TRISECTING AN ANGLE

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0. After students learn about bisecting an angle using straightedge and compasses, the question of a similar construction for trisecting an angle may come up, and some students may attempt to find a method. Once I was contacted by a middle school teacher, one of whose students thought he had succeeded.

The proposed construction was pleasantly simple. Let POQ be the (acute) angle to be trisected. From any point A on OP, drop a perpendicular to meet OQ at B. Construct an equilateral triangle ABC with side AB with O and the vertex C on opposite sides of AB. Then it is claimed that $\angle COB$ is equal to one third of $\angle POQ$. If you check it out with a protractor the method is not bad at all, with numerical evidence suggesting that the error is within one or two degrees. In fact, it works for one acute angle; it is not hard to identify and check this angle.

However, there is a pedagogical difficulty here. One could don the mantle of authority and simply tell the student that it was rigorously proved long ago that no such method exists. It is more satisfactory to find an explanation that involves mathematics accessible to the student. I pose two problems for the reader and suggest solutions for them that I think can be improved upon.

(1) Find an argument that the proposed trisection construction is faulty that involves facts of Euclidean geometry that the student might be expected to know. The more straightforward the argument the better.

(2) Using standard high school mathematics, provide an analysis that identifies the situations for which the method delivers a trisection.

1. We are asked to refute a construction that purports to produce a trisection for *every* acute angle. We employ a proof by contradiction: assume that the construction works for every angle and derive from this a false statement. All we have to do is to find *at least one* angle for which it does not work. The following argument will begin with the assumption that it works for both angles 30° and 60° and derive inconsistent conclusions. (In Section 3, you will see how the assumption that it works for $POQ = 60°$ leads to a contradiction.)

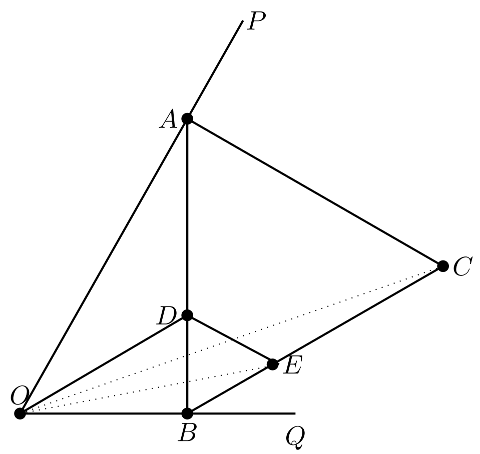
In the diagram below, $\angle POQ = 60°$ and $\angle DOB = 30°$. We will suppose that $AB = 3$, from which we find that $BD = 1$, $AD = 2$ and $OB = \sqrt{3}$. Triangles ABC and DBE are equilateral, and $\angle CBQ = \angle EBQ = 30°$. Since $AC \| DE$, $CE = AD = 2$.

Assuming the method is valid, $\angle EOB = 10°$, so $\angle OEB = \angle EBQ - \angle EOB = 20°$. Also, by hypothesis, $\angle COQ = 20°$, so $\angle COE = 10°$. Since $\angle OCE = \angle OEB - \angle COE = 10°$, triangle COE is isosceles with $OE = CE = 2$.

Consider triangle OBE, On the one hand, $\angle OBE$ is obtuse. On the other,

$$OB^2 + BE^2 = 3 + 1 = 4 = OE^2,$$

which can occur only if $\angle OBE = 90°$. Since these two statements are incompatible, the method fails for at least one of 30° and 60°.

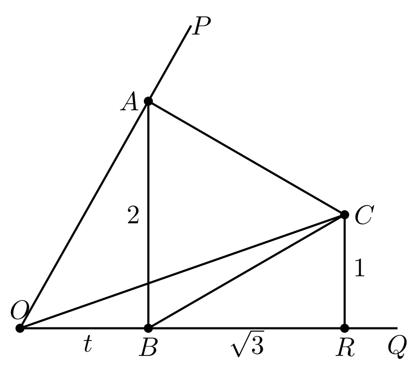


2. Now look at the general situation of a proper acute angle. In the diagram below, assume that $AB = 2$ and $OB = t$ with $t > 0$. From the diagram, we see that

$$\tan \angle COQ = \frac{1}{t + \sqrt{3}}$$

and

$$\tan \angle POQ = \frac{2}{t}.$$



It can be verified that

$$\tan 3\angle COQ = \frac{3t^2 + 6t\sqrt{3} + 8}{t^3 + 3t^2\sqrt{3} + 6t}.$$

Therefore

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\begin{aligned}
(t^3 + 3t^2\sqrt{3} + 6t)\cdot[\tan 3\angle COQ - \tan \angle POQ]& \\
&= (3t^2 + 6t\sqrt{3} + 8) - 2(t^2 + 3t\sqrt{3} + 6) \\
&= t^2 - 4 = (t-2)(t+2).
\end{aligned}
$$

This vanishes if and only if $t = 2$ and $\angle POQ = 45°$. (There is a degenerate situation when $\angle POQ = 90°$. Here $Q = B$ and $\angle COQ = 30°$.)

3. A different assignment of lengths suggested by J. Chris Fisher gives a more transparent relationship between the tangents of the angles POQ and COQ. Let $AB = \sqrt{3}$ and $OB = t$; then $CR = \sqrt{3}/2$, $BR = 3/2$ and $\tan \angle COQ = \sqrt{3}/(2t+3)$. Then

$$\tan 3\angle COQ = \frac{3\sqrt{3}(t^2 + 3t + 2)}{t(2t^2 + 9t + 9)} = \tan \angle POQ \left(\frac{3(t^2 + 3t + 2)}{2t^2 + 9t + 9}\right).$$

When $\angle POQ = 60°$, then $t = 1$ and

$$\tan 3\angle COQ = (9/10)\sqrt{3} = (9/10)\tan 60°.$$

In this case, the trisection method produces an angle of about 19.1°.

Let $f(t) = [3(t^2 + 3t + 2)]/[2t^2 + 9t + 9]$. Suppose that $\angle POQ = \theta$. If t is the value of OB that corresponds to θ, then $3/t$ is the value of OB that corresponds to the complement, $90° - \theta$. Then $f(3/t) \cdot f(t) = 1$. The function $f(t)$ increases from $2/3$ when $t = 0$ (and $\theta = 90°$) to $3/2$ when $t = \infty$ (and $\theta = 0$). However, the effect of the values of $f(t)$ in the accuracy of the trisection when θ is further from 45° and $f(t)$ is further from 1 is offset by the fact that when θ is close to 90° a large change in its tangent corresponds to a small change in the angle, and when θ is small, the error from a true trisection will also be small. For what angle is the deviation from a proper trisection maximum?