PROGRESSIVELY DIVISIBLE NUMBERS

A mathematical vignette

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Problem. Construct a 10-digit number $(a_1a_2a_2 \ldots a_{10})_{10}$ in base 10 such that no two of its ten digits are equal, a_1 is divisible by 1, $(a_1a_2)_{10}$ is divisible by 2, ..., $(a_1a_2 \ldots a_k)_{10}$ is divisible by k for each integer k with $1 \leq k \leq 10$.

The solution(s) can be found in progressive stages. Two background facts are useful: Any integer written to base 10 is divisible by 3 (resp. 9) if and only if the sum of its digits is divisible by 3 (resp. 9). Any number is divisible by 4 (resp. 8) if and only if the number comprised of its last three digits is divisible by 4 (resp. 8).

(1) What is the last digit a_{10} of the number?

(2) What is the fifth digit a_5 of the number?

(3) What is the parity of the digits in the even positions, reading from the left?

(4) What is the parity of the digits in the odd positions, reading from the left?

(5) What are the possible values of the fourth and eight digits, a_4 and a_8?

(6) What are the possible values of the second and sixth digits, a_2 and a_6?

(7) Explain why $a_1 + a_2 + a_3$ and $a_4 + a_5 + a_6$ are each multiples of 3.

(8) Determine the possible values of the middle three digits a_4, a_5, a_6.

(9) Why do we not have to worry about a_9?

We can now generalize the problem to an arbitrary positive integer base $b \geq 2$. The problem is now to find all $b-$digits numbers $(a_1a_2 \ldots a_b)_b$, written to base b whose digits are all distinct and for each k with $1 \leq k \leq b$, $(a_1a_2 \ldots a_k)_b$ is a multiple of k.

(10) Prove that the problem has no solution unless $b = 2m$ is even.

Henceforth, we will assume that $b = 2m$.

(11) Prove that the last digit is 0 and the mth digit is m.

(12) Prove that the even digits appear in the even positions reading from the left and the odd digits appear in the odd positions.

(13) Solve the problem for $b = 2, 4, 6, 8$.

Comments. For the base 10 problem, since $(a_3a_4)_{10}$ and $(a_7a_8)_{10}$ are multiples of 4 whose first digits are odd, a_4 and a_8 must be 2 and 6 in some order. Accordingly, a_2 and a_6 must be 4 and 8. Since $a_4 + a_5 + a_6$ is a multiple of 3, the only possibilities for (a_4, a_5, a_6) are $(2,5,8)$ and $(6,5,4)$. The main task is to try out possibilities for a_7, and we find that the only number that satisfies the conditions of the problem is 3816547290.

More generally, we have solutions $(10)_2$, $(1230)_4$, $(3210)_4$, $(143250)_6$, $(543210)_6$. There seem to be no solutions in base 8. For even bases $b \geq 12$, the challenge is to narrow down the possibilities efficiently. Are there other even bases for which no solutions exist?