The Farey Sequence

Jonathan Ainsworth, Michael Dawson,
John Pianta, James Warwick

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School of Mathematics
University of Edinburgh
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Abstract

The Farey sequence (of counting fractions) has been of interest to modern mathematicians since the 18th century. This project is an exploration of the Farey sequence and its applications. We will state and prove the properties of the Farey sequence and look at their application to clock-making and to numerical approximations. We will further see how the sequence is related to number theory (in particular, to the Riemann hypothesis) and examine a related topic, namely the Ford circles.

This project report is submitted in partial fulfilment of the requirements for the degree of BSc Mathematics except Pianta who is MA Mathematics

Contents

Chapter 1

Introduction to Farey sequences

1.1 The Ladies' Diary

Figure 1.1: Ladies Diary Cover 1747

"The Ladies Diary: or, the Woman's Almanack" was an annual publication printed in London from 1704 to 1841. It contained calendars, riddles, mathematical problems and other "Entertaining Particulars Peculiarly adapted for the Use and Diversion of the fair-sex" [1]

In the 1747 edition, the following mathematical question appeared.

III. QUESTION 281, *by Mr. J. May, jun. of Amsterdam.*

It is required to find (by a general theorem) the number of fractions of different values, each less than unity, so that the greatest denominator be less than 100?

The question of how many non-unique fractions occur with denominator ≤ 99 is a simple one and is given by the sum $\sum_{n=1}^{99} n = 4950$ (this sum can be quickly solved in your head using the method made famous by Gauss [2]). However, the question specifically asks for the number of fractions with *different values*, i.e. the unique fractions, a problem that took 4 years to answer.

The 1751 edition of the Diary published three solutions to this problem, two of which are flawed [3]. A writer going by the name of Flitcon provided the correct solution (see Appendix) that there are 3003 simplified fractions. In other words, 1947/4950 of the possible fractions cancel.

The topic of the next section will be an analysis of this solution and hence an answer to the original question.

1.2 Flitcon's solution

Some of the following proofs have been omitted in this report. For the proofs see, for example, Guthery [4].

Definition 1. *Euler's totient function is the function ϕ such that $\phi(n)$ is the number of integers $< n$ which are coprime to n.*

Definition 2. *A function, f, is* multiplicative *if $f(a).f(b) = f(a.b)$.*

Proposition 1. *Euler's totient function is multiplicative, provided that the integers a and b are coprime.*

Euler's totient function $\phi : n \mapsto$ "number of integers less than n which are coprime to n" can be computed as follows

Theorem 3. $\phi(n) = n.\prod_{i=1}^{k}(1 - \frac{1}{p_i})$ *where Π denotes the product and p_i are the k distinct prime factors of n (this used the well-known "Fundamental Theorem of Arithmetic").*

Now we can prove that Flitcon was correct, using the methods outlined in his solution.

Proposition 2 (Ladies' diary 1747)**.** *There are* 3003 *unique rationals of the form* $\frac{m}{n}$ *with* $m \leq n < 100$ *on the interval* $(0, 1)$*.*

Proof. Sketch proof (Flitcon's method):

- Construct a function, say ϕ, which maps n to "number of integers less than n which are coprime to n". We have defined Euler's totient function to do this. Flitcon used an equivalent function*.

- Construct a table with three columns. The first column contains the integers from 2 to 99. The second column contains the prime decomposition of n. The third column contains $\phi(n)$. When constructing this table it is helpful to notice (as Flitcon did) that the function ϕ is multiplicative.

- Sum the third column for all entries 2 to 99. This gives the total number of simplified fractions between 2 and 99.

Proof by Maple: See the Appendix for our Maple code which verifies Flitcon's original solution. The code can also be used to compute any Farey sequence and its length.

□

*Flitcon's method appears to use Euler's totient function (published 17 years previously, in 1734) however, Flitcon doesn't cite it and may have derived a non-general form of the function for himself.

Example 4. $\phi(36) = 36.(1 - \frac{1}{2})(1 - \frac{1}{3}) = 12$
Thus there are 12 unique fractions with denominator 36.

We would like to generalise the problem of how many simplified fractions there are between 0 and 1 given any restriction of the denominator. Such a sequence of numbers is called a Farey sequence.

1.3 Farey sequences

Definition 5. *A Farey sequence F_n is the set of rational numbers $\frac{p}{q}$ with p and q coprime, with $0 < p < q < n$, ordered by size.*

Example 6. $F_1 = \{\frac{0}{1}, \frac{1}{1}\}$
$F_2 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$
$F_3 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$
$F_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$

Remark 7. *The question in the Ladies' diary is equivalent to asking what is the length of F_{99}. Example 4 says that F_{36} has 12 more elements than F_{35}.*

1.4 Properties of the Farey Sequence

If we have two fractions $\frac{a}{b}$ and $\frac{c}{d}$ with the properties that $\frac{a}{b} < \frac{c}{d}$ and $|bc - ad| = 1$ Then the fractions are known as Farey neighbours, they appear next to each other in some Farey sequence. The mediant or Freshman's sum of these two fractions is given by

$$\frac{a}{b} \bigoplus \frac{c}{d} = \frac{a+c}{b+d}.$$

Theorem 8 (Mediant Property)**.** *If $\frac{a}{b} < \frac{c}{d}$ then their mediant $\frac{a+c}{b+d}$ lies between them, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.*

Proof.

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc-ad}{b(b+d)} > 0 \quad \text{and} \quad \frac{c}{d} - \frac{a+c}{b+d} = \frac{bc-ad}{d(b+d)} > 0$$

□

Proof comes from Tom M. Apostol [5].

A consequence of this is that if two fractions in a Farey sequence are Farey neighbours they will remain so until their mediant separates them in a later Farey Sequence. For example $\frac{0}{1} < \frac{1}{2}$ are Farey neighbours in F_2, and their mediant is $\frac{1}{3}$ which separates them in F_3. This is an important property that will feature throughout the report.

Remark 9. *The mediant always generates simplified fractions. So for example $\frac{2}{4}$ will not be generated by taking mediants.*

Theorem 10 (Neighbours Property). *Given $0 \leq \frac{a}{b} < \frac{c}{d} \leq 1$, $\frac{a}{b}$ and $\frac{c}{d}$ are Farey neighbours in F_n if and only if $bc - ad = 1$.*

Proof. If $\frac{p}{q}$, $\frac{a}{b}$ and $\frac{c}{d}$ are in some Farey sequence, with $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ and $bp - aq = qc - pd = 1$, then

$$\begin{aligned} bp + pd &= qc + aq \\ p(b+d) &= q(a+c) \\ \frac{p}{q} &= \frac{a+c}{b+d} \end{aligned}$$

Hence $\frac{a}{b}$ and $\frac{p}{q}$ are neighbours and $\frac{p}{q}$ and $\frac{c}{d}$ are neighbours.

We will prove the converse by induction.

First $F_1 = \{\frac{0}{1}, \frac{1}{1}\}$ and $|bc - ad| = |1 - 0| = 1$ so the result is true for $n = 1$. Assuming the result for F_n we next prove that the result follows for F_{n+1}. We have $F_n = \{\ldots \frac{a}{b}, \frac{c}{d} \ldots\}$ and $|bc - ad| = 1$. If $b + d \leq n + 1$ then $F_{n+1} = \{\ldots \frac{a}{b}, \frac{(a+b)}{(c+d)}, \frac{c}{d} \ldots\}$ and $|b(a+c) - a(b+d)| = |bc - ad| = 1$. The only other possibility is that $b + d > n + 1$ in which case $F_{n+1} = \{\ldots \frac{a}{b}, \frac{c}{d} \ldots\}$ with $|bc - ad| = 1$, so the result is true for $n + 1$.

Hence, by the axiom of induction the result is true for all positive integer values of n. □

Proof adapted from NRich [6].

1.5 Farey History

So how did this list of simple or "vulgar" fractions become known as the Farey sequence? As with many things in life, history may have been too kind to those who were simply remembered, and too harsh to those forgotten.

Whilst Mr Flitcon's solution got the precise number of elements of F_{99} (not including 0 and 1), it did not give us an explicit formula to find those elements, or indeed even list them.

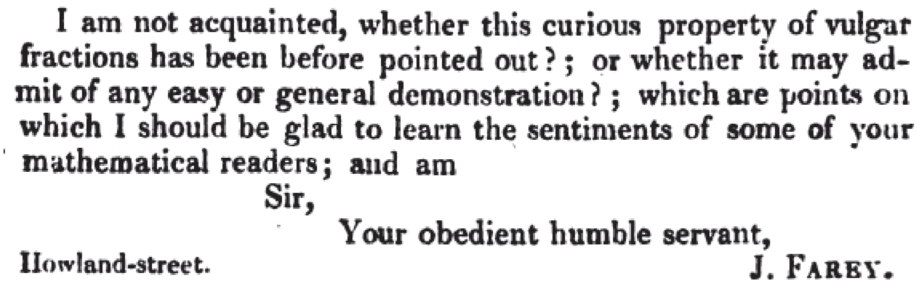
The French Revolution acted as an unlikely catalyst for the first ever publishing of F_{99}, but when France's new regime legislated that the whole country was to switch to the metric system instead of imperial measurements in 1791 it fell to Charles Haros to create a mathematical table to convert between fractions and decimals.

Published in *"Journal de l'Ecole Polytechnique"*, Haros' table contained every irreducible fraction with denominators from 2 to 99 and their decimal approximation. Aside from including $\frac{0}{1}$ and $\frac{1}{1}$, Charles Haros had formulated F_{99}. To do this he used the mediant property to find the fractions with higher denominators and even provided a sketch proof that it worked. He also noted that if two numbers $\frac{a}{b}$ and $\frac{c}{d}$ are neighbours in the table then $|bc - ad| = 1$.

From here the history of the Farey sequence travels to Britain, and to a man called Henry Goodwyn. Henry Goodwyn ran and owned a brewery and made mathematical tables in his spare time. In his retirement he set out (much like Charles Haros) to create a table of fractions and decimal equivalents. However, Goodwyn's tables were to contain every irreducible fraction with denominators between 1 and 1024. His paper *"The First Centenary of a Series of concise and useful Tables of all the complete decimal Quotients which can arise from dividing a Unit or any whole Number less than each Divisor by all Integers from 1 to 1024"* was presented to the Royal Society on 25th April 1816, having had a private printing circulated a year previously.

Less than a month after Henry Goodwyn's presentation to the Royal Society a geologist by the name of John Farey wrote into The Philosophical Magazine and Journal with a note entitled "On a curious Property of the vulgar Fractions". In the note, John Farey pointed out a "curious property", but offered no proof (see a later remark). He finished the letter as in Figure 1.2.

John Farey's note was then republished in the French magazine "*Bulletin de la Société Philomatique*". It was from here that Cauchy saw it and provided a proof that the mediant property holds (crediting John Farey) in August 1816 [7].



I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of any easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers; and am

Sir,

Your obedient humble servant,

Howland-street. J. FAREY.

Figure 1.2: Excerpt from Farey's Letter

1.6 Length of the Farey Sequence

Definition 11. *The length of a given Farey sequence is given by the recursion formula*

$$|F_n| = |F_{n-1}| + \phi(n)$$

Where $\phi(n)$ is Euler's Totient Function, see Theorem 3.

Theorem 12. *The length of the Farey sequence behaves asymptotically with*

$$|F_n| \sim \frac{3n^2}{\pi^2}$$

The following three graphs represent what the length of the Farey sequence behaves asymptotically with, what the length of the Farey sequence actually is along with the error between the two. A code in *Maple* was written to calculate the exact way that the function was behaving.

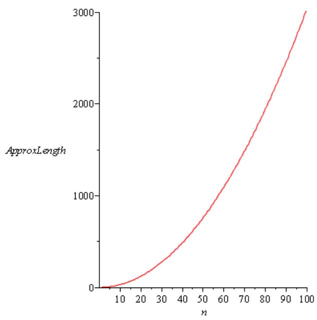


Figure 1.3: Approximated Length

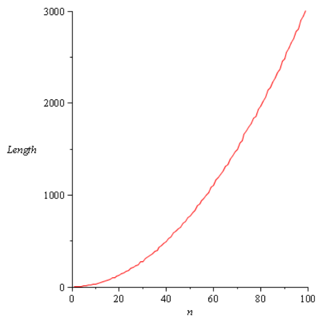


Figure 1.4: Exact Length

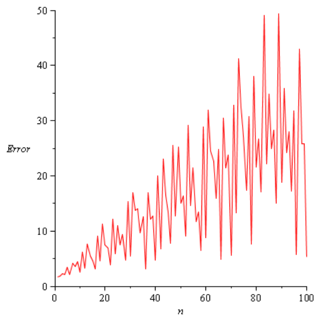


Figure 1.5: Error in Approximation

Chapter 2

Clocks and Farey

Mathematics has always been important in forming time-based instruments such as calendars and plays an important part in the making of clocks. Clockmakers not only borrowed some of the ideas from mathematics but actually developed some of their own that then became part of mathematics. This resulted in the creation of the Stern-Brocot tree, which is surprisingly useful in the construction of clocks.

2.1 Creation of the Stern-Brocot tree

The Stern-Brocot tree was first described by Moritz Stern in 1858, who explained it's relation to other areas in number theory. Clock maker Achielle Brocot, discovered the tree independently in 1861[8] and used it practically to approximate gear ratios, but never realised it to be of any other mathematical significance.

We saw earlier how the Farey sequence is constructed using Farey neighbours and mediants, when this process is extended to the whole real line we get the Stern-Brocot tree. A similar process of mediant insertion, starting with a different pair of interval endpoints $[\frac{0}{1}, \frac{1}{0}]$, may also be seen to describe the construction of the vertices at each level of the Stern-Brocot tree.[9]

To construct the Stern-Brocot tree, we need to recall the mediant of two rational numbers to be

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

We start to construct the tree in steps beginning with the two fractions $\frac{0}{1}$ and $\frac{1}{0}$. It is useful to think of $\frac{1}{0}$ here as representing infinity. We then continue the construction forming the next level by inserting the mediant of any two consecutive rational numbers that are already in the tree as in Figure 2.1.

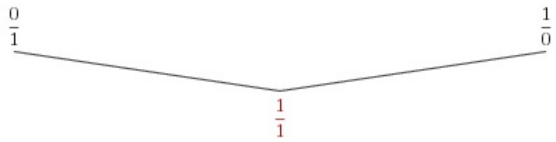


Figure 2.1

And we continue to add the mediants in Figure 2.2.

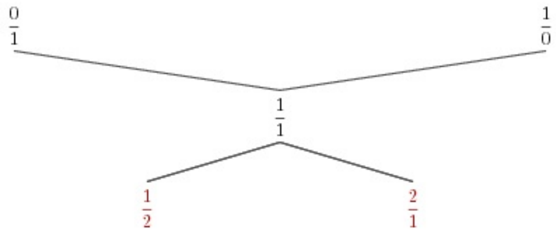


Figure 2.2

And eventually we end up with a tree like in Figure 2.3.

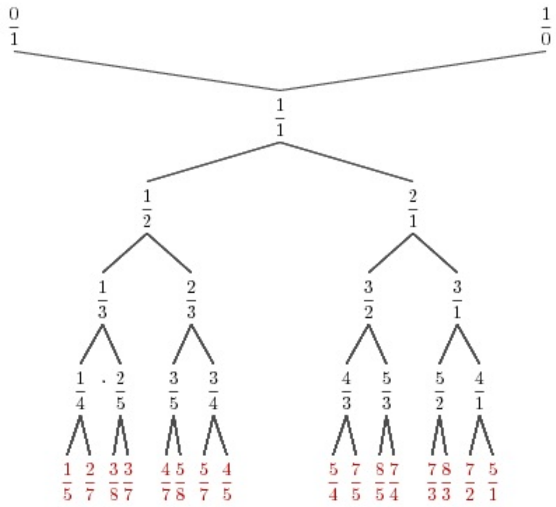


Figure 2.3

Notice that this has constructed a binary search tree with the Farey sequence between $\frac{0}{1}$ and $\frac{1}{1}$ and a periodic extension of the Farey sequence between $\frac{1}{1}$ and $\frac{1}{0}$. When a rational number r appears, it is the Farey neighbour of two rationals.

Definition 13. *An* Ancestor *of a number in the Stern-Brocot tree is any number that appears above it in the same branch.*

A Child *of a number in the Stern-Brocot tree is a number that appears directly below it. Each number has a* Right Child *(R) and a* Left Child *(L).*

Moreover, the Left Child of any number is the mediant of that number and its first Ancestor to the left (and first Ancestor to the right for the Right Child).

Example 14. $\frac{1}{3}$ *has Left Child* $\frac{1}{4}$*, Right Child* $\frac{2}{5}$ *and Ancestors* $\frac{1}{2}$, $\frac{1}{1}$, $\frac{0}{1}$ *and* $\frac{1}{0}$.

The Left Child of any number is the mediant of that number and its first Ancestor to the left (and first Ancestor to the right for the Right Child).

Example 15. *The Right Child of* $\frac{3}{4}$ *is the mediant of* $\frac{3}{4}$ *and* $\frac{1}{1}$ *(the first ancestor to its right)* $= \frac{4}{5}$.

Every rational number appears in this tree exactly once as the new rationals added are always between the consecutive numbers that have already appeared.

Thus the Stern-Brocot sequences differ from the Farey sequences in two ways. It includes all positive rationals, not just those within the interval [0,1], and at the nth step all mediants are included, not only the ones with denominator less than or equal to n. The Farey sequence of order n may be found by an in order traversal of the left subtree of the Stern-Brocot tree, backtracking whenever a number with denominator greater than n is reached.

2.2 Navigating the Tree

Now we will see how the tree can be navigated and show how it contains every number. We will start with an example.[10]

Where can we find $\frac{4}{7}$?

The construction always begins with the fractions

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{0}.$$

We will begin at the fraction $\frac{1}{1}$. Since $\frac{0}{1} < \frac{4}{7} < \frac{1}{1}$, we will move left to the Farey neighbour of $\frac{0}{1}$ and $\frac{1}{1}$, which is $\frac{1}{2}$ as shown in Figure 2.4. This is a move to the Left Child of $\frac{1}{1}$, denoted L.

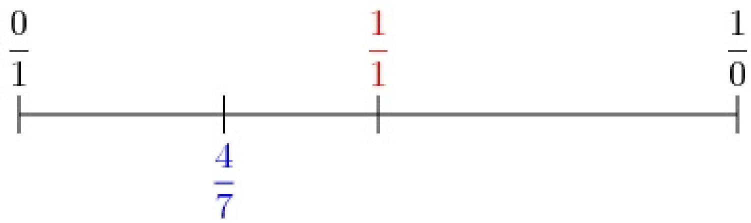


Figure 2.4

Now we have $\frac{1}{2} < \frac{4}{7} < \frac{1}{1}$ so we will move to $\frac{2}{3}$, as in Figure 2.5. We have moved to the Right Child (R) of $\frac{1}{2}$. The path from $\frac{1}{1}$ to $\frac{2}{3}$ can be described as LR.

In the same way we now have $\frac{1}{2} < \frac{4}{7} < \frac{2}{3}$. So we move to the left to $\frac{3}{5}$, and the path is LRL. With one more step to the left, we arrive at $\frac{4}{7}$, as in Figure 2.6, which we may represent as LRLL.

How does this show that every positive rational appears in the tree? If we are looking for the rational r. At every step, r is always between two rationals $\frac{p}{q}$ and

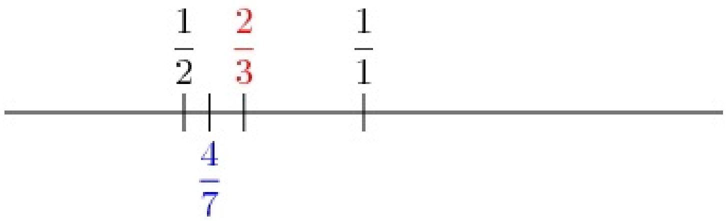


Figure 2.5

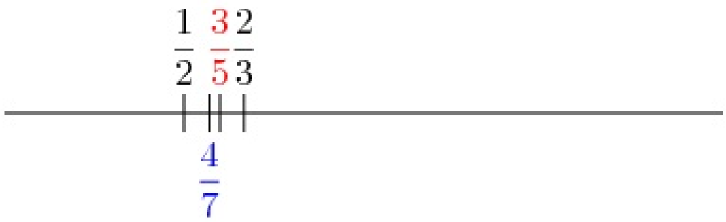


Figure 2.6

$\frac{p'}{q'}$. If r is equal to the mediant of $\frac{p}{q}$ and $\frac{p'}{q'}$, then we have found r. If not we move from the mediant to the left if r is less than the mediant and right otherwise. If we never find r then this process will continue indefinitely, and the denominators of the rationals $\frac{p}{q}$ and $\frac{p'}{q'}$ that bracket r grow will arbitrarily large.

Let's illustrate this geometrically. Since q and q' grow arbitrarily large, the points (q, p) and (q', p') are eventually to the right of r. And since $\frac{p}{q} < r < \frac{p'}{q'}$, it follows that r lies in the parallelogram as shown in Figure 2.7.

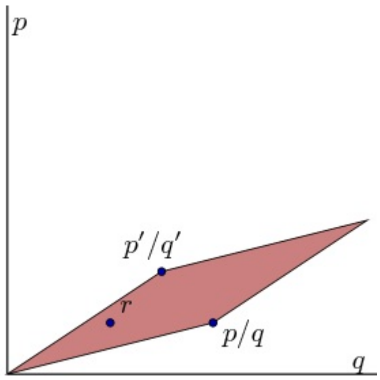


Figure 2.7

Remember that the area of the shaded parallelogram above is $qp' - pq' = 1$. Consider now the area of the parallelogram defined by $\frac{p}{q}$ and r, shaded darkly in

Figure 2.8.

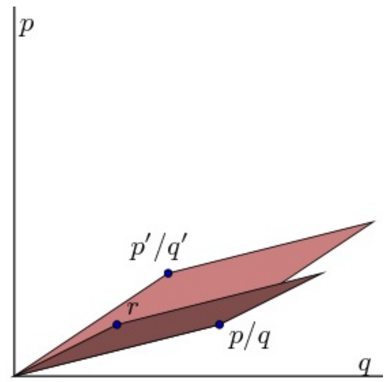


Figure 2.8

The area of the dark parallelogram must be a positive integer, obeying the mediant property, that is less than 1. Since this is impossible, r must have appeared somewhere earlier in the tree.

We now have a way to label strings of positive rational numbers with strings of right and left movements L's and R's.

If we consider infinite strings of L's and R's we see that they correspond to positive irrational numbers. If we have a positive irrational number α we can find it's corresponding string with the following algorithm:[11]

1. Begin at $r = \frac{1}{1}$ and let S be the empty string.

2. If $\alpha < r$, replace r by its Left Child and replace S by S=L. Otherwise, replace r by its Right Child and S by S=R.

3. Repeat step 2.

In this way we can find some well known strings, for example, S = RLRLRLRLRLR.... Here following the tree from $\frac{1}{1}$ we move to $\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}$,.... Since we are alternating from right to left moves we are always finding the mediant of the previous two terms in the tree and this should be recognised as ratios of consecutive Fibonacci numbers. So we can see that this converges to the golden ratio and it's string is Φ = RLRLRLRLRLR....

The key use of the Stern-Brocot tree in clock making is that any sequence found in this way is optimal in the following sense. Given any rational approximation to a rational number x, the sequence usually contains an element that is a more accurate approximation with a smaller numerator and denominator.

We will show why this property holds by considering the process in which we create a sequence that converges to x. At every step there are consecutive rationals $\frac{p}{q}$ and $\frac{p'}{q'}$ that bracket x. Since r is a rational number different from x, eventually r will not be bracketed by these two rational numbers.

Now consider the last step at which r is bracketed by $\frac{p}{q}$ and $\frac{p'}{q'}$. If $\frac{p''}{q''}$ is the next mediant obtained in the process, then it must lie between r and x.

Now r will be found in the Stern-Brocot tree under $\frac{p''}{q''}$, which means that the numerator of r is no smaller than p'' and the denominator is no smaller than q''. Thus, $\frac{p''}{q''}$ is closer to x than r is and the numerator and denominator are no larger than r's.

We now also explain why the rationals that appear in the tree are expressed in their simplest form. i.e If we have the ratio $\frac{p}{q}$, then p and q have no common factors. Suppose instead that the rational $\frac{sp}{sq}$ with $s > 1$. Suppose that the fraction $\frac{a}{b}$ is the rational that follows $\frac{sp}{sq}$ when it appears in the tree. Then we have $|sqa - spb| = s|qa - pb| = 1$ (as they are Farey neighbours) which is impossible if $s > 1$ which implies that the rationals in the tree are in their simplest form.

2.3 Gear Ratios

So why would a clock maker be interested in this tree?

Clocks typically have a source of energy, such as a spring, a suspended weight, or battery that turns a shaft at a fixed rate. If the clock has a minute hand and an hour hand we need some mechanism that will speed or slow down the motion of the shaft as it is transferred to the hands.

Gears are used to slow the motion of the shaft. In Figure 2.9, the small green gear, with 20 teeth, drives the blue gear, with 60 teeth. In the clock making language the smaller gear is called a pinion and the larger gear is a wheel.

Every time the green pinion advances by one tooth, so does the blue wheel. Therefore, one revolution of the green pinion produces $\frac{20}{60}$ or $\frac{1}{3}$ of a revolution of the blue wheel or the green wheel turns three times to turn the blue wheel once.

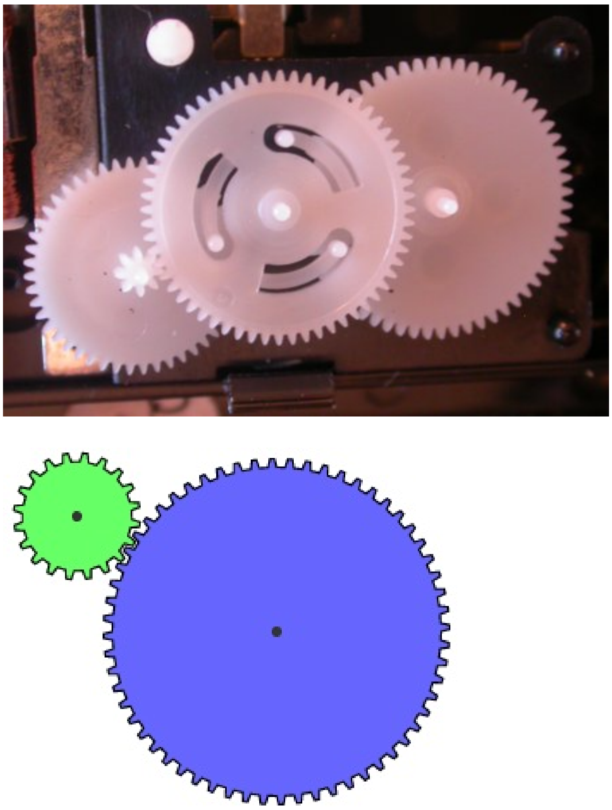


Figure 2.9

If we wish to have a shaft that rotates once a second that drives a minute hand, we could use a pinion and wheel whose ratio of teeth is $\frac{1}{60}$. But there is another option. Figure 2.10 illustrates a gear train; the green pinion turns the blue wheel slowing the speed by a factor $\frac{1}{6}$. However, the blue wheel turns the red pinion at the same rate, and this pinion, with 10 teeth, turns the gray wheel, with 100 teeth thus slowing the speed by another factor of $\frac{1}{10}$. Therefore, the overall ratio of the speed of the green pinion to the grey wheel is $\frac{1}{6} \times \frac{1}{10} = \frac{1}{60}$.

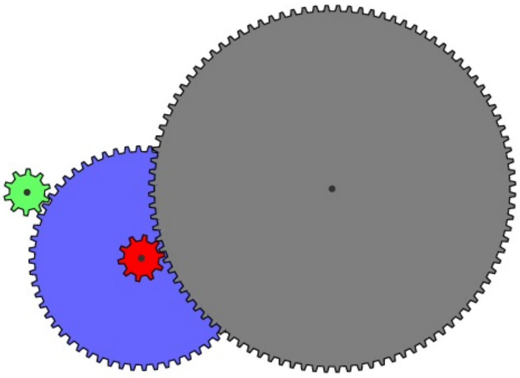


Figure 2.10

In theory any number of stages can be added to a gear train. For example we could add a pinion on the grey wheel that can drive another wheel.

To illustrate how this is used in clock making, imagine that we have a shaft that rotates once per minute and we wish to drive an hour hand, which rotates once every 12 hours.[9] The ratio of these speeds is 1/720. The most obvious solution is a single pinion with one tooth to drive a wheel with 720 teeth or a pinion with 10 teeth driving a wheel with 7200 teeth. But the larger gears would be impractical, so instead we use a gear train:

$$\frac{1}{720} = \frac{1}{10 \times 8 \times 9} = \frac{1}{10} \times \frac{1}{8} \times \frac{1}{9}$$

The key to this is that we can easily factor 720.

Consider the situation where we require a shaft that rotates once every 23 minutes to turn a wheel that rotates once every 191 minutes (an example Brocot used in his paper).[13] Since these are both prime, it is not possible to use this required ratio as a gear train.

First we shall approximate $\frac{23}{191}$ with $\frac{p}{q}$. So we have a pinion with p teeth on the shaft and it's turning a wheel with q teeth. The pinion makes one revolution every 23 minutes. This means that the wheel turns once every $23\frac{p}{q}$ minutes, making the error

$$23\frac{q}{p} - 191 = \frac{23q - 191p}{p} = \frac{E(\frac{p}{q}, \frac{23}{191})}{p} \text{minutes.}$$

Where E is the mediant property of the Farey sequence mentioned earlier, $E(\frac{a}{b}, \frac{c}{d}) = |bc - ad|$, which is being used in this case as a way to measure the error of an approximation.

We may easily compute $E(\frac{p}{q}, \frac{23}{191})$ as we descend the tree. Since $\frac{23}{191}$ is between $\frac{1}{8}$ and $\frac{1}{9}$, Brocot began by making a table like this:

p	q	E	p	q	E
1	9	16	1	8	-7
2	17	9	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
23	191	0			

This shows that if we were to approximate $\frac{23}{191}$ by $\frac{2}{17}$, the error will result in the wheel taking $E(\frac{2}{17}, \frac{23}{191})/p = \frac{23 \cdot 17 - 191 \cdot 2}{2} = \frac{9}{2} = 4.5$ too many minutes to rotate. Brocot then continued in this way until the completed table looks like this:

p	q	E	p	q	E
1	9	16	1	8	-7
2	17	9	4	33	-5
3	25	2	7	58	-3
13	108	1	10	83	-1
23	191	0			

Brocot's algorithm reveals that the closest approximations to $\frac{23}{19}$ are ratios of $\frac{10}{83}$ (which runs a tenth of a minute fast) and $\frac{13}{108}$ (a thirteenth of a minute slow). It is possible to do better than this using a gear train and Brocot originally did this taking mediants between the approximation and the exact fraction. However finding the fraction that had a nicely factorisable numerator and denominator was trial and error. He realised that all the work for this could be done beforehand and wrote a table showing all fractions with numerator and denominator less than 100, ordered in magnitude.

Here is an example of a basic problem from Camus' "A treatise on the teeth of wheels"[14]. "To find the number of the teeth...of the wheels and pinions of a machine, which being moved by a pinion, placed on the hour wheel, shall cause a wheel to make a revolution in a mean year, supposed to consist of 365 days, 5 hours, 49 minutes."

Multiplying out the days and years we find that we need a ratio of $\frac{720}{525949}$. The numerator factors well but we have a problem with the prime denominator. So we need to find an approximation to this fraction, but with that both the numerator and denominator that has small factors. Camus original solution to this was a series of repeated trials which Brocot thought was defective and so solved it similarly as above which is more efficient.

In this case the error in approximation is $E = q(720) - p(525949)$.

p	q	E	p	q	E
0	1	720	1	0	-525,949
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
33	24106	3	163	119069	-7
262	191387	2	**196**	**143175**	**-4**
491	358668	1	229	167281	-1
720	**525949**	**0**			

We choose the fraction to be $\frac{196}{143{,}175}$ as it can be factored nicely and both numbers in the fraction would have appeared the Brocot table mentioned earlier. We can then make this into a gear train of $\frac{2}{3} \times \frac{2}{25} \times \frac{7}{23} \times \frac{7}{83}$ and the error in the gear train is $\frac{4}{196}$. This results in a gear train that is just over a second too fast. Not a bad approximation at all.

Methods like this are hardly used these days as there is so much computing power. A gearing problem can be solved using brute force that would have been horrific in the times of Camus or Brocot. If you need to approximate a ratio have a computer try all pairs of gears with no more than 100 teeth. There are 10,000 combinations which a computer can churn out in seconds. For a two-stage gear train, running through the 100 million possibilities takes a computer minutes. As is said by Hayes in "On the teeth of wheels," "The whirling gears of progress have put the gear-makers out of work."[12]

Chapter 3

Approximation and Fibonacci

3.1 Approximating irrationals using Farey Sequences

Given an irrational on the interval $[0, 1]$, we can find a rational approximation using the Farey sequences, as mentioned in the previous section. We can choose to limit the size of the denominator in our approximation, by restricting the size of the Farey sequence, F_n, i.e. we restrict the value of n. For example, we are asked to find an approximation of $\frac{1}{\pi} = 0.318309886183791$ where the denominator is no larger than 100. The first and easiest approximation is probably $\frac{32}{100} = \frac{8}{25}$. But is there a more suitable approximation?

It is possible to use the Farey sequence as a tool for approximation by using the mediant.

As we saw with the Stern-Brocot section, given an irrational number between 0 and 1 it is possible to find a rational approximation using iterations of the mediant in the Farey sequence. See for example, Cook [15].

Example 16. *Look for the closest rational approximation of $\frac{1}{\pi}$ where the denominator is less than* 100*. One can iterate using Theorem 8 to find the closest rational approximation. That is to say that if given an interval in which the rational number is known to be present, the mediant of that interval will provide two smaller intervals. The rational number will then lie in one of the intervals upon which the mediant is carried out a further time. This provides a way of getting* *closer to the rational.*

The first interval is the first Farey sequence, in other words $\left\{\frac{0}{1}, \frac{1}{1}\right\}$. Its corresponding mediant is $\frac{1}{2}$. The next step is to decide whether $\frac{1}{\pi}$ lies in $\left\{\frac{0}{1}, \frac{1}{2}\right\}$ or $\left\{\frac{1}{2}, \frac{1}{1}\right\}$. Since $\frac{1}{\pi} < \frac{1}{2}$ the new interval becomes $\left\{\frac{0}{1}, \frac{1}{2}\right\}$ and the current approximation is $\frac{1}{2}$. With this interval a new mediant can be found and subsequently a new approximation. The new interval is $\left\{\frac{0}{1}, \frac{1}{2}\right\}$ and its corresponding mediant is $\frac{1}{3}$. These steps are repeated until the denominator of the new approximation lies just under 100 and a further iteration forces the denominator to be more than 100. After a few steps the rational approximation of $\frac{1}{\pi}$ is $\frac{29}{91}$. It may be noticed that $\frac{7}{22}$ is still a better approximation than $\frac{29}{91}$ and therefore shows that despite the fact that a new approximation is found after each step, the best approximation is not necessarily the newest fraction found after each iteration. The next approximation which is more accurate than $\frac{7}{22}$ is in fact $\frac{57}{179}$.

The one big drawback of this process is that in order to find the most suitable rational approximation the exact value of the irrational to be approximated needs to be known in order to choose the correct intervals. This process searches for fractions around the irrational but does not necessarily find a better one after each iteration.

As we have seen in the previous section, the Stern-Brocot tree has been made redundant using developments in computing. We wrote a *Matlab* code which takes as its input an irrational number to be approximated and the limit of the denominator at which to stop. The code is shown in the Appendix. Figure 6.3. The following is the output of the *Matlab* code:

```
>> fareyapprox(1/pi)
Maximum denominator size:  100
Elapsed time is 0.000183 seconds.

Fraction =

    29    91


StrnBrct =

LLRRRRRRRLLL


Bfraction =

     7    22


BStrnBrct =

LLRRRRRR
```

Figure 3.1: Output from Matlab for approximating $\frac{1}{\pi}$

This code outputs the limit of the iteration process, its path down the Stern-Brocot tree, the best fractional approximate, its path down the Stern-Brocot tree and how long the iteration took. It took *Matlab* only 0.000183 seconds to find an approximate for $\frac{1}{\pi}$. This code is a modern way of using the Stern-Brocot tree to approximate values. The code only considers the left-hand branch of the tree because it only looks at Farey fractions and hence fractions between 0 and 1.

3.2 Farey Sequences of Fibonacci Numbers

The Fibonacci sequence is given by $\varphi = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. The next term in the sequence is the sum of the previous two, $\varphi_m = \varphi_{m-1} + \varphi_{m-2}$.

Definition 17. *Define the sequence of Fibonacci fractions as:*
$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \ldots, \frac{\varphi_m}{\varphi_{m+2}}, \frac{\varphi_{m+1}}{\varphi_{m+3}}, \ldots$

We know from F_3 that $\frac{1}{2}$ and $\frac{1}{3}$ are Farey neighbours.

Theorem 18. *It can be shown that any two neighbouring fractions in the sequence of Fibonacci fractions are neighbours in the Farey sequence.*

We need to show that for all n, $\frac{\varphi_n}{\varphi_{n+2}}$ and $\frac{\varphi_{n+1}}{\varphi_{n+3}}$ are Farey neighbours. That is $|\varphi_{n+1}\varphi_{n+2} - \varphi_n\varphi_{n+3}| = 1$.

Proof by induction.

Take $n = 1$:

$|1 * 2 - 1 * 3| = 1$

Suppose it is true for $n = k$

Take $n = k + 1$:

$$|\varphi_{k+2}\varphi_{k+3} - \varphi_{k+1}\varphi_{k+4}|$$

Since $\varphi_n = \varphi_{n-1} + \varphi_{n-2}$

$$|\varphi_{k+2}\varphi_{k+3} - \varphi_{k+1}\varphi_{k+4}| = |(\varphi_{k+1} + \varphi_k)\varphi_{k+3} - \varphi_{k+1}(\varphi_{k+2} + \varphi_{k+3})|$$

$$= |\varphi_{k+1}\varphi_{k+3} + \varphi_k\varphi_{k+3} - \varphi_{k+1}\varphi_{k+2} - \varphi_{k+1}\varphi_{k+3}|$$

$$= |\varphi_k\varphi_{k+3} - \varphi_{k+1}\varphi_{k+2}|$$

Therefore by induction the statement is true for all n. *End of proof.* [17]

Chapter 4

Ford Circles

4.1 Introduction

The aim of this part of the project is to explore how circles - arguably the most studied objects in mathematics - are intimately linked to the rationals in a non-obvious way. The usual way to represent rationals geometrically is as points on the real numberline. We will look at a different geometric representation of the rationals, as described by Lester Ford [18]. We will discuss some of the properties of this geometry as well as using a group theoretic approach to describe it. We will see that these circles are in bijection with the Farey sequences.

4.2 Motivation and definition

Throughout this section we will continue to consider sets of rational numbers on the interval $[0, 1]$ in their simplified form. For each simplified rational number $\frac{p}{q}$ we can define a circle in the upper-half plane which is tangent to the point $\frac{p}{q}$ on the real line.

We can see that these circles intersect.

If we define the radius of the circles as $\frac{1}{2q^2}$ we get some interesting results - all the circles which intersect do so trivially, that is, any two circles are either disjoint or tangent.

Figure 4.3 is an image of the seven largest such circles with radius $\frac{1}{2q^2}$ on the

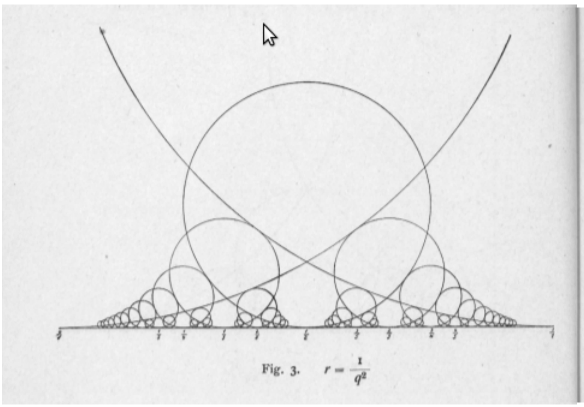


Figure 4.1: $r = 1/q^2$

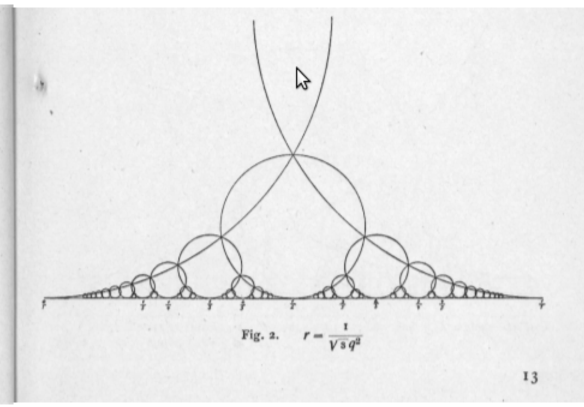


Figure 4.2: $r = 1/\sqrt{3}q^2$

interval $[0, 1]$.

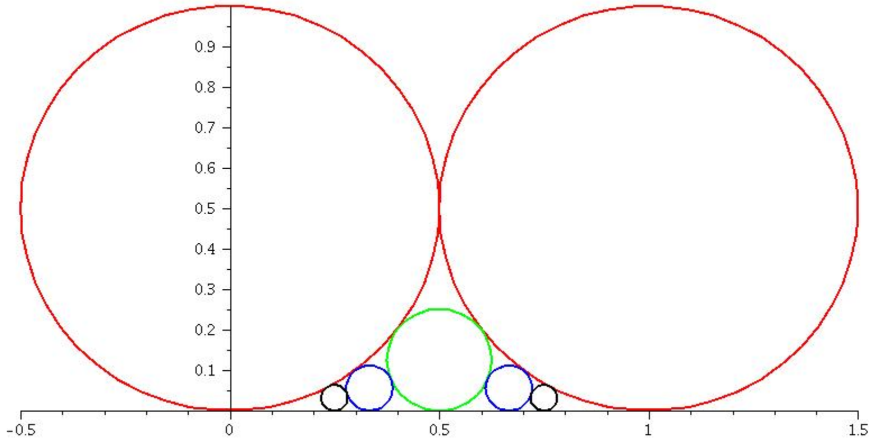


Figure 4.3: The seven largest Ford circles, taken from our Maple worksheet

We call sets of circles of this form, as in Figure 4.3, sets of *Ford circles*, named after the American mathematician Lester Ford, who worked at Edinburgh University in 1914 [20].

Remark 19. *The images from Züllig's paper [19] suggest a motivation for choosing the diameter to be $\frac{1}{q^2}$. We can see that as the radius approaches $\frac{1}{2q^2}$ the circles approach tangency. This is not the motivation that Ford had for working with these sets of circles - a brief description of this will come at the end of the chapter.*

We will use the obvious notation to represent a Ford circle:

Definition 20. *Let the circle $C(\frac{p}{q})$ be defined as the circle lying in the upper-half plane tangent to the point $\frac{p}{q}$ on the real line with radius $\frac{1}{2q^2}$. Each such circle is* *a Ford circle.*

Example 21. *The green circle in Figure 4.3 is called* $C(\frac{1}{2})$.

Definition 22. C_n *is the set of Ford circles which contains all the circles with diameter* $\geq \frac{1}{n^2}$.

Example 23. *Figure 4.3 represents the set of Ford circles* C_4.

4.3 Properties of Ford circles

We can see intuitively how Ford circles are related to the Farey sequence. For example the circles are symmetric about $\frac{1}{2}$, as are the elements in F_n. We will proceed to show that there exists a bijection between the Ford circles and the Farey sequence. This result lies close to the heart of mathematics - linking number theory and geometry.

Proposition 3. *The set of Ford circles* C_n *is in one-to-one correspondence with the Farey sequence* F_n.

Example 24. *If we look back to Figure 4.3 we can map the Ford circles one-to-one with the following Farey sequence (from left to right according to colour):*

$F_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$

Just as with the Farey sequence, we can generate the set of Ford circles iteratively ("in layers"). The set of circles C_n is precisely the set C_{n-1} in union with the set of circles of radius $\frac{1}{2n^2}$. In other words, the "new elements" in each Farey sequence (the ones in F_n which are not in F_{n-1}) are given by the set of Ford circles which have the smallest radius in the diagram for C_n.

Example 25. *Again, looking back to Figure 4.3 we can see the set of circles* C_1 *in red, the set* $C_2 - C_1$ *in green, the set* $C_3 - C_2$ *in blue and the set* $C_4 - C_3$ *in black.*

Here are the two main properties of the Farey sequences which we saw in Chapter 1 - the mediant property (Theorem 8) and the neighbours property (Theorem 10) expressed in geometric language of Ford circles:

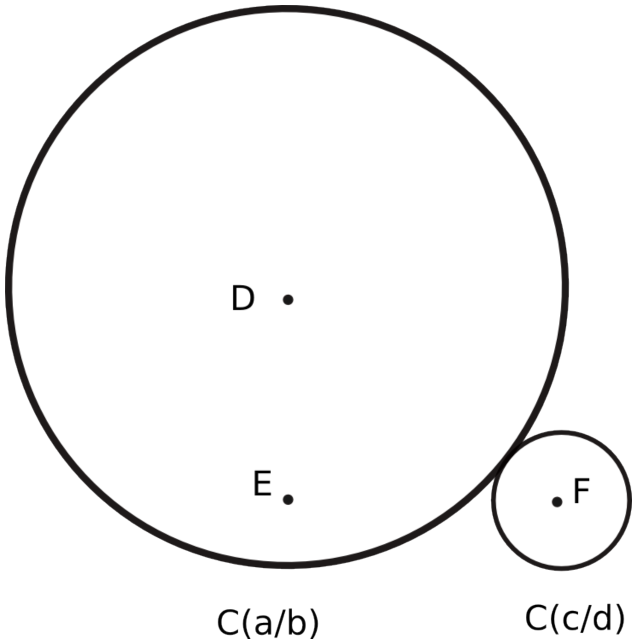


Figure 4.4: Geometric proof of neighbours

Theorem 26 (Neighbours property for Ford circles). *The intersection of two Ford circles is a singleton iff $|bc - ad| = 1$. That is, we have an analogous definition to Farey neighbours and thus we can describe two circles as Ford neighbours iff $|bc - ad| = 1$.*

We have seen the algebraic proof in the context of Farey sequences. Now let's look at a geometric proof.

Proof. Consider two tangent Ford circles $C(\frac{a}{b})$ and $C(\frac{c}{d})$ with $\frac{a}{b} < \frac{c}{d}$. As in Figure 4.4. Recall that we know the radii of the circles in terms of a,b,c, and d. Consider the right-angled triangle DEF, where D is the center of $C(\frac{a}{b})$ and F is the center of $C(\frac{c}{d})$ and E marks the intersection of the horizontal line through F and the vertical line through D. Since we know the radius of the circles we can calculate the coordinates of D, E and F. We have that length $E = \frac{1}{2b^2} - \frac{1}{2d^2}$, $D = \frac{1}{2b^2} + \frac{1}{2d^2}$, and $F = \frac{c}{d} - \frac{a}{b}$. From the Pythagorean Theorem we can generate the identity:

$F^2 + E^2 = D^2$ which can be written algebraically as $(\frac{c}{d} - \frac{a}{b})^2 + (\frac{1}{2b^2} - \frac{1}{2d^2})^2 = (\frac{1}{2b^2} + \frac{1}{2d^2})^2$ If we expand the brackets and cancel like terms we get $c^2b^2 + a^2d^2 - 2acbd - 1 = 0$ and we can factorise this to $(ad - bc)^2 = 1$ Hence it follows that $|bc - ad| = 1$, as required.

The argument is reversible. Assume that $|bc - ad| = 1$ holds. Using the usual notation for lines, DE and EF are defined according to the points described above and the line DF (the line connecting the centers of the two circles) will be equal to the sum of their radii. Which directly implies that the circles are tangent to each other. $\square$

The above proof is based on the proof which can be found on Cut the Knot [21].

Example 27 (Computing the lengths)**.** *Consider the right-angled triangle whose vertices are the center of $C(1)$, the center of $C(\frac{1}{2})$ and the point $(1, \frac{1}{8})$. Recall that $C(1)$ is tangent to the real line at 1 which in this example is $\frac{c}{d}$ and $C(\frac{1}{2})$ is tangent at $\frac{1}{2} = \frac{a}{b}$.*

We can compute the lengths of the sides of the triangle to be $\frac{4}{8}$, $\frac{3}{8}$, $\frac{5}{8}$.

Remark 28 (Ex-neighbours in the Farey sequence)**.** *We can see from the theorem above that the Ford circles contain more information than the Farey sequences. In the general Farey sequence F_n there is no way of telling from the written set of fractions whether $\frac{0}{1}$ and $\frac{1}{1}$ were once neighbours (they were, in F_1). In the diagram for the general set of Ford circles C_n we can see that $C(0)$ and $C(1)$ are tangent and hence were once neighbours in a Farey sequence. We could say that the information about "ex-neighbours" is preserved in the diagrams of Ford circles.*

Theorem 29 (Mediant property for Ford circles)**.** *If $C(\frac{a}{b})$ and $C(\frac{c}{d})$ are tangent Ford circles (i.e. they are Ford neighbours) then $C(\frac{a+c}{b+d})$ is the unique circle tangent to the real line and both the circles. That is, $C(\frac{a+c}{b+d})$ is the mediant Ford circle.*

Proof. For any fraction we can define a Ford circle. Hence the mediant Ford circle exists. Since $|bc - ad| = 1$ implies that both $(b + d)c - (a + c)d = 1$ (i.e. the equation obtained by plugging in the values of the two kissing circles $C(\frac{a+b}{c+d})$ and $C(\frac{c}{d})$ and $(a + c)b - (b + d)a = 1$ (i.e. the same equation again describing the kissing circles $C(\frac{a+b}{c+d})$ and $C(a/b)$, it is clear that if there is a mediant Ford circle then it touches the other two circles iff they are tangent to each other. $\square$

This proof is adapted from Cut the Knot [21].

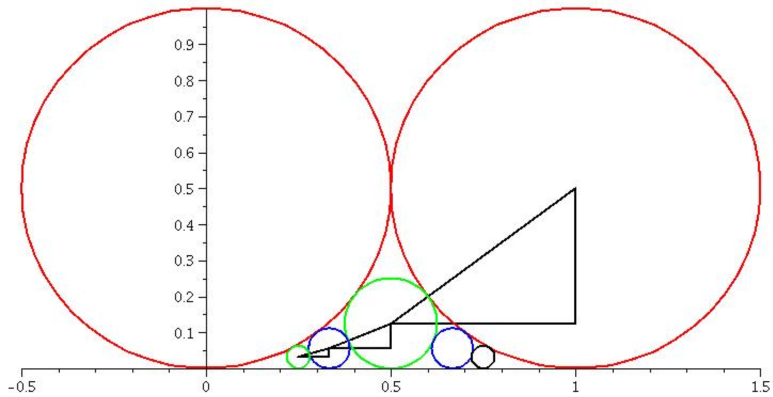


Figure 4.5: The right-angled triangles represent distinct pythagorean triples

Remark 30 (Pythagorean triples)**.** *If we look back to example 27, the sides of the triangle correspond to the primitive Pythagorean triple* $3, 4, 5$*.*

There is a unique (up to the symmetries of the diagram) primitive pythagorean triple for any pair of Ford neighbours. A description of how to find all primitive Pythagorean triples can be found at Cut the Knot [22].

Remark 31 (Ladies' diary and a geometric picture of $\mathbb{Q}$)**.** *Note that if we look back to the problem in the Ladies' Diary then by the neighbours property of Ford circles we have shown that each of the* 3005 *circles contained in* C_{99} *are either tangent or disjoint. In general, there is a Ford circle for every rational. Furthermore, recall the fact that the mediant generates all of the rationals and the mediant never produces a fraction which is not in simplified form e.g. we never produce* $\frac{2}{4}$*. If we start with* $C(1)$ *and* $C(0)$ *and a straight line then we can generate the rational points on the real line by constructing Ford circles.*

4.4 From algebra to geometry and back to algebra

In this section we will sketch a group theoretic approach to the analysis of Ford circles. We will see that this approach leads immediately to some new results. The following approach is described by Carnahan [23]. This section assumes the reader has some familiarity with Groups, Group actions and Möbius transformations.

Recall that the matrix group $\mathbb{SL}_2(\mathbb{Z})$ is the set of 2×2 matrices with integer

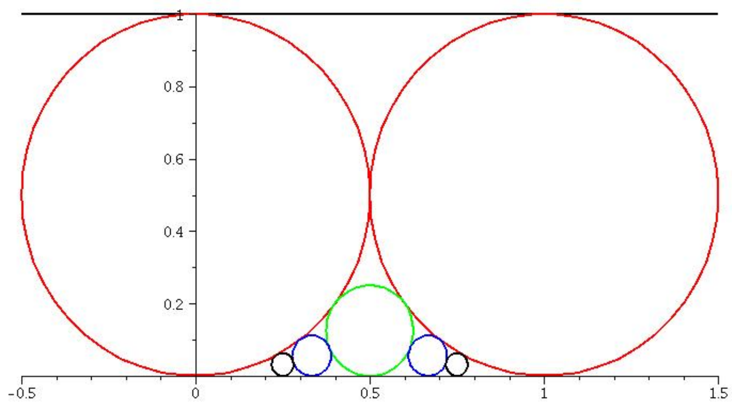


Figure 4.6: This image depicts the Ford circle $C(\frac{1}{0})$ in black

coefficients such that $|bc - ad| = 1$. The elements of the rows and columns of the matrices are coprime.

Definition 32. *The Ford circle of infinite radius $C(\frac{1}{0})$ is equal to the line $\mathbb{R} + i$.*

We will proceed to consider the Ford circles as curves in the complex plane rather than in the Cartesian plane as before.

Proposition 4. *Given a matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathbb{SL}_2(\mathbb{Z})$ the corresponding Möbius transformation maps $C(\frac{1}{0})$ to $C(\frac{a}{b})$.*

This proposition can be verified directly: take a point $x + i$ which lies on the line $\mathbb{R} + i = C(\frac{1}{0})$. The point maps to $z = \frac{ax+ai+c}{bx+bi+d}$, which lies on $C(\frac{a}{b})$ whose center is $(\frac{a}{b}, \frac{1}{2b^2})$ and whose radius is $\frac{1}{2b^2}$ hence the equation of the circle is given by $|z - (\frac{a}{b} + i(\frac{1}{2b^2}))|^2 = (\frac{1}{2b^2})^2$ and indeed it can be checked that the points lie on this circle, see Carnahan [23].

Proposition 5. *The action induced on two Ford circles $C(\frac{a}{b})$ and $C(\frac{c}{d})$ preserves $|bc - ad| = 1$*

The above propositions shows that Ford circles map to Ford circles. With these two results we can succinctly prove the two key properties of Ford circles (the neighbour and mediant properties). We will look at the latter here, the former is covered by Carnahan [23].

Theorem 33. *The mediant property holds for Ford Circles.*

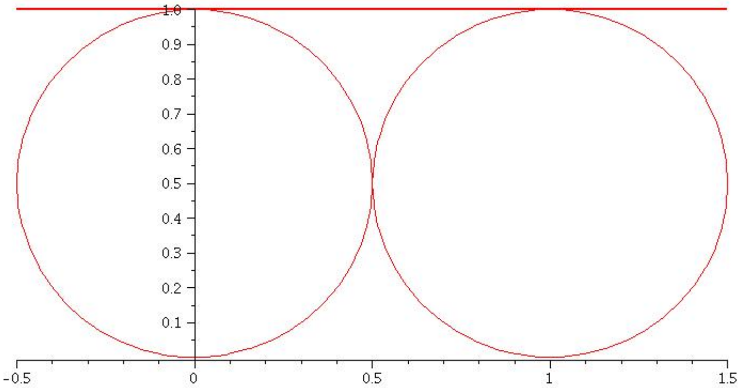


Figure 4.7: The three Ford circles used in the proof of Theorem 33

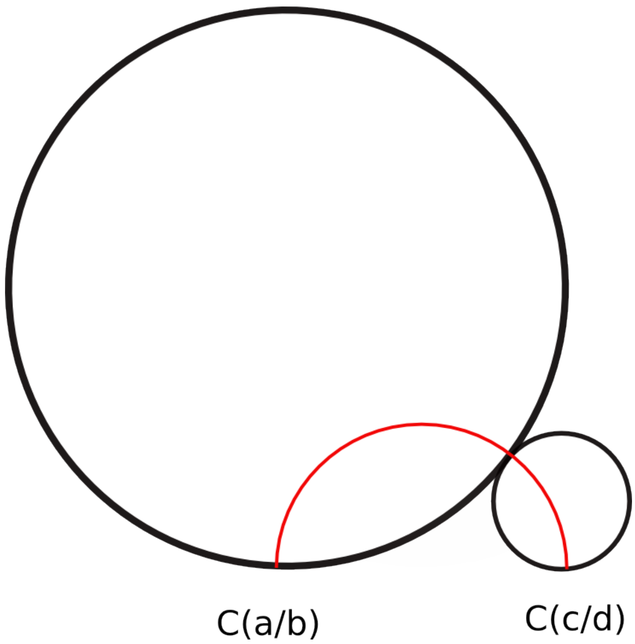


Figure 4.8: The semicircle whose diameter is $\frac{c}{d} - \frac{a}{b}$

Proof. We can consider the neighbours $C(\frac{1}{0})$ and $C(0)$ as well as their mediant $C(1)$. Since we are working with groups and groups actions it is enough to show that $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ maps these circles to $C(\frac{a}{b})$, $C(\frac{c}{d})$ and $C(\frac{a+c}{b+d})$ respectively, which we have already demonstrated in the proposition above. Hence we have the required result. $\square$

What are the advantages of this group-theoretic approach? There are three advantages:

1. We can find the points of intersection of any two intersecting Ford circles immediately. From observing that $C(1) \cap C(0) = i$ we can conclude that

$C(\frac{a}{b}) \cap C(\frac{c}{d}) = \frac{ai+c}{bi+d}$

2. We can see that the semicircle whose diameter is $\frac{c}{d} - \frac{a}{b}$ goes through the point of intersection of the two circles $C(\frac{a}{b})$ and $C(\frac{c}{d})$, see Figure 4.8. This ultimately gives rise to an interesting graph (the Farey Diagram). See, for example, Hatcher [24].

3. We can use Ford circles to express the irrationals by continued fraction chain of rational Ford circles. We have a natural way to do this which Ford discusses in his paper Fractions (1938) [18]. This is related to the earlier section in this report on irrational approximations. Ian Short provides a method for constructing these approximations and suggests a way to extend this approach to approximating irrational complex numbers with rational complex numbers [25].

When we construct irrational complex numbers from rational complex numbers in the way described above it requires a definition of Ford Spheres, which are described in Ford's paper, Fractions (1938) [18].

Ford first found the Ford spheres and then simplified these spheres to the two-dimensional version - the Ford circles. Which indeed provides the historically accurate motivation for the Ford circles, as promised earlier in the report. A more detailed account of Ford's discovery can be found in the introduction to his 1938 paper [18]. *Disclaimer! Ford circles were found in ancient Japanese Sangaku mathematics too [26] .*

Chapter 5

The Farey Sequence and The Riemann Hypothesis

The Riemann Hypothesis is one of the most famous mathematical problems in recent history. It has remained unproven since Bernhard Riemann (1826-1866) first conjectured it in his 1859 paper, "*Ueber die Anzahl der Primazahlen unter einer gegebenen Grösse*" (On the Number of Primes Less Than a Given Magnitude) and is one of the Clay Mathematics Institute's millennium problems, with $1,000,000 being awarded to anyone who can prove it. [27]

As is obvious from the title, Riemann wanted to find the number of primes less than a given number. To do this he turned to the zeta function (now called the Riemann zeta function).

Definition 34. *The Riemann Zeta Function is defined for all* $s \in \mathbb{C} \backslash \{1\}$ *as follows:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The Zeta function converges absolutely for all complex s with $Re(s) > 1$.

- $\zeta(1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$ (the harmonic series), thus $s = 1$ is a simple pole
- $\zeta(s) = 0$ when $s = -2, -4, -6, \ldots$, these are known as the trivial zeros

The trivial zeros occur at $s = -2, -4, -6, -8, \ldots$ because of the relation between the zeta function and Bernoulli numbers[28]:

$$\zeta(1-n) = \frac{-B_n}{n}, n \in \mathbb{N}, n \geq 2$$

Where $B_n = 0$ for odd $n \geq 3$.[29]

The Zeta function is related to the distribution of primes by equality to the Euler Product (note p is prime):

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdots$$

The Riemann Hypothesis *All the non-trivial zeros to the Riemann zeta function lie on the critical line* $s = \frac{1}{2} + it$

It is important to note that this does not mean *every* point on this line is a zero, but that every non-trivial zero lies on it for some t. 1,500,000,000 solutions to $\zeta(s) = 0$ have been found to lie on the critical line (and no where else), but a formal proof for all cases has never been found.[27]

Whilst having never been proved, equivalent statements to the Riemann hypothesis exist. One such statement is *Mertens conjecture*.

Notation (Little o): $f(x) = o(g(x))$ means $f(x)$ has a smaller rate of growth than $g(x)$. In other words, for all constants $C > 0$:

$$|f(x)| \leq C \cdot |g(x)|$$

Which is equivalent to

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

Mertens Conjecture

Before stating Mertens conjecture, the Möbius Function and Mertens Function must be defined.

Definition 35. *Möbius Function*

$$\mu(k) = \begin{cases} 0 & \textit{if } k \textit{ is not square free} \\ (-1)^p & \textit{if } k \textit{ is the product of } p \textit{ distinct primes} \end{cases}$$

A number is square-free if it does not contain any squares in its prime decomposition (all its prime factors are unique). For example $18 = 3^2 \cdot 2$, thus is not square-free.

- $\mu(1) = 1$
- $\mu(2) = (-1)^1 = -1$ as 2 is prime
- $\mu(4) = \mu(2^2) = 0$ as 4 is not square-free
- $\mu(6) = \mu(3 \cdot 2) = (-1)^2 = 1$ as 6 is the product of two distinct primes

Definition 36. *Mertens Function*

$$M(n) = \sum_{k \leq n} \mu(k) \tag{5.1}$$

- $M(2) = \mu(1) + \mu(2) = 1 + (-1) = 0$
- $M(3) = \mu(1) + \mu(2) + \mu(3) = 1 + (-1) + (-1) = -1$
- $M(6) = 1 + (-1) + (-1) + 0 + (-1) + 1 = -1$

Mertens function moves slowly, only able to increase or decrease by 1 at the most at each step, and there is no n such that $|M(n)| > n$.

In 1885, Dutch mathematician Thomas Joannes Stieltjes conjectured that there was no n such that $|M(n)| > n^{\frac{1}{2}}$, yet a hundred years later in 1985 this was proven false by Andrew Odlyzko and Herman te Riele.[30]

However, the Riemann Hypothesis is equivalent to the weaker conjecture that:

$$M(n) = o(n^{\frac{1}{2}+\epsilon}) \tag{5.2}$$

$\forall \epsilon > 0$. In other words, for all C:

$$M(n) \leq Cn^{(\frac{1}{2}+\epsilon)}$$

Meaning $|M(n)|$ is bounded by $n^{\frac{1}{2}+\epsilon}$. It is using this that the equivalent statement to the Riemann hypothesis using the Farey sequence is drawn.

5.1 Equivalent statement to the Riemann Hypothesis using the Farey Sequence

As with Mertens conjecture, before we can draw the equivalent statement, it is necessary to make a few definitions.

Definition 37. *Let $L(n)$ be the length of the Farey Sequence F_n and r_v be the v^{th} Farey term. We define the difference, δ_v in the following way:*

$$\delta_v = r_v - v/L(n) \tag{5.3}$$

Example 38.

$$\begin{aligned}
F_5 &= \{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\} \\
L(5) &= 11 \\
\delta_1 &= \frac{0}{1} - \frac{1}{11} = -\frac{1}{11} \\
\delta_2 &= \frac{1}{5} - \frac{2}{11} = \frac{1}{55} \\
&\vdots \\
\delta_{11} &= \frac{1}{1} - \frac{11}{11} = 0
\end{aligned}$$

In 1924, the Franel-Landau theorem was published[4], stating that:

$$\sum_{v=1}^{L(n)} |\delta_v| = o(n^{\frac{1}{2}+\epsilon}) \tag{5.4}$$

$\forall\epsilon > 0$ and where n refers to F_n, which is equivalent to the Riemann hypothesis.

This is the link between the Farey sequence and the Riemann hypothesis. To show its equivalence to the Riemann hypothesis, the paper contained the proof that:

$$\sum_{v=1}^{L(n)} |\delta_v| = o(n^{1/2+\epsilon}) \Leftrightarrow M(n) = o(n^{1/2+\epsilon}) \tag{5.5}$$

Thus, it is true if and only if Mertens conjecture is true, which is equivalent to the Riemann hypothesis. We will follow the proof of this statement from H. M. Edwards' *"Riemann's Zeta Function"*[32], following it from left to right, then right to left.

The key to the proof is to relate δ_v to $M(x)$. If that is achieved then we can start to prove 5.5.

To do this, we first look at a real-valued function defined on the interval $[0, 1]$. Let r_v denote the elements of the Farey sequence, as above. The sum of the function at the points in the Farey sequence can be related to Mertens function by the following equality:

$$\sum_{v=1}^{L(n)} f(r_v) = \sum_{k=1}^{\infty}\sum_{j=1}^{k} f(\frac{j}{k})M(\frac{n}{k}) \tag{5.6}$$

5.2 Forward Proof of Equivalent Statement

In this section, we want to show:

$$\sum_{v=1}^{L(n)} |\delta_v| = o(n^{1/2+\epsilon}) \Rightarrow M(n) = o(n^{1/2+\epsilon})$$

Taking formula 5.6 and applying it to the function $f(u) = e^{2\pi iu}$ gives

$$\sum_{v=1}^{L(n)} e^{2\pi iu} = \sum_{k=1}^{\infty}\sum_{j=1}^{k} e^{2\pi ij/k}M(\frac{n}{k}) \tag{5.7}$$

Lemma 39. *$\sum_{j=1}^{k} e^{2\pi ij/k} = 0$ unless $k = 1$, in which case it equals 1.*

This simplifies the previous equation to:

$$
\begin{aligned}
M(n) &= \sum_{v=1}^{L(n)} e^{2\pi i r_v} \\
&= \sum_{v=1}^{L(n)} e^{2\pi i[(v/L(n))+\delta_v]} \\
&= \sum_{v=1}^{L(n)} e^{2\pi iv/L(n)}(e^{2\pi i\delta_v}-1) + \sum_{v=1}^{L(n)} e^{2\pi iv/L(n)} \\
\Rightarrow |M(n)| &\leq \sum_{v=1}^{L(n)} |(e^{2\pi i\delta_v}-1)| + 0
\end{aligned}
$$

Rearranging the right hand side gives:

$$
\begin{aligned}
|M(n)| &\leq \sum_{v=1}^{L(n)} |e^{\pi i\delta_v}||(e^{\pi i\delta_v} - e^{-\pi i\delta_v})| \\
&= 2\sum_{v=1}^{L(n)} |sin(\pi\delta_v)| \\
&\leq 2\pi\sum_{v=1}^{L(n)} |\delta_v|
\end{aligned}
$$

Using 5.4 and letting $2\pi \cdot K(\epsilon) = K'(\epsilon)$ we gain the inequality:

$$|M(n)| \leq K'(\epsilon)n^{1/2+\epsilon}$$

Therefore:

$$M(n) = o(n^{1/2+\epsilon})$$

$\forall\epsilon > 0$

5.3 Backward Proof of Equivalent Statement

This time we want to show:

$$M(n) = o(n^{1/2+\epsilon}) \Rightarrow \sum |\delta_v| = o(n^{1/2+\epsilon})$$

Definition 40. *Bernoulli Polynomials.*

The n^{th} Bernoulli polynomial $B_n(u)$ satisfies the equation:

$$\int_x^{x+1} B_n(u)du = x^n \tag{5.8}$$

Bernoulli polynomials can be extended to be periodic with period 1, denoted here as $\bar{B}_n(u)$.

Example 41.

$$B_1(u) = u + \frac{1}{2} \tag{5.9}$$

$$\bar{B}_1(u) = u - \lfloor u \rfloor + \frac{1}{2} \tag{5.10}$$

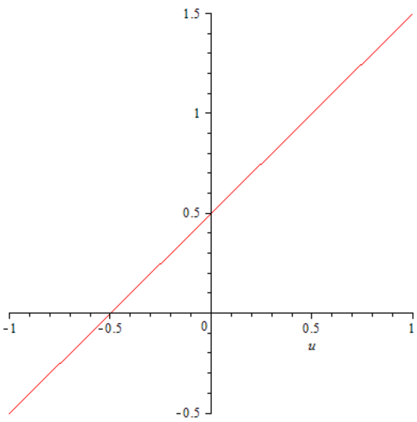


Figure 5.1: Graph of $B_1(u)$

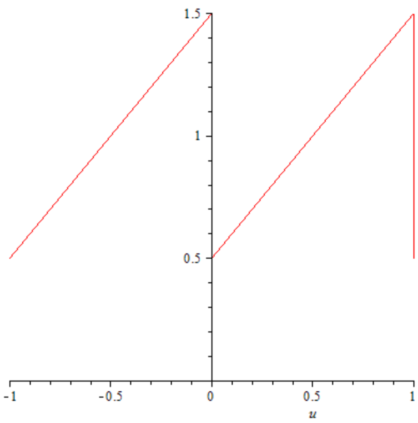


Figure 5.2: Graph of $\bar{B}_1(u)$

A property of Bernoulli polynomials that will be made use of later is the following:

$$B_n(ku) = k^{n-1}[B_n(u) + B_n(u + \frac{1}{k}) + ... + B_n(u + \frac{k-1}{k})] \tag{5.11}$$

It should be noted that this identity also holds for periodic Bernoulli polynomials.

Define the function G to be:

$$G(u) = \sum_{v=1}^{L(n)} \bar{B}_1(u + r_v) \tag{5.12}$$

And I as:

$$I = \int_0^1 [G(u)]^2 du \tag{5.13}$$

Using the form 5.6, it is possible to write G as:

$$G(u) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} \bar{B}_1(u + \frac{j}{k}) M(\frac{n}{k})$$

And then applying the identity 5.11:

$$G(u) = \sum_{k=1}^{\infty} \bar{B}_1(ku) M(\frac{n}{k}) \tag{5.14}$$

Which gives two expressions for G, and thus two ways to evaluate I.

First consider equation 5.12. This form shows that at every Farey fraction, r_v, the function jumps down by 1 and increases with a gradient of $L(n)$ between r_v and r_{v+1}.

Between r_v and r_{v+1}, G acts like the function $G(u) = L(n)u - v - 1/2$, as the following graphs illustrate.

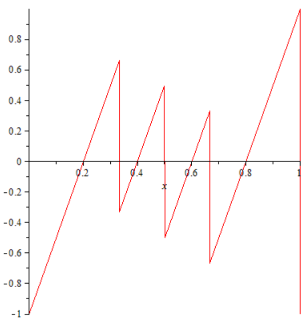


Figure 5.3: G(u) using elements of F_3

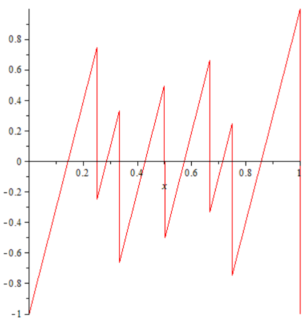


Figure 5.4: G(u) using elements of F_4

This gives us a new representation of I

$$\begin{aligned} I &= \sum_{v=1}^{L(n)} \int_{r_{v-1}}^{r_v} (L(n)u - v - \frac{1}{2})^2 du \\ &= \sum_{v=1}^{L(n)} \frac{1}{L(n)} \frac{(L(n)u - v + \frac{1}{2})^3}{3} |_{r_{v-1}}^{r_v} \end{aligned}$$

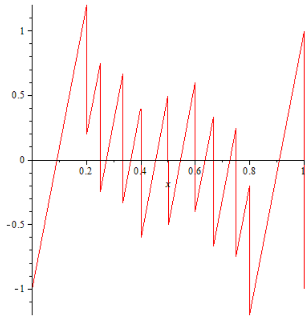


Figure 5.5: G(u) using elements from F_5

As the first term of every Farey sequence is 0, $r_0 = 0$.

$$I = \sum_{v=1}^{L(n)} \left(\frac{1}{L(n)} \frac{(L(n)r_v - v + \frac{1}{2})^3}{3} - \frac{1}{L(n)} \frac{(L(n)r_{v-1} - v + \frac{1}{2})^3}{3}\right)$$

Substituting $L(n)r_v = L(n)[r_v - \frac{v}{L(n)} + \frac{v}{L(n)}] = L(n)\delta_v + v$:

$$\begin{aligned} I &= \frac{1}{3L(n)} \sum_{v=1}^{L(n)} [(L(n)\delta_v + \frac{1}{2})^3 - (L(n)\delta_{v-1} - \frac{1}{2})^3] \\ &= \frac{1}{3L(n)} \sum_{v=1}^{L(n)} [(L(n)\delta_v + \frac{1}{2})^3 - (L(n)\delta_v - \frac{1}{2})^3] \\ &= \frac{1}{3L(n)} \sum_{v=1}^{L(n)} [2 \cdot 3(L(n)\delta_v)^2 \cdot \frac{1}{2} + (\frac{1}{2})^3] \end{aligned}$$

$$= L(n) \sum_{v=1}^{L(n)} \delta_v^2 + \frac{1}{12} \tag{5.15}$$

Which provides us with an explicit formula for I.

Using $G(u)$ from 5.14 and the fact that the sum is finite, we can evaluate I as follows:

$$I = \sum_{a=1}^{L(n)} \sum_{b=1}^{L(n)} M(\frac{n}{a}) M(\frac{n}{b}) \int_0^1 \bar{B}_1(au)\bar{B}_1(bu)du \tag{5.16}$$

Let $I_{ab} = \int_0^1 \bar{B}_1(au)\bar{B}_1(bu)du$, which can be evaluated explicitly.

To find an explicit formula for I_{ab}, it is essential to consider three cases.

1. $b = 1$

2. b and a are coprime

3. a and b are not coprime

At first it may seem strange to consider these cases, but once the formula for $b = 1$ is obtained, the other two soon follow, and give us the formula for *any* values of a and b.

Case 1: $b = 1$

$$I_{a1} = \int_0^a \bar{B}_1(au)\bar{B}_1(u)du \tag{5.17}$$

Substituting $v = au \Rightarrow du = a^{-1}dv$

$$= \int_0^a \bar{B}_1(v)\bar{B}_1(\frac{v}{a})a^{-1}dv$$

Substituting $v = k + t \Rightarrow dv = dt$

$$= a^{-1}\sum_{k=0}^{a-1}\int_0^1 \bar{B}_1(k+t)\bar{B}_1(\frac{k}{a}+\frac{t}{a})dt$$

Using the periodicity of $\bar{B}_1$ and 5.11.

$$\begin{aligned} &= a^{-1}\int_0^1 \bar{B}_1(t)\bar{B}_1(a\cdot\frac{t}{a})dt \\ &= a^{-1}\int_0^1 (t-\frac{1}{2})^2 dt \\ \Rightarrow I_{a1} &= (12a)^{-1} \end{aligned}$$

Case 2: a and b are coprime.

Following the same steps as with I_{a1}, the formula quickly becomes:

$$\begin{aligned} I_{ab} &= a^{-1}\int_0^1 \bar{B}_1(t)\bar{B}_1(a\cdot\frac{bt}{a})dt \\ &= a^{-1}\int_0^1 \bar{B}_1(t)\bar{B}_1(bt)dt \end{aligned}$$

Which is in the same form as 5.17, but with $a = 1$ instead. Therefore:

$$\begin{aligned}
&= a^{-1} I_{1b} \\
&= a^{-1}(12b)^{-1} \\
&= (12ab)^{-1}
\end{aligned}$$

Case 3: a and b are not coprime. Let $c = (a, b)$, the greatest common divisor or a and b. Then $a = \alpha c$ and $b = \beta c$.

$$\begin{aligned}
I_{ab} &= \int_0^1 \bar{B}_1(\alpha c u)\bar{B}_1(\beta c u)du \\
&= c^{-1}\int_0^c \bar{B}_1(\alpha t)\bar{B}_1(\beta t)dt \\
&= I_{\alpha\beta} \\
&= (12\alpha\beta)^{-1}
\end{aligned}$$

Substituting $\alpha\beta = \frac{ab}{c^2}$

$$= \frac{c^2}{12ab}$$

Now we have an explicit evaluation of I_{ab} for any values of a and b. If a and b are coprime, we set $c = 1$, if they are not, we let $c = gcd(a, b)$ as before.

This leads to the final equation for I:

$$I = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} M(\frac{n}{a})M(\frac{n}{b})\frac{c^2}{12ab} \tag{5.18}$$

Using $M(n) = o(n^{1/2+\epsilon})\forall\epsilon > 0$, then $\exists C$ such that $M(n) < Cn^{(1/2)+\epsilon}$. Substituting this value for $M(n)$ gives:

$$\begin{aligned}
I &< \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} C(\frac{n}{a})^{(1/2)+\epsilon}C(\frac{n}{b})^{(1/2)+\epsilon}\frac{c^2}{12ab} \\
&= n^{1+2\epsilon}\frac{C^2}{12}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{c^2}{(\alpha c\beta c)^{(3/2)+\epsilon}} \\
&= n^{1+2\epsilon}\frac{C^2}{12}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}\sum_{c=1}^{\infty}\frac{c^2}{\alpha^{3/2}\beta^{3/2}c^{1+2\epsilon}}
\end{aligned}$$

Letting $K = \frac{C^2}{12}$, taking our explicit formula for I from 5.15, and the fact that the infinite sum above converges to 0:

$$L(n)\sum_{v=1}^{L(n)} \delta_v^2 \quad < \quad Kn^{1+2\epsilon}$$

Taking the square root of each side gives:

$$L(n)^{1/2}(\sum_{v=1}^{L(n)}(\delta_v^2))^{1/2} < K^{1/2}n^{1/2+\epsilon}$$

The final step is to show:

$$\sum_{v=1}^{L(n)} |\delta_v| \leq L(n)^{1/2}(\sum_{v=1}^{L(n)}(\delta_v^2))^{1/2} \tag{5.19}$$

This is done by applying the Cauchy-Schwarz inequality:

$$\begin{aligned}
\sum_{v=1}^{L(n)} |\delta_v| &= |\sum_{v=1}^{L(n)}(\pm 1)\delta_v| \\
&\leq (\underbrace{1+1+\cdots+1}_{\text{L(n) times}})^{1/2}(\sum_{v=1}^{L(n)}(\delta_v)^2)^{1/2} \\
&= L(n)^{1/2}(\sum_{v=1}^{L(n)}(\delta_v)^2)^{1/2}
\end{aligned}$$

Thus we obtain

$$\sum_{v=1}^{L(n)} |\delta_v| < K^{1/2}n^{1/2+\epsilon}$$

for all $\epsilon > 0$ Therefore:

$$\sum_{v=1}^{L(n)} |\delta_v| = o(n^{1/2+\epsilon})$$

which concludes the proof of statement 5.5.[32]

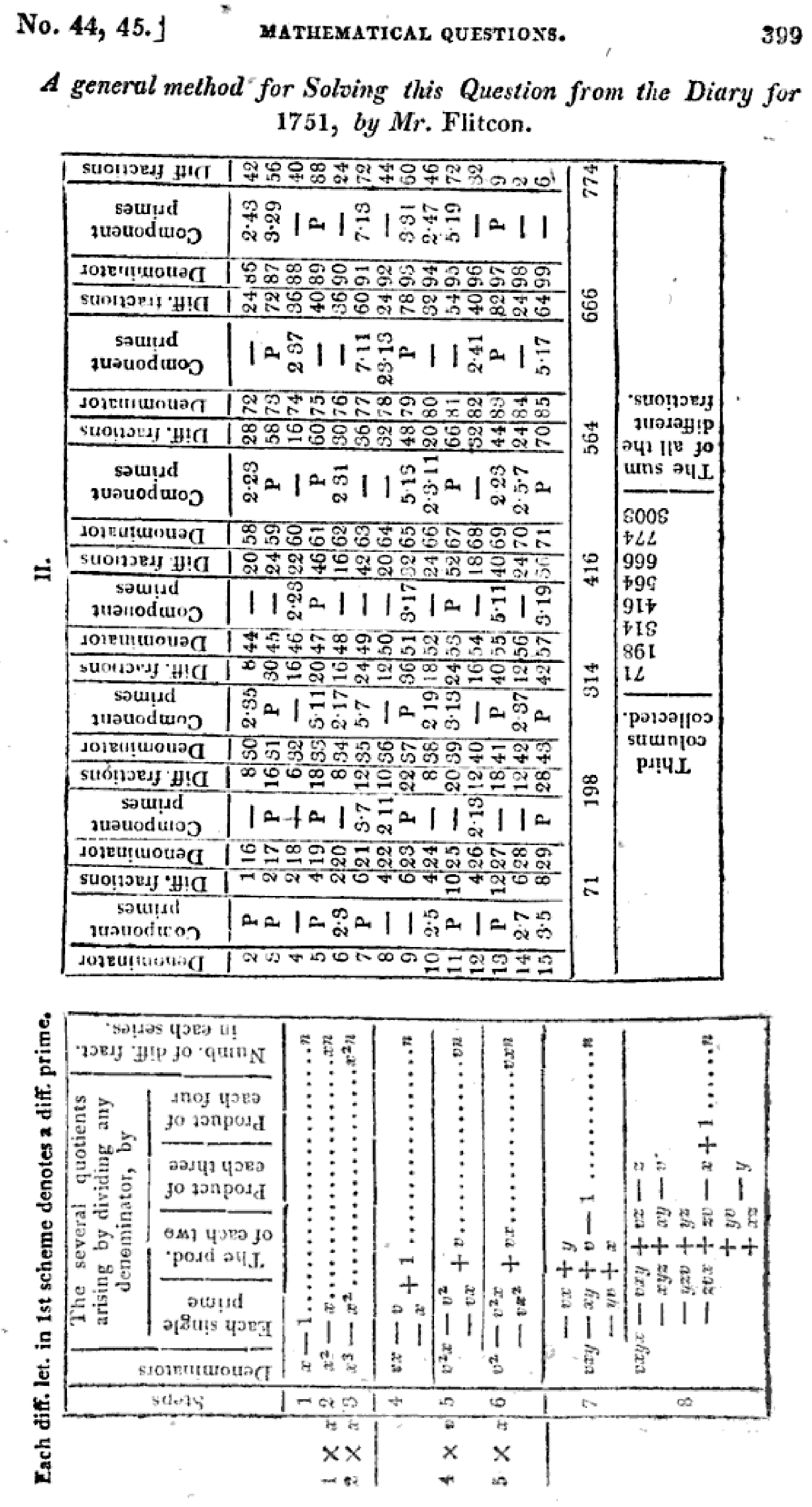
Chapter 6

Appendix

```
> F := proc(N) local a,b; sort([op({seq(seq(a/b,a=0..b),b=1..N)})]) end proc;
> nops(F(100));
> L := proc (n)
   seq(nops(F(i)), i = 1 .. n)
   end proc;
> A := seq(3*n^2/Pi^2, n = 1 .. 100);
> g := proc (x)
  nops(F(x))
  end proc;
> plot(3*n^2/Pi^2, n = 1 .. 100);
> g(12);
> B := NULL:
  for i to 100 do
   B := B, [i, nops(F(i))]:
  end do:
  B;
> plot([B], style = LINE);
> Error2 := proc (n)
   evalf(nops(F(n))-3*n^2/Pi^2)
  end proc;
> Error2(15);
> Errorcoords := NULL:
   for i to 100 do
  Errorcoords := Errorcoords, [i, Error2(i)]
  end do:
>
> Errorcoords;
> plot([Errorcoords], style = LINE);
```

Figure 6.1: Our Maple code

We based our code on JacquesC's function [33] for a Farey sequence.



No. 44, 45.] MATHEMATICAL QUESTIONS. 399

A general method for Solving this Question from the Diary for 1751, by Mr. Flitcon.

II.

Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions
2	P	1	16	—	8	30	2·35	8	44	—	20	58	2·23	28	72	—	24	86	2·43	42
3	P	2	17	P	16	31	P	30	45	—	24	59	P	58	73	P	72	87	3·29	56
4	—	2	18	—	6	32	—	16	46	2·23	22	60	—	16	74	2 37	36	88	—	40
5	P	4	19	P	18	33	5·11	20	47	P	46	61	P	60	75	—	40	89	P	88
6	2·3	2	20	—	8	34	2·17	16	48	—	16	62	2 31	30	76	—	36	90	—	24
7	P	6	21	3·7	12	35	5·7	24	49	—	42	63	—	36	77	7·11	60	91	7·13	72
8	—	4	22	2 11	10	36	—	12	50	—	20	64	—	32	78	23·13	24	92	—	44
9	—	6	23	P	22	37	P	36	51	3·17	32	65	5·13	48	79	P	78	93	3·31	60
10	2·5	4	24	—	8	38	2 19	18	52	—	24	66	2·3·11	20	80	—	32	94	2·47	46
11	P	10	25	—	20	39	3·13	24	53	P	52	67	P	66	81	—	54	95	5·19	72
12	—	4	26	2·13	12	40	—	16	54	—	18	68	—	32	82	2·41	40	96	—	32
13	P	12	27	—	18	41	P	40	55	5·11	40	69	2 23	44	83	P	82	97	P	9
14	2·7	6	28	—	12	42	2·37	12	56	—	24	70	2·5·7	24	84	—	24	98	—	2
15	3·5	8	29	P	28	43	P	42	57	3·19	36	71	P	70	85	5·17	64	99	—	6
		71			198			314			416			564			666			774

Third columns collected.	The sum of all the different fractions.
71	
198	
314	
416	
564	
666	
774	
	3003

Each diff. let. in 1st scheme denotes a diff. prime.

	Steps	Denominators	The several quotients arising by dividing any denominator, by Each single prime	The prod. of each two	Product of each three	Product of each four	Numb. of diff. fract. in each series.
	1	x	-1				n
$1 \times x$	2	x^2	$-x$				xn
$2 \times x$	3	x^3	$-x^2$				x^2n
	4	vx	$-v$ $-x$	$+1$			n
$4 \times v$	5	v^2x	$-v^2$ $-vx$	$+v$			vn
$5 \times x$	6	v^2	$-v^2x$ $-vx^2$	$+vx$			vxn
	7	vxy	$-vx$ $-xy$ $-yv$	$+y$ $+v$ $+x$	-1		n
	8	$vxyz$	$-vxy$ $-xyz$ $-yzv$ $-zvx$	$+vz$ $+xy$ $+yz$ $+zv$ $+yv$ $+xz$	$-z$ $-v$ $-x$ $-y$	$+1$	n

Figure 6.2: Flitcon's solution, 1751

```
function fareyapprox( x )
format long
a=0;     %set the initial Farey Sequence
b=1;
c=1;
d=1;
A=0;
B=1;
C=1;
D=1;
counter=0;
Terror=10;
StrnBrct = [];
    bound = input('Maximum denominator size:  ');
        %to make sure the the value we are working with is less than 1
    y = x - floor(x);
    tic
    while (b+d<bound)
        %if y is larger than the freshman's addition
    if y > ((a+c)/(b+d))
        %this is to make sure that the process stops before the denominator
        %reaches the maximum
        if b+d+d>bound
            break
        else
            StrnBrct = [StrnBrct, 'R']; %Stern-Brocot path Right
            counter = counter + 1;
            a=a+c;
            b=b+d;
            if Terror> abs(x -((floor(x)*(b+d)+(a+c))/(b+d)))
                Terror = abs(x -((floor(x)*(b+d)+(a+c))/(b+d)))
                A=a;
                B=b;
                C=c;
                D=d;

            end
        end
        %if y is smaller than the freshman's addition
    elseif y < ((a+c)/(b+d))
        if b+b+d>bound
            break
        else
            StrnBrct = [StrnBrct, 'L']; %Stern-Brocot path Left
            counter = counter + 1;
            c = a+c;
            d = b+d;
            if Terror> abs(x -((floor(x)*(b+d)+(a+c))/(b+d)))
                Terror = abs(x -((floor(x)*(b+d)+(a+c))/(b+d)))
                A=a;
                B=b;
                C=c;
                D=d;

            end
        end



    end
    end
    toc
Input = x                          %compares the values
Fapprox = (floor(x)*(b+d)+(a+c))/(b+d)
Fraction = [floor(x)*(b+d)+(a+c), b+d]
counter
Bfraction=[floor(x)*(B+D)+(A+C), B+D]
Terror
Ferror = abs(x -((floor(x)*(b+d)+(a+c))/(b+d)))
StrnBrct
end
```

Figure 6.3: Matlab code for approximation of irrationals using the Farey sequence.

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