CONSECUTIVE INTEGER SUMS

A mathematical vignette

Ed Barbeau, Toronto, ON

In this vignette, we consider sums of the form

$$1 * 2 * 3 * \cdots * (n-1) * n. \qquad (1)$$

where each $*$ stands for either $+$ or $-$. In particular, we look at situations in which the sum can be made to vanish.

Exercise 1. For which values of n is it possible to replace the $*$'s in such a way that the sum is 0?

In what follows, we add the additional condition that

$$1 * 2 * \cdots * k \geq 0$$

for all integers k for which $1 \leq k \leq n$. Thus $1 + 2 - 3 = 0$ is acceptable, but $1 - 2 - 3 + 4 = 0$ is not. We say that the positive integer is *up-acceptable* if such a representation of 0 exists.

Exercise 2. Verify that 3 and 8 are up-acceptable, but that 1, 2, 5, 6, 7 and 9 are not.

If there are integers r and s for which $r * (r+1) * (r+2) * \cdots * s = 0$ for suitable choise of signs, then, if $n = r - 1$ is positively acceptable, then so also is s.

Exercise 3. Prove that $n = m^2 - 1$ is positively acceptable for each integer $m \geq 2$.

For any two integers r and s with $r \leq s$, let $[r, s]$ denote the sum $r + (r+1) + \cdots + s$. Thus $[1, n] = t_n$. The equation $[r, s] = [s+1, t]$ for $r \leq s < t$ is equivalent to $r + (r+1) + (r+2) + \cdots + s - (s+1) - (s+2) - \cdots - t = 0$.

Exercise 4. (a) Verify that, if $[r, s] = [s+1, t]$, then the number $t - s$ of entries in $[s+1, t]$ is strictly less than the number $s - r + 1$ of entries in $[r, s]$.

(b) Find all examples where $s - r = t - s$, *i.e.*, $[r, s]$ has exactly one more summand that $[s+1, t]$.

(c) Investigate the possibilities that $(s - r) - (t - s)$ is equal to 1, 2, 3, *etc.*

(d) One way to find a family of such equations is to start with the obvious

$$(-r) + (-r+1) + \cdots + (-2) + (-1) + 0 + 1 + 2 + \cdots + (r-1) + r = 0.$$

This can be considered as an example where the left side has $2r - 1$ terms and the right side has zero terms (so it is an "empty" sum equal to 0). Can you, by suitably increasing the values and the number of terms, find other situation where the number of terms on the left exceeds the number of terms on the right by $2r + 1$?

Exercise 5. Determine other sums $r * (r+1) * (r+2) * \cdots * s$ that vanish for suitable choices of sign and for which all the partical sums are nonnegative.

We can also look at

$$n * (n-1) * (n-2) * \cdots * 3 * 2 * 1.$$

We say that the integer n is *down-acceptable* if the signs are chosen so that $n * (n-1) * \cdots * k$ are nonnegative for $n \geq k \geq 2$ and $n + (n-1) * \cdots * 2 * 1 = 0$.

Exercise 6. Which integers n between 2 and 16 inclusive are down-acceptable? Are there infinitely many down-acceptable integers?

Investigation. Which permutations of the numbers $1, 2, \ldots, n$ are such that, if you arrange the numbers in order according to the permutation and admit $+$ or $-$ signs so that the sum is 0 and all partical sums are nonnegative. For example, $4 - 3 + 1 - 2 = 0$.

Hints, solutions and comments.

Exercise 1. Note that any sum formed must have the same parity as $t_n = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. Note that for any integer m, $m - (m+1) - (m+2) + m + 3 = 0$.

Exercise 2. Since t_n is odd for $n = 1, 2, 5, 6$, these integers can be ruled out. We have $1 + 2 - 3 = 0$ as well as

$$0 = 1 + 2 - 3 + 4 + 5 + 6 - 7 - 8 = 1 + 2 + 3 - 4 + 5 - 6 + 7 - 8.$$

Exercise 3. We have $1 + 2 - 3 = 0$, $4 + 5 + 6 - 7 - 8 = 0$, and, in general,

$$\begin{aligned}(m^2 + (m^2+1) + \cdots m^2 + m) - (m^2 + m + 1) + \cdots + (m^2 + 2m) \\ = m^2 - [(m^2+m+1) - (m^2+1)] - [(m^2+m+2) - (m^2+2)] - \cdots - [(m^2+2 \\ = m^2 - m - m - \cdots - m = m^2 - m \times m = 0.\end{aligned}$$

Exercise 4. (b) Wolog, we may assume that

$$(m-k) + (m-k+1) + \cdots + m = (m+1) + (m+2) + \cdots + (m+k),$$

whereupon

$$m = 2(1 + 2 + \cdots + k) = k(k+1).$$

This gives us the situation identified in Exercise 3.

(c) We have some examples:

$$4 + 5 + 6 + 7 + 8 = 9 + 10 + 11 = 30; 12 + 13 + \cdots + 17 + 18 \quad = 19 + 20 + 21 + 22 + 23 = 105; 2 + 3 + \cdots + 7 =$$

One way to find these is to start with $[1, n] - [n+1, n+k]$ where k is large enough to make this negative, and note that by increasing each integer by 1, you increase this by $n - k$. Is it possible to increase the sum to exactly 0?

There are various ways of experimenting that will lead to the general equality for any integers k and r:

$$[(2k^2{-}1)r{+}k^2, (2k^2{+}2k{+}1)r{+}k(k{+}1)] = [(2k^2{+}2k{+}1)r{+}(k^2{+}k{+}1), (2k^2{+}4k{+}1)r{+}(k^2{+}2k)],$$

or, alternatively,

$$[(2k^2{-}1)r{+}k^2, (k^2{+}(k{+}1)^2)r{+}k(k{+}1)] = [(k^2{+}(k{+}1)^2)r{+}k(k{+}1){+}1, (2(k{+}1)^2{-}1)r{+}((k{+}1)^2{-}1)].$$

Note that the common sum of the two sides is equal to

$$\tfrac{1}{2}(2r+1)^2[k(k+1)(2k+1).$$

The left expression has $(k+1)(2r+1)$ entries and the right side has $k(2r+1)$ entries. Finally, note that the last term on the right for k is one less that the first term on the left for $k+1$.

Exercise 6. There are at least two possible ways of approaching this exercise. The first is to look at up-accepatable integers and the choice of signs that implement the representation of 0, and see if you can derive a representation in the other direction by changing each of the signs. For example, $1+2-3=0$ leads to $3-2-1=0$. In this way, one can see that $n=m^2-1$ is down-acceptable for all values of n.

The second way is to build a backward tree of possibilities. Since $n*\cdots*2 \geq 0$, the sign in front of 1 must be $-$. Similarly, the sign in front of 2 must be -1. The sum must end in either $3-2-1$ in which case $n*\cdots*4=0$ or end in $-3-2-1$ in which case $n*\cdots+4=6$. Backtracking along this tree, we find that we can get to zero if and only if $n*\cdots*9$ is one of the following numbers

$$0, 2, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 36.$$

In this way, we find the representations:

$$11-10+9+8-7-6+5-4-3-2-1=0;$$
$$12-11+10-9+8-7+6-5+4+3-2-1=0;$$
$$16-15+14-13+12-11+10+9-8+7-5-4-3-2-10.$$