A SELF-REFERENTIAL SEQUENCE

A mathematical vignette

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1. I first learned of the following problem from Anthony Gardiner, a British mathematician interested in education of young people. Determine all the numbers with at most ten digits for which the first digit on the left records the number 0s, the second digit the number of 1s, and so on. Thus, written to base 10, $(w_0w_1w_2\dots w_n)_{10}$ has w_i digits equal to i for $0 \leq i \leq 10$. Play around with this for small values of n.

This can be generalized to vectors $(w_0, w_1, w_2, \dots, w_n)$, where for each k, w_k is the number of appearances of k. There are a few immediate observations. None of the entries can exceed n; in fact, n itself cannot be an entry. The first entry w_0 has to be positive. The sum $w_0 + w_1 + \cdots + w_n$ is equal to the number of terms, namely $n+1$. If w_i is positive for some i, then i is in the sequence somewhere w_i times, which suggests that most later terms in the sequence are likely to be 0.

2. One way to approach the problem is to realize that the number of terms in the vector can be counted in three ways. It is clearly $n+1$. It is also $w_0+w_1+\cdots+w_n$. Finally, since there are w_0 zeros, w_1 1s, w_2 2s, and so on, the sum of the terms can be written as $w_1 + 2w_2 + 3w_3 + \cdots + nw_n$. Since, equating these two expressions leads to

$$w_0 = w_2 + 2w_3 + \cdots + (n-1)w_n. \tag{1}$$

If $w_0 = 1$, then $w_2 = 1$ and $w_k = 0$ for $k \geq 3$. This leads to the possibility $(1, 2, 1, 0)$. If $w_0 = 2$, then either $w_2 = 2$ or $w_3 = 1$ with all the other w_k, except possibly w_1, vanishing. In the first instance, we are lead to $(2, 0, 2, 0)$ and $(2, 1, 2, 0, 0)$. However, if $w_3 = 1$, then a 3 has to appear somewhere, but it cannot possibly be w_1.

So let us suppose that $w_0 = r \geq 3$. Then $w_r \geq 1$. Then

$$r = w_0 \geq (r-1)w_r,$$

so that $w_r \leq r/(r-1) < 2$. Therefore $w_r = 1$ and equation (1) forces $w_2 = 1$, $w_1 = 2$ and $w_k = 0$ for $k \neq 0, 1, 2, r$. Therefore

$$n + 1 = w_0 + w_1 + w_2 + w_r = r + 2 + 1 + 1 = r + 4,$$

whence $r = n - 3$. Since $r \geq 3$, $n \geq 6$ and we obtain the sequence

$$(n-3, 2, 1, 0_{n-6}, 1, 0, 0, 0),$$

where 0_{n-6} represents a string of $n-6$ zeros. In particular, we obtain the vectors $(3, 2, 1, 1, 0, 0, 0)$, $(4, 2, 1, 0, 1, 0, 0, 0)$ and $(5, 2, 1, 0, 0, 1, 0, 0, 0)$.

3. Another approach to solving the problem is to use a method of "successive approximation". Given a positive integer n, write down any vector $\mathbf{w} = (w_0, w_1, \dots, w_n)$ where each of the entries is a nonnegative integer not exceeding n. From this, we construct a second vector $\mathbf{w}' = (w'_0, w'_1, \dots, w'_n)$ with w'_i equal to the number of times that i appears in the first sequence. We write $\mathbf{w} \to \mathbf{w}'$. If we keep doing this, one of three things will happen: (1) we will arrive at a sequence that goes to itself, yielding a solution to the problem; (2) we will arrive at a periodic orbit of sequences; (3) we will arrive at a sequence of with one entry equal to $n+1$ and all the rest 0. When $n \geq 3$, the last will happen under circumstances that are easy to avoid by our original choice of sequence.

A little experimentation reveals that the tendency under the operation is for the number of positive entries in $\mathbf{w}$ to decrease. However, if the sequence of images is eventually periodic, then, at some point, $\mathbf{w}'$ must have at least as many positive entries as $\mathbf{w}$. Let us see when this might occur. Let n be a positive integer exceeding 2. We will use the notation $\langle v_0, v_1, \ldots, v_n \rangle$ to refer to any vector of nonnegative integers v_i with $\sum_i v_i = n+1$ and $0 \leq v_i \leq n$, where the same numbers in different orders are identified. (The parentheses will have the integers in a particular order.)

Let $\mathbf{w}$ be a vector in which there are k distinct positive entries a_i $(1 \leq i \leq k)$ where a_i appears with positive frequency b_i, along with r zeros. Suppose b_1 is the maximum frequency. Then

$$n + 1 = \sum_{i=1}^{k} b_i a_i = \sum_{i=1}^{k} b_k + r.$$

Then $\mathbf{w}'$ has at most $k+1$ entries consisting of the b_i and possibly r. If $\mathbf{w}'$ has at least as many positive integer as $\mathbf{w}$, then

$$b_1 + b_2 + \cdots + b_k \leq k + 1,$$

or

$$(b_1 - 1) + (b_2 - 1) + \cdots + (b_k - 1) \leq 1.$$

There are two possibilities. In the first instance $b_1 = b_2 = \cdots = b_k = 1$, and $\mathbf{w}'' = \langle s, k, 0, 0, \ldots \rangle$ where s is the number of zeros in $\mathbf{w}'$. In the second instance, $b_1 = 2$ and $b_2 = b_3 = \cdots = b_k = 1$ and $\mathbf{w}''$ has 1, 2 or 3 nonzero entries.

In any case, the orbit must arrive a vector with one, two or three nonzero entries. With one distinct positive entry, we have $\langle a, 0_n \rangle$ with $a = n + 1$ (not acceptable), $\langle a, a, 0_{n-1}$ with $a = \frac{1}{2}(n + 1$ and $\langle a, a, a, 0_{n-2} \rangle$ with $a = \frac{1}{3}(n + 1)$.

If $n \geq 4$, we obtain the orbit

$$\begin{aligned} \langle a, a, 0_{n-1} \rangle \to (n-1, 0, \ldots, 0, 2, 0) &\to (n-1, 0, 1, 0_{n-4}, 1, 0) \\ (n-2, 2, 0_{n-3}, 1, 0) &\to (n-2, 1, 1, 0_{n-5}, 1, 0, 0) \\ (n-3, 3, 0_{n-3}, 1, 0, 0) &\leftrightarrow (n-2, 1, 0, 1, 0_{n-7}, 1, 0, 0, 0) \end{aligned}$$

When $n = 3$, then $a + a = 4$, so $a = 2$ and we get the orbit

$$\langle 2, 2, 0, 0 \rangle \to (2, 0, 2, 0),$$

ending in a fixed point.

If $n \neq 5$, we get the orbit

$$\begin{aligned}\langle a,a,a,0_{n-2}\rangle &\to (n-2,0,\ldots,0,3,0,\ldots,0) \to (n-1,0,0,1,0_{n-5},1,0)\\ &(n-2,2,0_{n-3},1,0) \to (n-2,1,1,0_{n-5},1,0,0)\\ &(n-3,3,0_{n-3},1,0,0) \leftrightarrow (n-2,1,0,1,0_{n-7},1,0,0,0)\end{aligned}$$

When $n = 5$, the $a = 2$ and we get the orbit

$$\begin{aligned}\langle 2,2,2,0,0,0\rangle &\to (3,0,3,0,0,0) \to (4,0,0,2,0,0) \to (4,0,1,0,1,0)\\ &\to (3,2,0,0,1,0) \to (3,1,1,1,0,0) \leftrightarrow (2,3,0,1,0,0).\end{aligned}$$

When $a \neq b$, we get

$$\begin{aligned}\langle a,b,0_{n-1})\rangle &\to \langle n-1,1,1,0_{n-2}\rangle \to (n-2,0,2,0_{n-4},1,0)\\ &\to (n-2,1,1,0_{n-5},1,0,0) \to (n-2,3,0_{n-1}) \to (n-1,0,0,1,0_{n-6},1,0,0)\\ &(n-2,2,0_{n-3},1,0) \to \ldots\end{aligned}$$

When $n \neq 4$ and $a \neq b$, we have

$$\begin{aligned}\langle a,a,b,0_{n-2})\rangle &\to \langle n-2,2,1,0_{n-2}\rangle \to (n-2,1.1,0_{n-4},1,0)\\ &\to (n-2,3,0_{n-1}) \to (n-1,0,0,1,0_{n-6},1,0,0)\\ &\to (n-2,2,0_{n-3},1,0) \to \ldots\end{aligned}$$

For $n = 4$, then $2a + b = 5$ and $(a,b) = (2,1),(1,3)$. We get the orbits

$$(2,2,1,0,0) \to (2,1,2,0,0),$$

ending in a fixed vector, and

$$(1,1,3,0,0) \to (2,2,0,1,0) \to (2,1,2,0,0).$$

Finally, we have, with a,b,c distinct positive integers,

$$\langle a,b,c,0_{n-2}\rangle \to \langle n-2,1,1,1,0_{n-3}\rangle.$$

We treat the case $n = 4,5,6$ separately:

$$\begin{aligned}\langle 2,1,1,1,0\rangle &\to \langle 1,3,1,0,0\rangle\\ &\to (2,2,1,0,0) \to (2,1,2,0,0).\end{aligned}$$

$$(3,1,1,1,0,0) \leftrightarrow (2,3,1,0,0,0).$$

$$\begin{aligned}\langle 4,1,1,1,0,0,0\rangle &\to (3,3,0,0,1,0,0) \to (4,1,0,2,0,0,0)\\ &\to (4,1,1,0,1,0,0) \to (3,3,0,0,1,0,0).\end{aligned}$$

When $n \geq 7$, then

$$\langle n-2,1,1,1,0_{n-3}\rangle \to (n-3,3,0_{n-4},1,0,0) \leftrightarrow (n-2,1,0,1,0_{n-4},1).$$

4. To sum up, we have the fixed vectors $(1, 2, 1, 0), (2, 0, 2, 0), (2, 1, 2, 0, 0)$ and

$$(n-3, 2, 1, 0_{n-6}, 1, 0, 0, 0)$$

for $n \geq 6$. This includes (3, 2, 1, 1, 0, 0, 0), (4, 2, 1, 0, 1, 0, 0, 0) and (5, 2, 1, 0, 0, 1, 0, 0, 0).

We have the following period-2 orbits:

$$(3, 1, 1, 1, 0, 0) \leftrightarrow (2, 3, 0, 1, 0, 0)$$

and

$$(n-2, 1, 0, 1, 0_{n-7}, 1, 0, 0, 0) \leftrightarrow (n-3, 3, 0_{n-4}, 1, 0, 0)$$

for $n \geq 7$.

Finally, there is one period-3 orbit:

$$(4, 1, 0, 2, 0, 0, 0) \rightarrow (4, 1, 1, 0, 1, 0, 0) \rightarrow (3, 3, 0, 0, 1, 0, 0) \rightarrow \ldots.$$