OBLONG NUMBERS

A mathematical vignette

An *oblong number* is the product of two consecutive integers: 2, 6, 12, 20, 30, 42, ... The purpose of this vignette is to suggest investigations that might be given at the middle school or early secondary level to introduce some aspects of mathematical usage:

(1) the use of data to suggest mathematical patterns, make conjectures, formulate them algebraically, use algebra to establish their truth;

(2) looking at situations from different perspectives, each of which may offer its own insight and directions for further investigation.

Can an oblong number be a perfect square?

This question can be looked at from different points of view, and, indeed, when this is done with a group of students, it is possible for each of these to emerge.

It can be looked at in arithmetic terms. The key observation is to note that the two terms of the product yielding the oblong number differ by 1, and so are coprime. Therefore, if the product is square, each of its terms must be so, so that you have two positive squares differing by 1.

Most of the time, students will accept that two positive squares cannot differ by 1, but on one occasion, a student questioned this. When I sought an explanation, the class was hard pressed to come up with a reason. One approach was to observe that the difference between two consecutive squares increased: $4 - 1 = 3; 9 - 4 = 5; \dots$; however, logically, this is slightly cumbersome because there is a hidden additional step: if you are given two squares, the larger one is at least as great as the successor square to the lower.

However, the question can be tackled neatly using algebra. Suppose $x > y > 0$. Then $x^2 - y^2 = (x + y)(x - y)$. If this product of two positive integers is equal to 1, then each term of the product must be 1, so that $x + y = 1$ and $x - y = 1$. The only solution of this linear system is $(x, y) = (1, 0)$ which we have excluded.

Note that this is a rigorous proof that shows algebra as a proof technique that applies an algebraic identity.

Comment. A question that might follow on from this is what possible values can a difference of squares of integers have. This can be looked at from the factorization of $x^2 - y^2$ or from the remainders when squares are divided by 4.

The product of two consecutive oblong numbers.

Consider the sequence of oblong numbers: 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, ...

Looking at the sequence, one might conjecture that the product of consecutive oblong numbers occur later in the sequence. There are at least two ways of approaching this conjecture. One is to look closely at the numbers, try to find a pattern, express it algebraically and check the identity that is found. The second is to express the oblong numbers and their product algebraically and try to manipulate into the form of a product of two consecutive oblong numbers.

In the first instance, we have:

$$\begin{aligned} 2 \times 6 = 12 &= 3 \times 4; \\ 6 \times 12 = 72 &= 8 \times 9; \\ 12 \times 20 = 240 &= 15 \times 16; \\ 20 \times 30 = 600 &= 24 \times 25. \end{aligned}$$

Let the lower oblong number in the product is $n(n+1)$. Then when $n = 1$, the product of consecutive oblongs is 3×4; when $n = 2$, it is 8×9; when $n = 3$, it is 15×16. If the student recognizes that these all involve squares, then the question arises as to *which squares*, and we are led to conjecture

$$[n(n+1)] \times [(n+1)(n+2)] = [(n+1)^2 - 1] \times (n+1)^2.$$

This can be checked by simply multiplying everything out, but an astute student will notice immediately that $(n+1)^2$ is a factor on both sides and it is just a matter of checking the $n(n+2) = (n+1)^2 - 1$.

However, if one takes the consecutive oblong numbers to be $(n-1)n$ and $n(n+1)$, then multiplying these together gives immediately $(n^2 - 1)n^2$, which is recognizably oblong. Notice this approach involves judgment in selecting variables, and recognition that the product has the right pattern.

Comments. (1) More generally, one can seek other pairs of oblong numbers whose product is oblong.

(2) One can generalize oblong numbers to the product of pairs that all differ by some number $d \geq 2$:
$d = 2$: 3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, ...
$d = 3$: 4, 10, 18, 28, 40, 54, 70, 88, 108, 130, 154, 180, ...
$d = 4$: 5, 12, 21, 32, 45, 60, 77, 96, 127, 140, 165, 192, ...

In the case of $d = 2$, one can notice the connection with the rather delicious result that the product of four consecutive integers is always one less than a perfect square:

$$\begin{aligned} n(n+1)(n+2)(n+3) &= [n(n+2)][(n+1)(n+3)] = [(n+1)^2 - 1][(n+2)^2 - 1] \\ &= [n(n+3)][(n+1)(n+2)] = [n^2 + 3n][n^2 + 3n + 2] \\ &= [(n^2 + 3n + 1) - 1][(n^2 + 3n + 1) + 1] = (n^2 + 3n + 1)^2 - 1. \end{aligned}$$

By the way, we also have the result that the the product of four consecutive integers differs from the third greater square by a perfect square:

$$(n^2 + 3n + 3)^2 - n(n + 1)(n + 2)(n + 3) = (2n + 3)^2.$$

Summing reciprocals of oblong numbers.

Finding the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 + \frac{1}{2} + \cdots + \frac{1}{n(n+1)}$$

of the reciprocals of the first n oblong numbers. An examination of the this sum for small values of n leads to the conjecture that the sum of $1 - \frac{1}{n+1}$. This result can be established either by an induction argument or by summing by differences. The latter involves creating a telescopic series by expressing each term as a difference and allowing for a broad cancellation of all terms except the end ones.

There is a very useful identity that students can be introduced to:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Now we can obtain the result:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} \\ &= 1 + \sum_{k=2}^{n} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

As n increases, the sum gets closer and closer to 1, and we can say that the limit of the sum as n tends to infinity is 1. This gives meaning to the equation

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

This fact can be kept as a special treat for particularly engaged students. They will meet this idea of an infinite series later when they study geometric series.

"Pythagorean" triples of oblong numbers.

In examining the list of oblong numbers, it is not hard to locate triples of oblong numbers for which the sum of the smallest two is equal to the third. For example, $30 + 42 = 72$. As for regular Pythagorean triples, one can search for others and, if possible, find patterns and perhaps a formula for such triples.

Suppose

$$u(u+1)+v(v+1)=w(w+1).$$

Multiplying by 4, we obtain the equivalent equation

$$(2u+1)^2+(2v+1)^2=(2w+1)^2+1.$$

So finding triples of oblong numbers is equivalent to solving the equation $x^2+y^2=z^2+1$, where x,y,z are all odd integers.

Some examples are given in the following table, where triples (u,v,w), and the corresponding oblong triples and triples

$$(x,y,z)=(2u+1,2v+1,2w+1)$$

are given.

(u,v,w)	$(u(u+1),v(v+1),w(w+1)$	$(2u+1,2v+1,2w+1)$
(2, 2, 3)	(6, 6, 12)	(5, 5, 7)
(3, 5, 6)	(12, 30, 42)	(7, 11, 13)
(5, 6, 8)	(30, 42, 72)	(11, 13, 17)
(6, 9, 11)	(42, 90, 132)	(13, 17, 23)
(8, 10, 13)	(72, 110, 182)	(17, 21, 27)
(11, 14, 18)	(132, 210, 342)	(23, 29, 37)
(14, 14, 20)	(210, 210, 420)	(29, 29, 41)

Futher analysis of the Diophantine equation $x^2+y^2=z^2+1$ is given in the appendix to this section.

It is interesting to note that the triples $(u,v,w)=(2,2,3),(5,6,8),(8,10,13)$ have the property that the first and last entries are consecutive Fibonacci numbers and the middle entry is twice the previous Fibonacci number. But this does not seem to be part of a regular pattern.

Appendix: $x^2+y^2=z^2+1$

(1) The equation can be rewritten as $x^2-1=(z+y)(z-y)$. Fix a value of x, and let y and z be the variables. Then the equation has the obvious solution $(y,z)=(1,x)$. However, if x^2-1 has two factors g and h of the same parity which differ by more than 2, then the system

$$z+y=g \qquad z-y=h$$

can be solved for integers z and y and another solution obtained. For example, $5^2-1=24=12\times 2$ and we get $(x,y,z)=(5,5,7)$. Since $7^2-1=48=24=12\times 4$, we get solutions $(x,y,z)=(7,11,13),(7,4,8)$.

We can get families of solutions in this way. Suppose that $x=2r+1$, so that

$$x^2-1=4r^2+4r=2(2r^2+2r)=4r(r+1).$$

From the factorization $2(2r^2+2r)$ we obtain the solution

$$(x,y,z)=(2r+1,r^2+r-1,r^2+r+1).$$

When $r = 2s$, then

$$x^2 - 1 = 16s^2 + 8s = 2(8s^2 + 4s) = 4(4s^2 + 2s),$$

and we obtain the solutions

$$(x, y, z) = (4s + 1, 4s^2 + 2s - 1, 4s^2 + 2s + 1), (4s + 1, 2s^2 + s - 2, 2s^2 + s + 2).$$

When $r = 2s - 1$, then

$$x^2 - 1 = 16s^2 - 8s = 2(8s^2 - 4s) = 4(4s^2 - 2s),$$

and we obtain the solutions

$$(x, y, z) = (4s - 1, 4s^2 - 2s - 1, 4s^2 - 2s + 1), (4s - 1, 2s^2 - s - 2, 2s^2 - s + 2).$$

When the values of x, y, z are all odd, we can use these to get three oblong numbers, the largest of which is the sum of the other two.

(2) Another way to find solutions of the equation is to note that $z^2 + 1$ is the sum of two squares. This representation as the sum of two squares is unique when $z^2 + 1$ is either a prime or twice a prime. However, it can be expressed as the sum $x^2 + y^2$ in other ways than $(x, y) = (z, 1)$ if it has a prime factorization consisting of primes congruent to 1 modulo 4 and other primes raised to an even power.

Using the fact that $(a+bi)(c+di) = (ac-bd, ad+bc)$ and equating the absolute value of the two sides yields the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

. For example

$$8^2 + 1 = 65 = 5 \times 13 = (1^2 + 2^2)(2^2 + 3^2) = 4^2 + 7^2,$$

and we obtain the solution $(x, y, z) = (4, 7, 8)$. Also

$$12^2 + 1 = 145 = 5 \times 29 = (1^2 + 2^2)(2^2 + 5^2) = 8^2 + 9^2,$$

and we obtain the solution $(x, y, z) = (8, 9, 12)$.

(3) Somewhat related to (2), we can note that at least one of x and y must be odd. Setting $x = 2s + 1$, we can rewrite the equation as

$$4s(s + 1) = z^2 - y^2.$$

Every multiple of 4 can be written as the difference of two squares, often in more than one way, and each such representation will yield a solution of $x^2 + y^2 = z^2 + 1$.

(4) One can generalize special cases. If $u = v$, this leads to $x = y$ and the equation $z^2 = 2x^2 = -1$, which is a Pell's equation with solutions given by

$$z_k + x_k\sqrt{2} = (1 + \sqrt{2})(3 + 2\sqrt{2})^k$$

for $k \geq 1$. Thus

$$(z, x) = (7, 5), (41, 29), (239, 169), \ldots.$$

If $v = u + 1$, then $y = x + 2$, and we are led to the equation

$$2(x + 1)^2 + 2 = x^2 + (x + 2)^2 = z^2 + 1$$

or $z^2 - 2(x+1)^2 = 1$. The solutions of this Pell's equation are given by

$$(z_k + (x_k+1)\sqrt{2}) = (3+2\sqrt{2})^k$$

for $k \geq 1$. Thus

$$(x,y,z) = (1,3,3), (11,13,17), (69,71,99), (407,409,577), \ldots,$$

and, corespondingly,

$$(u,v,w) = (0,1,1), (5,6,8), (34,35,49), (203,204,288), \ldots.$$

We get the oblong triples

$$(0,2,2), (30,42,72), (1190,1260,2450), (41412,41820,83232), \ldots,$$

where the sum of the first two entries is equal to the third.

Twice oblong numbers.

One possible route for further investigation is to ask when twice an oblong number is a perfect square and more generally, by how much it differs from the next higher square. Consider this table of values, where for the nth oblong number, the next square not less than it is given by m:

n	$2n(n+1)$	$m^2 - 2n(n+1)$	$(m+1)^2 - 2n(n+1)$
1	4	**0**	5
2	12	**4**	13
3	24	**1**	**25**
4	40	**9**	24
5	60	**4**	21
6	84	**16**	37
7	112	**9**	32
8	144	**0**	**25**
9	180	**16**	45
10	220	5	**36**
11	264	**25**	60
12	312	12	**49**
13	364	**36**	77
14	420	21	**64**
15	480	**49**	96

Looking at the first few entries in this table, it seems that each double of an oblong number differs from the next square up by a square, but this pattern is broken at $n = 10$. However, a closer examination suggests that something more subtle is going on. When the difference for the next square up is not a square, then the following difference is a square. In fact, which occurs depends on the parity of n when $n \geq 7$.

This suggests that we look at the oblong numbers for $n = 2k - 1$ and $n = 2k$ separately. In the first instance, we can see a pattern and verify that

$$2n(n+1) = 2(2k-1)(2k) = (3k-1)^2 - (k-1)^2.$$

This can be done expeditiously by factoring the right side as a difference of squares.

This is not the end of the story. The next square below $(3k-1)^2$ is $(3k-2)^2$. As k grows larger, this square depends more strongly on its leading term $9k^2$. However, the leading terms of $2n(n+1) = 2(2k-1)(2k)$ is $8k^2$, so that as k increases, $(3k-2)^2$ is going to eventually surpass $2(2k-1)(2k)$. We can check this.

$$(3k-2)^2 - 2(2k-1)(2k) = k^2 - 8k + 4 = k(k-8) + 4,$$

which is positive if and only if $k \geq 8$. So it seems that we did not continue our table down far enough:

n	$2n(n+1)$	$m^2 - 2n(n+1)$	$(m+1)^2 - 2n(n+1)$
16	544	33	**81**
17	612	13	**64**
18	684	45	**100**
19	760	24	**81**
20	840	**1**	60
21	924	37	**100**
22	1012	12	77
23	1104	52	**121**

Now let $n = 2k$, so that $2n(n+1) = 2(2k)(2k+1) = 8k^2 + 4k$. Then

$$(3k+1)^2 - 2(2k)(2k+1) = (k+1)^2.$$

However, the next lower square will eventually surpass $2n(n+1)$ when n gets larger because of the leading terms $9k^2$ and $8k^2$. Indeed,

$$(3k)^2 - 2(2k)(2k+1) = k^2 - 4k = k(k-4),$$

which is nonnegative when $k \geq 4$. Also

$$(3k-1)^2 - 2(2k)(2k+1) = k^2 - 10k + 1 = k(k-10) + 1,$$

which is positive when $k \geq 10$.

This means that focussing on m^2 as the next square not less than $2n(n+1)$ is a bit of a red herring, since the representation of m shifts as n increases. However, we have established that the value of $2n(n+1)$ augmented by a suitable square will yield a square value.

One issue that comes up is a qualitative analysis of how algebraic expressions vary, the idea that the value of a polynomial is largely determined by its leading term when the variable takes large values, and by its terms of lower degree when the variable is close to zero. A similar issue comes up when comparing, say, exponential and polynomial growth.

This investigation will not be suitable for every algebra class, but it can be carried to different levels depending on the interests and abilities of the students. For some, one can go further and ask when $2n(2n+1)$ is itself a square or, given some square number r^2 when $2n(2n+1) = r^2$ is equal to a square number m^2. The condition $2n(n+1) = m^2$ leads to the Pell equation

$$(2n+1)^2 - 2m^2 = 1,$$

with solutions $(n, m) = (1, 1), (8, 12), (48, 70), \ldots$. The condition $2n(n+1)+1 = m^2$ leads to the Pell equation

$$(2n+1)^2 - 2m^2 = -1,$$

with solutions $(n, m) = (0, 1), (3, 5), (20, 29), \ldots$. This is a rich vein for those students who need greater motivation.

Note that asking for those values of n for which $2n(n+1)$ is square is the same as asking for those for which

$$1 + 2 + \cdots + n = \binom{n+1}{2},$$

is square, a problem that has been well worked over.