# Farey Series and Pick's Area Theorem

Maxim Bruckheimer and Abraham Arcavi

## Introduction

In this article, we compare the historical notes and references in well-known texts with what would seem to be the reality as exhibited by the original sources, in the case of Farey series, Pick's area theorem, and the connection between them. Although one might suppose that we have chosen a particularly "unfortunate" example, in which the historical notes and texts are almost totally misleading, it is our experience in preparing historical activities for the classroom that, more often than not, the information readily available to nonprofessional historians is unreliable. There are signs that history is playing a greater role in the mathematics classroom, and there is a need for readily available reliable historical information relevant to the school curriculum. Errors in printed histories are relatively costly to correct, and the significance of the error relative to the whole justifies neither the expense nor the effort, and, thus, the errors achieve more or less permanent status. Perhaps in these days of flexible electronic data handling and storage, some historians will devise an electronic historical retrieval system, to which corrections and additions can be made as they are discovered — a sort of electronic Tropfke [1].

## The "Textbook" Farey

The sequence of all non-negative reduced proper fractions with denominator not exceeding $n$, arranged in increasing order, is called the Farey sequence of order $n$, $\boldsymbol{f}_n$. To understand the discussion one needs to know two fundamental properties of $\boldsymbol{f}_n$:

I. If $a/b$ and $c/d$ are two adjacent terms of $\boldsymbol{f}_n$, then $bc - ad = 1$.
II. If $a/b$, $e/f$, and $c/d$ are three adjacent terms of $\boldsymbol{f}_n$, then $e/f = (a+c)/(b+d)$.

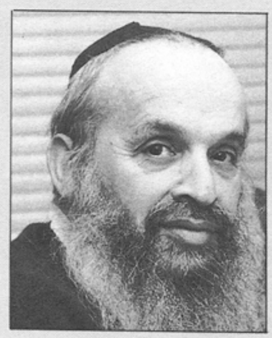
Our interest in Farey series[1] began as a result of some work for students on Egyptian unit fractions. In Beck, Bleicher, and Crowe [2], pp. 416 ff. we found that Farey series could be used to express any fraction between 0 and 1 as the sum of distinct unit fractions. So we began reading.

> In 1816 a minerologist [sic] named Farey published a mathematical paper in which he discussed the properties of ... [what] have since been called the *Farey sequences of order n*, although he was not the first to consider them....

For further information on this last point the authors refer to the book by Dickson [3]. Why should a mineralogist be interested in fractions? Sufficiently interested to publish "a mathematical paper"?

Before we referred to Dickson, we turned to another of the books immediately at hand — by Hardy and Wright [4], pp. 36–37.

[1] Farey series are not really series but sequences, but everyone (including Beck, *et al.*) calls them Farey series.



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The history of 'Farey series' is very curious. Theorems 28 and 29 [properties I and II above] seem to have been stated and proved first by Haros in 1802: see Dickson, *History*, i, 156. Farey did not publish anything on the subject until 1816, when he stated Theorem 29 in a note in the *Philosophical Magazine*. He gave no proof, and it is unlikely that he had found one, since he seems to have been at the best an indifferent mathematician.

Assuming the statement about Haros to be true, we note again an oft-recurring phenomenon, which can be described in the spirit of May [5] as follows:

If Theorem $X$ bears the name of $Y$, then it was probably first stated and/or proved by $Z$.

We were also left wondering whether Farey had *claimed* to have a proof.

To return to Hardy and Wright:

Cauchy, however, saw Farey's statement, and supplied the proof (*Exercices de mathématiques*, i, 114–16).[2] Mathematicians generally have followed Cauchy's example in attributing the results to Farey, and the series will no doubt continue to bear his name.

Farey has a notice of twenty lines in the *Dictionary of national biography*, where he is described as a geologist.[3] As a geologist he is forgotten, and his biographer does not mention the one thing in his life which survives.

But if Farey was an "indifferent mathematician," then why should he get a mention in the *Dictionary of national biography* (*DNB*), just because Cauchy attached his name to a result which he did not prove and which he was not the first to notice?[4]

Our two "sources" so far agree on one thing — that one should refer to Dickson. He says,

C. Haros proved the results rediscovered by Farey and Cauchy.

Then follows a description of what Farey stated, which is essentially property II stated above. Dickson continues,

Henry Goodwyn mentioned this property on page 5 of the introduction to his "tabular series of decimal quotients" of 1818, published in 1816 for private circulation..., and is apparently to be credited with the theorem.

[2] The date of this reference is 1826 and it is a reprint of the original published in 1816 (immediately after the appearance of a French translation of Farey's letter) in the *Bulletin des Sciences par la Société Philomatique de Paris* 3 (1816), 133–135.

[3] Apparently Farey as a mineralogist and geologist is not *completely* forgotten, as we are informed by Dr. Hugh Torrens of Keele University, England.

[4] Even worse, in our view, is Hardy's remark in *A Mathematician's Apology* (p. 81): "... Farey is immortal because he failed to understand a theorem which Haros had proved perfectly fourteen years before;..." From where does Hardy know that Farey "failed to understand" — or that Farey even knew of Haros's paper. Glaisher in 1879 does not mention it! And then "a theorem which Haros had proved perfectly" — poetic licence, perhaps, to which Hardy seems to have succumbed more than once in this book.

Why should Goodwyn be credited with the theorem if Haros proved the result 14 years earlier?

Later (p. 157), Dickson states,

J. W. L. Glaisher gave some of the above facts on the history of Farey series. Glaisher treated the history more fully....

But even Glaisher is at best a secondary source. With so many doubts and unanswered questions, only primary sources can resolve them.

## The "Real" Farey

The whole is so short that we can let the original Farey [6] speak for himself.

*On a curious Property of vulgar Fractions.*
*By Mr.* J. Farey, *Sen. To Mr. Tilloch*

SIR. — On examining lately, some very curious and elaborate Tables of "Complete decimal Quotients," calculated by Henry Goodwyn, Esq. of Blackheath, of which he has printed a copious specimen, for private circulation among curious and practical calculators, preparatory to the printing of the whole of these useful Tables, if sufficient encouragement, either public or individual, should appear to warrant such a step: I was fortunate while so doing, to deduce from them the following general property; viz.

If all the possible vulgar fractions of different values, whose greatest denominator (when in their lowest terms) does not exceed any given number, be arranged in the order of their values, or quotients; then if both the numerator and the denominator of any fraction therein, be added to the numerator and the denominator, respectively, of the fraction next but one to it (on either side), the sums will give the fraction next to it; although, perhaps, not in its lowest terms.

For example, if 5 be the greatest denominator given; then are all the possible fractions, when arranged, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$; taking $\frac{1}{3}$, as the given fraction, we have $\frac{1+1}{5+3} = \frac{2}{8} = \frac{1}{4}$ the next smaller fraction than $\frac{1}{3}$; or $\frac{1+1}{3+2} = \frac{2}{5}$, the next larger fraction to $\frac{1}{3}$. Again, if 99 be the largest denominator, then, in a part of the arranged Table, we should have $\frac{15}{52}$, $\frac{28}{97}$, $\frac{13}{45}$, $\frac{24}{83}$, $\frac{11}{38}$, &c.; and if the third of these fractions be given, we have $\frac{15+13}{52+45} = \frac{28}{97}$ the second: or $\frac{13+11}{45+38} = \frac{24}{83}$ the fourth of them: and so in all the other cases.

I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of any easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers; and am

Sir,
Your obedient humble servant,

J. Farey.
Howland-street.

We now know what Farey did and did not do. He did not write a "mathematical paper," and not only is it "unlikely that he had found one" but it would seem certain that he did not have a proof.

What remains is Haros's "claim" to priority. Glaisher [7], p. 335 was apparently unaware of Haros, but seems to have been suitably cautious:

It seems curious that so elementary and remarkable a property of fractions should not have been discovered until 1816. It may of course be found that it had been published previously; but supposing the discovery to be due to Mr. Goodwyn and Mr. Farey, an explanation might be afforded by the fact that the 'Tabular Series' is probably the earliest Table of the kind, and that the property would not be likely to present itself to anyone who had not arranged a complete series of proper fractions having denominators less than a given number in order of magnitude.

The fact is that Haros [8] anticipated both Goodwyn and Farey in a certain sense, as just the title of his paper indicates[5]:

Tables pour évaluer une fraction ordinaire avec autant de décimales qu'on voudra; et pour la fraction ordinaire la plus simple, et qui approche sensiblement d'une fraction décimale.

[Tables for evaluating a common fraction with as many decimals as desired; and the simplest good approximation by a common fraction of a decimal fraction.]

In the first part, he discusses the conversion of a fraction into decimal form. After stating some of the properties, he announces that he has calculated a new table yielding the decimal expansion of any irreducible fraction with denominator not exceeding 99. Unfortunately, he does not give the table, and this makes his description somewhat difficult to follow. However, it is the second part of the paper which interests us here.

His aim is to enable one to evaluate best approximations to decimal numbers by fractions with a low denominator. For this, he wants to arrange all fractions with denominator $\leq 99$ in order of size, for then he will be able to rely on their already calculated decimal values.

In other words, Haros proposes to write down the sequence $\boldsymbol{f}_{99}$.

He begins with the sequence

$$\frac{1}{99}, \frac{1}{98}, \frac{1}{97} \cdots \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \cdots \frac{96}{97}, \frac{97}{98}, \frac{98}{99},$$

in which, as he shows, each fraction differs from its neighbour by the reciprocal of the product of their denominators. Now comes the crux of his argument:

It remains to intercalate between the foregoing all other irreducible fractions with denominator less than 100. In this process, intermediate fractions must follow in order of size, and the difference of a fraction from its neighbour must always be one over the product of their denominators; for then any fraction in the sequence will be irreducible and will give as simply as possible the approximate value of one or the other of the two fractions between which it lies.

This falls far short of proving the first of the fundamental properties of Farey series. (The two properties are equivalent, see [4].)

In the following, Haros shows that if $a/b$ and $c/d$ satisfy the condition $bc - ad = 1$, and $x/y$ is a fraction between them satisfying the same condition with regard to its neighbours, then $x/y = (a + c)/(b + d)$.

What Haros seems to have done is to give a method for finding fractions belonging to $\boldsymbol{f}_{99}$, between those already listed — but how does one know that one will get them all? And it does not prove the more general result noted by Farey that if $a/b$, $e/f$, and $c/d$ are any three consecutive fractions in a Farey series, then $e/f = (a+c)/(b+d)$.

Clearly, Dickson overstated the case when he wrote that Haros proved the results rediscovered by Farey and Cauchy — and understated the case when he devoted relatively many lines to Goodwyn's tables, without a mention of those of Haros described in the 1802 paper. As Glaisher surmised, it was tables of fractions that made people notice the remarkable property of three consecutive fractions in what have come to be called Farey series. Farey did no more, but Haros deduced this property in special circumstances from the fundamental property of the difference of two neighbouring fractions. However, not until Cauchy saw Farey's letter were both results stated and proved satisfactorily.

## Pick's Area Theorem

That would have been the end of the story. But a couple of years later we decided to develop an activity for students around Pick's area theorem and, as usual, we wanted to include some historical background. So we began our search in textbooks again, starting with one by Coxeter [9] — and at once we were back in Farey land. There, opposite Pick's area theorem (p. 209), heading the section was the misleading quote from Hardy and Wright about Farey's entry in the *DNB*. The connection between Pick and Farey obviously had to be explored, both historically and for the activity we wanted to develop.

The relevant bits from Coxeter are as follows.

According to Steinhaus ... it was G. Pick in 1899, who discovered the following theorem:
*The area of any simple polygon whose vertices are lattice points is given by the formula*

$$\tfrac{1}{2}b + c - 1,$$

*where b is the number of lattice points on the boundary while c is the number of lattice points inside.*

"According to Steinhaus" would suggest that Coxeter is being careful — or why not quote Pick, as cited in Ref. 10 (p. 260), directly. Perhaps because he had not seen Pick's paper.

From Pick's area theorem, Coxeter deduces that if a triangle, whose vertices are the lattice points $(0, 0)$, $(b, a)$, $(d, c)$, contains no other lattice points within or on its sides, then $bc - ad = 1$.

Now if we represent any fraction $a/b$ in $\boldsymbol{f}_n$ by the lattice point $(b, a)$ then because the fractions are reduced, any two adjacent fractions $a/b$ and $c/d$ in $\boldsymbol{f}_n$ together with

[5] We are grateful to Dr. Baruch Schwarz of the Hebrew University at Jerusalem for help with understanding the French and for checking what we have written about Haros's paper.

the origin form an otherwise lattice-point-free triangle as above.[6] Hence, we have $bc - ad = 1$, one of the two fundamental properties of Farey series proved in a most elegant fashion. Coxeter attributes this proof of the Farey property to Pólya [11].

To fill in the historical background a little, we obtained a few biographical details of Pick from Poggendorf [12], p. 569 and ordered copies of the papers by Pick and Pólya. It appears that Georg Alexander Pick was born in 1859 in Vienna and died in 1943 (?) in the Theresienstadt concentration camp. He spent most of his working life at the German University in Prague, and Kline [13], p. 1131, in connection with Einstein's work on the theory of general relativity, notes,

> However, to make progress ... he [Einstein] discussed it in Prague with a colleague, the mathematician Georg Pick, ....

To analysts, Pick is well remembered for interpolation of analytic functions; see [14].[7]

To return to Pick's area theorem. The impression we were left with from Coxeter was that Pick discovered his theorem and Pólya applied Pick to Farey.[8] However, the facts as they appear from the original articles are somewhat different. Thus, Pick [16] begins his article by citing the widespread use of plane lattices "for visualisation and as heuristic aids in number theory" going back to Gauss.[9] His own aim, he says, is rather to put the elements of number theory on a geometric basis, by use of an area formula for lattice polygons which "in spite of its simplicity seems to have gone unnoticed till now."

The surprise, however, comes in the third section of his article, where he derives the above fundamental property of Farey series (and some more) in exactly the same way as in [9], where Coxeter as we saw attributes this to Pólya.[10]

And so to Pólya and the introduction to his paper.

> A beautiful geometric treatment of the well-known principal property of Farey series goes back to Sylvester. As this treatment seems to have been generally forgotten, and as, moreover, Sylvester's inferences are not irreproachable, it is probably in order to go through the matter briefly here.

Thus, Pólya himself attributes the lattice approach to the Farey property to Sylvester, in a paper originally published in 1883. Pólya does not present it as his own method.[11] Sylvester's paper does not contain Pick's area theorem; all he needs, as we have noted above, is the area of a lattice-point-free triangle. However, Pólya does give Pick's area theorem (in a slightly variant form) but does not attribute it to Pick—or to anyone else.

It would seem that the application of the geometry of lattices to Farey series probably dates back to Sylvester in 1883. Pick published his theorem in 1899 and, apparently unaware of Sylvester, applied a special case to Farey series again. This was repeated in 1925 by Pólya, apparently unaware of Pick, but based on Sylvester. By his own account, Pólya does not "deserve" the historical credit given him by Coxeter.[12]

[6] For the details, see Ref. 9, p. 211.

[7] In a recent paper on Pick's theorem, Grünbaum and Shephard [15] write: "He [Pick] made significant contributions to analysis and differential geometry." Perhaps we may say that among geometers Pick is remembered almost exclusively for a relatively minor, if extremely beautiful, result.

[8] A development similar to that of Coxeter, but without Pick's area theorem, can already be found in [4], where in their note on the appropriate sections (3.5–3.7) Hardy and Wright write, "Here we follow the lines of a lecture by Professor Pólya," thus strengthening the impression that the application of lattices to Farey series is due to Pólya.

[9] See also, for example, [17], p. 35.

[10] The mathematics in Pick's paper is discussed by the present authors in "A visual approach to some elementary number theory", *Mathematical Gazette* (to appear).

[11] And presumably did not do so either in the "lecture" referred to by Hardy and Wright.

[12] And Hardy and Wright.

## References


1. J. Tropfke, *Geschichte der Elementarmathematik*, 4th ed., Berlin: de Gruyter, (1980).
2. A. Beck, M. N. Bleicher, and D. W. Crowe, *Excursions into Mathematics*, New York: Worth (1969).
3. L. E. Dickson, *History of the Theory of Numbers*, Vol. I, New York: Chelsea (1952).
4. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 2nd ed., London: Clarendon Press, 1945.
5. K. O. May, Historiographic vices I. Logical attribution, *Historia Math.* 2 (1975), 185–187.
6. J. Farey, On a curious property of vulgar fractions, *Philos. Mag. J.* 47 (1816), 385–386.
7. J. W. L. Glaisher, On a property of vulgar fractions, *London, Edinburgh, Dublin Philos. Mag.* 7 (1879), 321–336.
8. C. Haros, Tables pour évaluer une fraction ordinaire avec autant de décimals qu'on voudra; et pour trouver la fraction ordinaire la plus simple, et qui approche sensiblement d'une fraction décimale. *J. Ecole Polytechn.* 4 (1802), 364–368.
9. H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., New York: Wiley (1969).
10. H. Steinhaus, *Mathematical Snapshots*, 2nd ed., Oxford: Oxford University Press (1950).
11. G. Pólya, Über eine geometrische Darstellung der Fareyschen Reihe, *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio Scientiarum Mathematicarum* 2 (1925), 129–133.
12. J. C. Poggendorff, *Biographisch-Literarisches Handwörterbuch der Exakten Naturwissenschaften*, Band VIIa, Teil 3, Akademie-Verlag (1959).
13. M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford: Oxford University Press (1972).
14. John B. Garnett, *Bounded Analytic Functions*, New York: Academic Press (1981).
15. B. Grünbaum and G. C. Shephard, Pick's theorem, *Amer. Math. Monthly* 100 (1993), 150–160.
16. G. Pick, Geometrisches zur Zahlenlehre, *Zeit. Vereines 'Lotos'* 19 (1899), 311–319.
17. F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Vol. I, Heidelberg: Springer-Verlag (1926).



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