admit addition then subjecting one of them, for instance C_1, to a small displacement we obtain curves C_1' and C_2 admitting addition. If moreover $C_1' \sim C_2$, then we write $C_1 \sim C_2$. This definition is legitimate since the relation $C_1 \sim C_2$ is independent of the choice of curve C_1'. The relation is reflexive, symmetric and transitive, so that the curves on the surface M^2 are divided into homology classes. We denote the totality of these classes by $\Delta^1 = \Delta^1(M^2)$. We introduce an addition operation into Δ^1. If z_1, z_2 are two elements of Δ^1 and $C_1 \in z_1$, $C_2 \in z_2$ are curves admitting addition, then the class z containing the curve $C_1 + C_2$ is, by definition, the sum of the classes z_1 and z_2, $z = z_1 + z_2$. The group Δ^1 is called the *connectivity group* of the surface M^2. All its non-zero elements are of order 2. A finite system of curves $C_1, \cdots, C_q$ on M^2 is called a *homology basis* if, given any curve C on the surface M^2, we have a relation

$$C \sim \sum_{i=1}^{q} \epsilon_i C_i,$$

where $\epsilon_i \equiv 0$ or 1 (mod 2) and if it follows from the relation

$$C \sim 0$$

that the residue classes ϵ_i are zero.

C) Let C_1 and C_2 be two curves on M^2 admitting addition. We denote by $J(C_1, C_2)$ the number of points of intersection of C_1 and C_2, reduced mod 2, and call $J(C_1, C_2)$ the *intersection index*. It is easy to see that

$$J(C_1 + C_2, C_3) = J(C_1, C_3) + J(C_2, C_3)$$

and that $C_1 \sim 0$ implies $J(C_1, C_2) = 0$. From this it follows that if $C_1 \sim D_1$, $C_2 \sim D_2$, then $J(C_1, C_2) = J(D_1, D_2)$. Thus, putting $J(z_1, z_2) = J(C_1, C_2)$, where $C_1 \in z_1$, $C_2 \in z_2$, we obtain a well-defined *intersection index* of two homology classes. It turns out that, on any surface M^2, there exists a homology basis consisting of curves $A_1, \cdots, A_p, B_1, \cdots, B_p$ such that

$$J(A_i, A_j) = J(B_i, B_j) = 0;\ J(A_i, B_j) = \delta_{ij};\ i, j = 1, \cdots, p. \tag{1}$$

Such a basis is called *canonical*. It immediately follows that for any homology class $z \in \Delta^1$ we have the relation

$$J(z, z) = 0,$$

and further that, if z_1 is a non-zero class, then there exists a class z_2 such that

$$J(z_1, z_2) = 1.$$

If M^2 is connected we may take for the curves $A_1, \cdots, A_p, B_1, \cdots, B_p$ curves giving a *canonical dissection* of the surface M^2. In this case p is the genus of the surface. If M^2 is not connected the required homology basis may be obtained as the union of bases of the components; in this case p is the sum of the genera of the components of M^2.

Theorem 22. *Let (M^2, U) be an orthonormally framed surface in the oriented Euclidean space E^{n+2} with basis $e_1, \cdots, e_{n+2}$ yielding the orientation, let $U(x) = \{u_1(x), \cdots, u_n(x)\}$, and let $C = f(M^1)$ be an oriented curve on M^2. Let $y \in M^1$. We denote by $\hat{u}_{n+2}(y)$ the unit vector tangent at $f(y)$ to the curve $f(M^1)$ and oriented as the curve and we denote by $\hat{u}_{n+1}(y)$ the unit vector tangent to M^2 at $f(y)$, orthogonal to $\hat{u}_{n+2}(y)$ and directed so that the vectors $u_1(f(y)), \cdots, u_n(f(y))$, $\hat{u}_{n+1}(y), \hat{u}_{n+2}(y)$ give the positive orientation to E^{n+2}. For convenience of notation we also put $\hat{u}_i(y) = u_i(f(y))$, $i = 1, \cdots, n$. We have*

$$\hat{u}_i(y) = \sum_{j=1}^{n+2} h_{ij}(y)\, e_j;\ i = 1, \cdots, n+2,$$

where $h(y) = \|h_{ij}(y)\|$ is an orthogonal matrix with positive determinant, so that h is a continuous map of the manifold M^1 into the group H_{n+2}. We set

$$\delta(M^2, U, C) = \delta(C) \equiv \beta(h) + r(C) + s(C), \tag{2}$$

where $\beta(h)$ is defined in proposition (D) of § 12, $r(C)$ is the number of components of the curve C, and $s(C)$ is the number of its double points. It turns out that $\delta(C)$ is an invariant of the homology class $z \in \Delta^1$ containing the curve C, so that we may set $\delta(M^2, U, z) = \delta(z) = \delta(C)$. Further it turns out that for two arbitrary homology classes z_1 and z_2 of M^2 we have

$$\delta(z_1 + z_2) = \delta(z_1) + \delta(z_2) + J(z_1, z_2). \tag{3}$$

Proof. We prove first that the residue class $\beta(h)$ does not depend on the basis $e_1, \cdots, e_{n+2}$ nor on the orientation of the curve $C = f(M^1)$. If we take another basis $e_1', \cdots, e_{n+2}'$ in place of $e_1, \cdots, e_{n+2}$ then we have

$$e_j = \sum_{k=1}^{n+2} l_{jk} e_k'; \; j = 1, \cdots, n+2,$$

where $l = \|l_{jk}\|$ is an orthogonal matrix with positive determinant. By the change of basis the matrix $h(y)$ is replaced by the matrix $h'(y) = H(y)l$. Since H_{n+2} is connected the maps h and h' are homotopic and so it follows that $\beta(h)$ does not depend on the choice of basis $e_1, \cdots, e_{n+2}$. If now we reverse the orientation of the connected curve $C = f(S^1)$ it becomes necessary to replace the vectors $\hat{u}_{n+1}(y)$ and $\hat{u}_{n+2}(y)$ by $-u_{n+1}(y)$ and $-u_{n+2}(y)$. The matrix $h(y)$ is thereby transformed into the matrix $h'(y) = l \cdot h(y)$, where $l_{ij} = 0$, for $i \neq j$,

$$l_{11} = \cdots = l_{nn} = 1, \; l_{n+1,n+1} = l_{n+2,n+2} = -1.$$

Since the matrix $l = \|l_{ij}\|$ belongs to H_{n+2}, the maps h and h' are homotopic, and so the value of $\beta(h)$ does not depend on the orientation of the connected curve.

Evidently the same conclusion is valid for arbitrary curves.

To prove that $\delta(C)$ is an invariant of the homology class z containing the curve C, we introduce a reconstruction operation on the oriented curve C near a double point a, as a result of which the curve is transformed into the oriented curve $C_a = f_a(M_a^1)$. The map of M_a^1 into H_{n+2} corresponding to the curve C_a will be denoted by h_a. The reconstruction operation will be defined in such a way that the curve C_a has one double point fewer than C and moreover fulfills the conditions

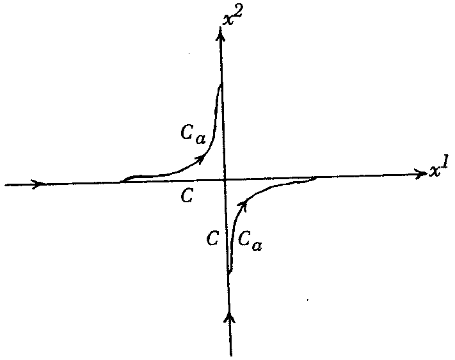

Fig. 2.

$$C_a \sim C, \;\; \delta(C_a) = \delta(C).$$

In view of proposition (C) and (D) of § 11 it may be supposed that near the double point a the surface M^2 coincides with the plane E^2, the curve C with two

intersecting straight lines and the vectors $u_i(x)$ with the vectors e_i, $i = 1, \cdots, n$. We take these lines as axes of a coordinate system x^1, x^2 defined near the point a on M^2. We choose the directions of the axes so that movement along the curve in the positive direction corresponds to an increase in the coordinates. We will take the curve C_a to coincide with C outside a neighbourhood of the point a and to be given near the point a by the equation $x^1 \cdot x^2 = -\epsilon$ where $\epsilon > 0$ (see Fig. 2). In this way the orientation of C is naturally transferred to C_a. It is easy to verify that if both branches of the curve C passing through a belong to one of its components then, after the reconstruction, instead of this component there appear two distinct components of C_a. On the other hand, if the two branches of the curve C passing through a belong to distinct components of the curve C then, as a result of the reconstruction, instead of these two components there appears a single component of the curve C_a. Thus in both cases $r(C) + s(C) \equiv r(C_a) + s(C_a) \pmod 2$.

We show that $\beta(h) = \beta(h_a)$. Indeed the transformation hf^{-1} maps a neighbourhood of the point a on the curve C into two points of the circle H_2 (see Theorem 12), and the transformation $h_a f_a^{-1}$ transforms the part of C_a near the point a into the circle H_2 with degree 0. It follows from this that $\delta(C) = \delta(C_a)$. It is evident that the curves C and C_a are homologous.

As a result of a finite number of such reconstructions, we obtain from C a curve $O(C)$ without double points and fulfilling the conditions

$$O(C) \sim C, \;\; \delta(O(C)) = \delta(C). \tag{4}$$

We show now that if the curve C is without double points and is nullhomologous on M^2, then

$$\delta(C) = 0. \tag{5}$$

Let $C = \Delta G$; then $\bar{G}$ is a smooth surface bounded by the curve C. It is easy to define on $\bar{G}$ a smooth function χ, positive and less than 1 in G and zero on C, whose derivative does not vanish on C. In the strip $E^{n+2} \times I$, where I is the interval $0 \leq t \leq 1$, we consider the surface P^2 given by the equation

$$t = +\sqrt{\chi(x)}, \qquad x \in \bar{G}.$$

It is easy to see that P^2 has the curve $C \times 0$ as its boundary and is orthogonal to the boundary set $E^{n+2} \times 0$ of the strip $E^{n+2} \times I$. Since the surface P^2 is homeomorphic to the orientable surface $\bar{G}$ we may regard it as oriented. Since the vectors of the system $U(x)$ are orthogonal to the surface $\bar{G}$ at the point x, the vectors of the system $U(x) \times t$ are orthogonal to the surface at the point $x \times t$. We adjoin to the system $U(x) \times t$ a unit vector $u_{n+1}(x, t) \times t$ so that the system $\tilde{U}(x, t) \times t$ obtained provides an orthonormal frame for the oriented surface P^2 in the oriented strip $E^{n+2} \times I$. The vector $u_{n+1}(x, 0)$ so obtained is orthogonal to the curve C and touches the surface M^2 at the point x. Thus by adjoining the vector $u_{n+1}(x, 0)$ to the system $U(x)$ we obtain a frame $V(x)$ for the curve C such that the framed curve (C, V) is nullhomologous. Further by adjoining to the system $V(x)$ a vector $u_{n+2}(x)$, tangent to the curve C at the point x, we obtain a system $\hat{U}(x)$ on the basis of which we may calculate the residue class $\delta(C)$. By comparing the construction of the residue class $\delta(C)$, given here, with the construction of the residue class $\delta(C, V)$ (see Theorem 20) we see that

$$\delta(C) = \delta(C, V).$$

Since the framed manifold (C, V) is nullhomologous, $\delta(C) = \delta(C, V) = 0$ (see Theorem 20). Thus relation (5) is proved.

Let C_1 and C_2 be two arbitrary curves on M^2 admitting addition. We have [1)]

$$s(C_1 + C_2) \equiv s(C_1) + s(C_2) + J(C_1, C_2) \pmod 2,$$

$$r(C_1 + C_2) = r(C_1) + r(C_2),$$

$$\beta(C_1 + C_2) = \beta(C_1) + \beta(C_2).$$

Hence it follows that

$$\delta(C_1 + C_2) = \delta(C_1) + \delta(C_2) + J(C_1, C_2). \tag{6}$$

If in particular $C_1 \sim C_2$, then $J(C_1, C_2) = 0$ and from relations (6), (5) we obtain

$$\delta(C_1) + \delta(C_2) = \delta(C_1 + C_2) = \delta(O(C_1 + C_2)) = 0.$$

It is thus proved that $\delta(C)$ is a homology invariant. From this and from relation (6) applied to arbitrary curves C_1 and C_2 admitting addition the validity of formula (3) follows.

Thus Theorem 22 is proved.

Theorem 23. *Let* (M^2, U) *be an orthonormally framed submanifold of Euclidean space* E^{n+2} *and*

$$A_1, \cdots, A_p, B_1, \cdots, B_p \tag{7}$$

an arbitrary canonical basis of the surface M^2. *It turns out that the residue class*

$$\delta = \delta(M^2, U) = \sum_{i=1}^{p} \delta(A_i)\delta(B_i) \tag{8}$$

does not depend on the special choice of canonical basis (7) *and is an invariant of the framed manifold* (M^2, U).

Proof. We consider an arbitrary canonical basis

$$A_1', \cdots, A_p', B_1', \cdots, B_p' \tag{9}$$

of the surface M^2 and show that

$$\sum_{i=1}^{p} \delta(A_i)\delta(B_i) = \sum_{i=1}^{p} \delta(A_i')\delta(B_i'). \tag{10}$$

A direct proof of equation (10) for arbitrary canonical bases (7) and (9) presents considerable computational difficulties, so we will consider 3 special types

1) Translator's note: The author permits himself to write '$\beta(C)$' for '$\beta(h)$' in view of what has already been proved.

of transformation of canonical bases and show, fairly easily, that for each separate type of transformation formula (10) is valid. In conclusion it will be proved that any transformation from one arbitrary canonical basis (7) to another (9) may be effected by a sequence of applications of the special transformations. Thus the invariance of δ will be completely demonstrated.

Transformation 1. Let j be an integer not exceeding p. We put

$$A_j' = B_j,\ B_j' = A_j;\ A_i' = A_i,\ B_i' = B_i,\ i \neq j. \tag{11}$$

It is obvious that the basis $A_1', \cdots, A_p', B_1', \cdots, B_p'$ defined by these relations is again canonical and that relation (19) holds in this case.

Transformation 2. We put

$$A_i' = \sum_{k=1}^{p} a_{ik}A_k, \quad i = 1, \cdots, p, \tag{12}$$

$$B_j' = \sum_{k=1}^{p} b_{jk}B_k, \quad j = 1, \cdots, p, \tag{13}$$

where a_{ik} and b_{jk} are residue classes mod 2. For the basis (12) – (13) to be canonical it is necessary that the matrix $a = \|a_{ij}\|$ be non-singular, that is, have determinant 1, and that the matrix $b = \|b_{ij}\|$ should be connected to the matrix a by the relation

$$\sum_k a_{ik}b_{jk} = \delta_{ij}, \tag{14}$$

that is, denoting the unit matrix by e and the transpose of b by b', $ab' = e$ or $a^{-1} = b'$, whence $b'a = e$. The last relation gives

$$\sum_{i=1}^{p} a_{ij}b_{ik} = \delta_{jk}. \tag{15}$$

Thus

$$\sum_i \delta(A_i')\delta(B_i') = \sum_i \delta\left(\sum_{j=1}^{p} a_{ij}A_j\right)\delta\left(\sum_{k=1}^{p} b_{ik}B_k\right) =$$

$$= \sum_{i,j,k=1}^{p} a_{ij}b_{ik}\delta(A_j)\delta(B_k) = \sum_{j,k=1}^{p} \delta_{jk}\delta(A_j(\delta(B_k) = \sum_{j=1}^{p} \delta(A_j)\delta(B_j)$$

(see (3)) and relation (10) holds for transformation 2. We remark that transformation 2 is uniquely determined by the matrix a giving the equations (12). The equations (13) are, in view of formula (14), uniquely determined by the equations (12). We will say that the equations (12) and (13) are *compatible* if relation (14) holds.

Transformation 3. We put

$$A_i' = A_i + \sum_k c_{ik}B_k, \quad i = 1, \cdots, p, \tag{16}$$

$$B_i' = B_i, \qquad i = 1, \cdots, p. \tag{17}$$

In order that $J(A_i', A_j') = 0;\ i, j = 1, \cdots, p$, it is necessary and sufficient that

$$c_{ij} = c_{ji}. \tag{18}$$

In fact,

$$J(A_i', A_j') = J(A_i, \sum_k c_{jk}B_k) + J(\sum_k c_{ik}B_k, A_j) =$$
$$= \sum_k c_{jk}\delta_{ik} + \sum_k c_{ik}\delta_{kj} = c_{ji} + c_{ij}.$$

If the relation (18) is fulfilled then the basis (16) – (17) is canonical. We prove that for transformation 3 the relation (10) holds. We have

$$\sum_i \delta(A_i')\delta(B_i') = \sum_i \delta(A_i + \sum_k c_{ik}B_k)\delta(B_i) =$$
$$= \sum_i (\delta(A_i) + \sum_k c_{ik}\delta(B_k) + \sum_k c_{ik}J(A_i, B_k))\delta(B_i) =$$
$$= \sum_i \delta(A_i)\delta(B_i) + \sum_{i,j} c_{ik}\delta(B_k)\delta(B_i) + \sum_{i,k} c_{ik}\delta_{ik}\delta(B_i).$$

Further

$$\sum_{i,k} c_{ik}\delta(B_i)\delta(B_k) = \sum_i c_{ii}\delta(B_i)\delta(B_i) = \sum_i c_{ii}\delta(B_i)$$

(since the calculations are taken mod 2) and

$$\sum_{i,k} c_{ik}\delta_{ik}\delta(B_i) = \sum_i c_{ii}\delta(B_i).$$

Thus relation (10) is fulfilled.

We consider now an arbitrary transformation from the canonical basis (7) to an arbitrary canonical basis (9). We have

$$A_i' = \sum_j r_{ij}A_j + \sum_k s_{ik}B_k. \tag{19}$$

The rank of the rectangular matrix of p rows and $2p$ columns defined by this transformation is equal to p, that is, one of its p-rowed minors is non-zero. By applying to the basis (7) sufficiently often a transformation of type 1 we may arrive at a new basis, which we again denote by $A_1, \cdots, A_p, B_1, \cdots, B_p$, such that in formula (19) the minor $|r_{ij}|$ is non-zero. Applying to the basis $A_1, \cdots, A_p, B_1, \cdots, B_p$ a transformation of type 2 with matrix $\|a_{ij}\| = \|r_{ij}\|$, we bring the transformation (19) into the form (16). We introduce now a canonical basis $A_1'', \cdots, A_p'', B_1'', \cdots, B_p''$ by applying transformation 3:

$$A_i'' = A_i + \sum_{k=1}^{p} s_{ik}B_k, \quad B_i'' = B_i.$$

The transition from this basis to the basis (9) is given by the formulae

$$A_i' = A_i'',$$
$$B_i' = \sum_{j=1}^{p} r_{ij}'A_j'' + \sum_{k=1}^{p} s_{ik}'B_k''. \tag{20}$$

The relation $J(A_i', B_j') = \delta_{ij}$ gives $\sum_k s_{jk}'\delta_{ik} = \delta_{ij}$ or $s_{ij}' = \delta_{ij}$. Thus the transformation (20) assumes the form

$$B_i' = B_i'' + \sum_{j=1}^{p} r_{ij}'A_j'',$$

that is, it is a transformation of type 3 in which the sets of curves A_i and B_i have changed roles. Thus the transition from the basis (7) to the basis (9) has been accomplished by a sequence of applications of transformations 1 – 3.

Theorem 23 is thereby proved.

Theorem 24. *If two framed submanifolds* (M_0^2, U_0) *and* (M_1^2, U_1) *of the Euclidean space* E^{n+2} *are homologous then we have*

$$\delta(M_0^2, U_0) = \delta(M_1^2, U_1) \tag{21}$$

(see (8)). *In this way we associate with each element* π *of the group* Π_n^2 *the residue class* $\delta(\pi) = \delta(M^2, U)$, *where* (M^2, U) *is a framed manifold in the class* π. *It turns out that, for* $n \geq 2$, δ *is an isomorphism of* Π_n^2 *on to the group of residue classes mod 2. From this assertion it follows that there exist exactly two homotopy classes of maps of the sphere* Σ^{n+2} *into the sphere* S^n, $n \geq 2$.

Proof. We first prove relation (21). Let (M^3, U) be a framed submanifold of of the strip $E^{n+2} \times I$ realising the homology $(M_0^2, U_0) \sim (M_1^2, U_1)$, as constructed in Lemma 1 of §14. We set $M_t^2 \times t = M^3 \cap (E^{n+2} \times t)$. If the point $(x, t) \in M^3$ is not a critical point of M^3 then a neighbourhood of the point x in the set M_t^2 is a smooth surface, so that for non-critical values of t the set M_t^2 is a surface. If (x_0, t_0) is a critical point of the manifold M^3, the set M_t^2 is given, for small values of $|t - t_0|$ near the point x_0, by the equations

$$\sigma^1(x^1)^2 + \sigma^2(x^2)^2 + \sigma^3(x^3)^2 = t - t_0; \quad x^4 = \cdots = x^{n+2} = 0, \tag{22}$$

(see §14, formula (3)). If the point $(x, t) \in M^3$ is not a critical point of M^3, then the image under orthogonal projection of the frame $U(x, t)$ on to the hyperplane $E^{n+2} \times t$ is a linearly independent system of vectors. We denote the system obtained from it by orthogonalisation by $V_t(x) \times t$. For non-critical values of the parameter t the system V_t constitutes an orthonormal frame for the manifold M_t^2. When the parameter t increases continuously without passing through a critical value the framed manifold (M_t^2, V_t) undergoes continuous deformation and it follows from general continuity considerations that the residue class $\delta(M_t^2, V_t)$ does not in this case change. Thus to prove (21) it is sufficient to show that the residue class $\delta(M_t^2, V_t)$ does not vary as t passes through the critical value $t = t_0$. We do this, distinguishing two distinct cases.

Case 1. Let $\sigma^1 = \sigma^2 = \sigma^3$. For definiteness we take $\sigma^1 = \sigma^2 = \sigma^3 = +1$. Under this hypothesis the manifold M_t^2 acquires after passing through a critical value a new component, consisting of a small sphere, while in the remaining components it is deformed continuously together with its frame. Since the adjunction of a sphere as a separate component does not affect the genus of the surface the canonical basis may be taken as unchanged and so $\delta(M_t^2, V_t)$ does not vary.

Case 2. Let the numbers $\sigma^1, \sigma^2, \sigma^3$ be not all equal. For definiteness we take $\sigma^1 = \sigma^2 = +1$, $\sigma^3 = -1$. Under this hypothesis the surface M_t^2 has for $t < t_0$ the form near x_0 of a hyperboloid of two sheets and for $t > t_0$ the form of a hyperboloid of one sheet. This alteration is equivalent to gluing a tube to the surface M_t^2,

$t < t_0$, are connected by the tube then the basis of the surface M_t^2 does not change on passing through t_0 and so $\delta(M_t^2, V_t)$ remains invariant. If the tube is glued to one component, then the basis of the surface must be supplemented by two curves. We examine this in detail. Let $A_1, \cdots, A_p, B_1, \cdots, B_p$ be a canonical basis of M_t^2, $t < t_0$. We will assume that the curves constituting this basis are situated far from the point x_0, so that the basis varies continuously as t passes through the critical value t_0, whence the residue classes $\delta(A_i)$ and $\delta(B_i)$, $i = 1, \cdots, p$ remain unchanged. As the curve A_{p+1} on M_t^2 we take the circle cut out of the part of M_t^2 near to x_0 by the hyperplane $x^3 = \epsilon$, where ϵ is a small positive number. For $t < t_0$ it is evidently nullhomologous on M_t^2 and since the frame on it varies continuously as t passes through the value t_0, it follows that $\delta(A_{p+1}) = 0$. Now let B'_{p+1} be an arbitrary curve on M_t^2, $t > t_0$, having intersection index 1 with A_{p+1}; such a curve evidently exists. We put

$$B_{p+1} = B'_{p+1} + \sum_{i=1}^{p} J(B_i, B'_{p+1}) A_i + \sum_{i=1}^{p} J(A_i, B'_{p+1}) B_i.$$

It is evident that the curves $A_1, \cdots, A_{p+1}, B_1, \cdots, B_{p+1}$ form a canonical basis for the surface M_t^2, $t > t_0$ and since $\delta(A_{p+1}) = 0$, then $\delta(A_{p+1})\delta(B_{p+1}) = 0$. Thus $\delta(M_t^2, V_t)$ is preserved unchanged as the parameter t passes through the critical value t_0, so that $\delta(M_0^2, V_0) = \delta(M_1^2, V_1)$. Since $U_0 = V_0$, $U_1 = V_1$, the relation (21) holds.

From this assertion it follows that δ is a map of the group Π_n^2 into the group of residue classes mod 2. By virtue of the definition of addition in Π_n^2 it is plain that δ is a homomorphism.

We now show that δ is a homomorphism on to the whole group of residue classes mod 2. For this it is sufficient to show that there exists a framed manifold (M^2, U) such that $\delta(M^2, U) = 1$. Since evidently,

$$\delta(E(M^2, U)) = \delta(M^2, U), \quad n \geq 2,$$

where E is the suspension operatiom, it is sufficient to consider the case $n = 2$.

Let E^4 be Euclidean vector space with orthonormal basis e_1, e_2, e_3, e_4 and corresponding coordinates x^1, x^2, x^3, x^4; let E^3 be the linear subspace defined by the equation $x^4 = 0$, and let M^2 be an ordinary metric torus lying in E^3 and having the axis e_3 as its axis of rotation. On the torus M^2 we introduce the usual circular coordinates ϕ, ψ and we determine the surface M^2 by the equations

$$\left.\begin{aligned} x^1 &= (2 + \cos\phi)\cos\psi, \\ x^2 &= (2 + \cos\phi)\sin\psi, \\ x^3 &= \sin\phi. \end{aligned}\right\} \quad (23)$$

We designate by A_1 the curve on M^2 given by the equation $\psi = 0$ and by B_1 the curve given by $\phi = 0$. Obviously the system A_1, B_1 constitutes a canonical basis for M^2. We denote by $v_1(x)$ the unit vector in E^3 normal to M^2 at the point $x = (\phi, \psi)$ and directed outwards from M^2, and we denote by $v_2(x)$ the vector issuing from x parallel to e_4. We define the frame $U(x) = \{u_1(x), u_2(x)\}$ by the relations

$$\begin{aligned} u_1(x) &= v_1(x)\cos(\phi - \psi) - v_2(x)\sin(\phi - \psi), \\ u_2(x) &= v_1(x)\sin(\phi - \psi) + v_2(x)\cos(\phi - \psi), \end{aligned} \quad (24)$$

and show that

$$\delta(M^2, U) = 1. \quad (25)$$

Let C be any simple closed curve on M^2. We denote by $v_4(x)$ the unit tangent vector to C at the point $x = (\phi, \psi) \in C$ and by $v_3(x)$ the unit vector tangent to M^2 at x and orthogonal to $v_4(x)$. We add to the relations (24) the relations

$$u_3(x) = v_3(x), \qquad u_4(x) = v_4(x). \quad (26)$$

The relations (24) and (26) together give the transition from the system $V(x)$ to the system $U(x)$. We denote the matrix of this transition by $f(x)$, $x \in C$. It is easy to see that for $C = A_1$ or B_1 we have

$$\beta(f) = 1. \quad (27)$$

Further for $C = A_1$ we have

$$x = (\phi, 0);\ v_1(x) = e_1\cos\phi + e_3\sin\phi;\ v_2(x) = e_4;$$

$$v_3(x) = e_2;\ v_4(x) = -e_1\sin\phi + e_3\cos\phi,$$

so that the transition from the system e_1, e_2, e_3, e_4 to the system $V(x)$ is given by an orthogonal matrix $g(x)$ for which

$$\beta(g) = 1. \quad (28)$$

For $C = B_1$ we have similarly

$$x = (0, \psi);\ v_1(x) = e_1\cos\psi + e_2\sin\psi;\ v_2(x) = e_4;$$

$$v_3(x) = -e_3;\ v_4(x) = -e_1\sin\psi + e_2\cos\psi,$$

so that the transition from the system e_1, e_2, e_3, e_4 to the system $V(x)$ is given by a matrix $g(x)$ for which

$$\beta(g) = 1. \quad (29)$$

In both cases $C = A_1$ and $C = B_1$ the transition from the system e_1, e_2, e_3, e_4 to the system $U(x)$ is given by a matrix $h(x) = g(x)f(x)$, for which

$$\beta(h) = \beta(g) + \beta(f) = 0$$

(see (27) – (29) and §12, D). Thus in view of formulae (2) and (8) we have

$$\delta(A_1) = 1, \quad \delta(B_1) = 1, \quad \delta(M^2, U) = 1,$$

and relation (25) is proved.

We prove finally that δ is an isomorphism. For this it is sufficient to show that the group Π_n^2 has no more than two elements since it is mapped on to the whole group of residue classes mod 2. It follows from Theorems 11 and 16 that for each framed manifold (M^2, U) in the Euclidean space E^{n+2} we have

$$(M^2, U) \sim E^{n-2}(N^2, V),$$

where (N^2, V) is a framed manifold of 4-dimensional Euclidean space and E^{n-2} is the $(n-2)$-fold suspension operation. Thus it is sufficient to show that Π_2^2 has no more than two elements, that is, that there exist no more than two classes of maps of S^4 into S^2. In view of Lemma 2 of §13, the number of classes of maps of S^4 into S^2 does not exceed the number of classes of maps of S^4 into S^3, but this last is equal to two by virtue of Theorem 21.

Thus Theorem 24 is proved.

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CONTENTS

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INVESTIGATIONS IN THE HOMOTOPY THEORY OF CONTINUOUS MAPPINGS

III. General theorems of extension and classification*

M. M. POSTNIKOV

This paper is devoted to answering questions concerning the search for criteria for two given continuous maps of an arbitrary cellular polyhedron into an arbitrary arcwise connected topological space to be homotopic. The paper consists of two paragraphs divided into sections. In Section 1 this question is reduced to the analogous question for simplicial maps of semi-simplicial complexes. In Section 2 the reduced question is completely solved.

The reduction described in Section 1 requires the preliminary computation of the natural systems (or at least of some segment) of the spaces considered. Thus, the results of this paper have essentially a conditional character. The extension and classification theorems proved here can be applied to concrete spaces only in so far as their natural systems are known.

A considerable amount of work has been devoted recently to computing natural systems (mainly the first non-trivial factor). Combining the results of that work with the results of the present paper leads to concrete extension and classification theorems. The author does not know a single concrete extension and classification theorem which it would be impossible to obtain in the way indicated, by an automatic computation from the general theorems proved here.

In an appendix several purely algebraic propositions are proved with the help of geometric considerations. It would be interesting to find algebraic proofs of them.

This paper is a direct continuation of the author's work [1]. Because of this, the numbering of sections and theorems of this paper continues the numbering of sections and theorems of that work. In references to this work we indicate the section (or proposition), and chapter (introduction, I and II). A short announcement of this work was published in D. A. N. S. S. S. R. [2].

CONTENTS

* For parts I and II see Transl. Amer. Math. Soc. (2) 7 (1957), 1–134.