NUMBER RINGS

A mathematical vignette

Ed Barbeau, University of Toronto

Arrange the numbers 1, 2, 3, 4, 5, 6 in a ring in any order whatsoever. Multiply together the numbers of each adjacent pair and add the products together. What order should the numbers be placed in order that the result be (a) as large as possible, (b) as small as possible?

For example, suppose the numbers are placed in the order (1, 5, 3, 4, 2, 6) with the 6 adjacent to 1 as well as to 2. The six products are 5, 15, 12, 8, 12, 6, and their sum is 58. Try to get a result higher than 58 and lower than 58. When you play around with the numbers, you should begin to get a feel for how they might be arranged to aschieve your goal.

Here is a similar problem. Taking the same ring of numbers, find the squares of the distances between adjacent pairs and add them together. For the example (1, 5, 3, 4, 2, 6), the six differences are $5-1=4$, $5-3=2$, $4-3=1$, $4-2=2$, $6-2=4$ and $6-1=5$. The sum of their squares is $4^2+2^2+1^2+2^2+4^2+5^2=66$. How should the numbers be arranged to make the sum of the squares of the differences (a) as large as possible, (b) as small as possible.

Do not read further until you have explored the situation a bit and come up with a few ideas. One way to proceed in a problem like this is to start with a configuration and look at the sort of modification that might bring you closer to your goal. For example, what happens when you switch an adjacent pair.

If, instead of (1, 5, 3, 4, 2, 6), we look at (1, 3, 5, 4, 2, 6), we get for the sum of the pairwise products $3+15+20+8+12+6=64$ and for the sum of the squares of the differences $4+4+1+4+16+25=54$. Notice that switching a single pair affects at most two of the numbers that have to be added together. Does switching an adjacent pair always increase the first sum and decrease the second?

What happens if you interchange a nonadjacent pair? If at the same time you reverse the order of the numbers between the two members of the pair, then once again you affect at most two of the numbers that have to be added together.

At this point, one may get some understanding of the mechanisms that make the sum either large or small, but it is difficult without the use of algebra to pin it down. So this particular example might be an occasion to show students the power that algebra can bring to formulating and explaining a situation. Let us move to the general situation with any number of integers.

Suppose that we have the n numbers 1, 2, 3, $\cdots$, n in a ring in some order:

$$(\cdots, a, b, c, d, \cdots, e, f, g, h, \cdots).$$

Now switch b and g and reverse the order of the numbers in between to get

$$(\cdots, a, g, f, e, \cdots, d, c, b, h \cdots).$$

The only adjacent pairs that get altered are $(a, b) \rightarrow (a, g)$ and $(g, h) \rightarrow (b, h)$. The other adjacent pairs stay the same, albeit in different positions.

The difference between the sum of products of the two arrangments is $(ab+gh)-(ag+bh) = (a-h)(b-g)$. The first sum is larger if and only if both $a > h$ and $b > g$ or both $a < h$ and $b < g$.

Similarly, the difference hetween the sum of squares of differences for the two arrangments is

$$(a-b)^2 + (g-h)^2 - (a-g)^2 - (b-h)^2 = 2(-ab - gh + ag + bh) = -2(a-h)(b-g).$$

Using these equations, we can arrive at a necessary condition for an arrangement that the sum cannot be increased or cannot be decreased. For example, if we want an arrangement for which the sum of the products of adjacent pairs cannot be increased, we need to arrange the numbers such that when part of the arrangement is $(p, q, \cdots, r, s)$ then either both $p < s$ and $q < r$ or both $p > s$ and $q > r$.

When $n = 4$, the sum of the products is maximized by $(1, 2, 4, 3)$; when $n = 5$, it is maximized by $(1, 2, 4, 5, 3)$.

For the general case, here are some arrangements to look at:

$$(2, 1, 3, 4, 5, 6, \cdots, n-2, n, n-1)$$

$$(1, 3, 5, 7, \cdots, 2m-1, 2m, 2m-2, \cdots, 4, 2)$$

$$(1, 3, 5, 7, \cdots, 2m-1, 2m+1, 2m, 2m-2, \cdots, 4, 2)$$

$$(\cdots, n-3, 3, n-1, 1, n, 2, n-2, \cdots).$$

We can note that replacing by first n natural numbers by a set $\{x_i : 1 \leq i \leq n\}$ of positive real numbers with $x_1 < x_2 < \cdots < x_n$ does not make the problem more difficult, because the criterion for the size of the sum involves only the ordering relation of the numbers rather than their actual values.