POWERS OF 2, 5 AND 10

A mathematical vignette

In the table below, we have listed the first few powers of 2 and 10. As the exponent n gets larger, so do the corresponding powers, and every once in a while, the power gets longer by one digit. Every time we increase the exponent by 1, **exactly one** of the two powers gets longer by one digit. Both powers cannot increase their length together, nor can they both keep the same length.

Table 1

n	2^n	5^n
1	2	5
2	4	25
3	8	125
4	16	625
5	32	3125
6	64	15625
7	128	78125
8	256	390625
9	512	1953125
10	1024	9765625

There is a related phenomenon. Express the powers of 10 to base 2 and to base 5. For example, since

$$\begin{aligned}1000 &= 512 + 256 + 128 + 64 + 32 + 8 \\ &= 1 \times 2^9 + 1 \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 \\ &\qquad + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 0 \times 1,\end{aligned}$$

we can write 1000 in base 2 numerations as $(1111101000)_2$, where it has 10 digits. Since

$$1000 = 625 + 375 = 1 \times 5^4 + 3 \times 5^3 + 0 \times 5^2 + 0 \times 5 + 0 \times 1,$$

we can write 1000 in base 5 numeration as $(13000)_5$ with five digits.

If we write out all the powers of 10 in these two bases, for each whole number greater than 1, there is a power of 10 that has a representation with that number of digits in **exactly one** of the two bases 2 and 5. Successive powers of 10 in base 5 have 2, 3, 5, 6, 8, 9, 11, ... digits, while successive powers of 10 in base 2 have 4, 7, 10, ... digits ($10 = (1010)_2$, $10^2 = (1100100)_2$).

Table 2

n	10^n in base 2	10^n in base 5
1	1010	20
2	1100100	400
3	1111101000	13000
4	10011100010000	310000
5		11200000
6		
7		
8		
9		
10		

The reasons behind these phenomena are based on the fact that, in base b numeration, the number n has d digits if and only if $b^{d-1} \leq n < b^d$.

Consider the situation where 2^n and 5^n are written in base 10. Suppose that 2^n and a digits and 5^n has b digits. Then

$$10^{a-1} < 2^n < 10^a \qquad \text{and} \qquad 10^{b-1} < 5^n < 10^b.$$

Multiplying these inequalities together, we find that

$$10^{a+b-2} < 2^n \times 5^n = 10^n < 10^{a+b}.$$

Therefore $n = a + b - 1$ or $a + b = b + 1$. If we increase n by 1, then $a + b$ also increases by 1. This is possible if and only if exactly one of a and b increases by 1.

We now look at the powers of 10 written to bases 2 and 5. We first show that there are not two powers of 10 for which one has the same number of digits in base 2 as the other does in base 5. For, suppose the contrary: the 10^m has k digits in base 2 while 10^n has k digits in base 5. Then

$$2^{k-1} < 10^m < 2^k \qquad \text{and} 5^{k-1} < 10^n < 5^k.$$

Multiplying these two inequalities yields

$$10^{k-1} < 10^{m+n} < 10^k,$$

an impossibility since 10^{m+n} cannot lie stricly between two consecutive powers of 10.

The number 10^n has k digits in base 2 if and only if $2^{k-1} < 10^n < 2^k$. This is he value of n where, in Table 1, 2^n changes its number of digits. For example, 10^3 has 10 digits in base 2 corresponding to the fact that between $n = 9$ and $n = 10$, 2^n gains one more digit, with $2^9 < 10^3 < 2^{10}$.

Likewise, 10^n has k digits in base 5 if and only 5^n gains one more digits at stage n in Table 1. We know that this cannot happen for the same n. Hence the same number of digits in bases 2 and 5 cannot occur for values of 10. However, every number of digits will be represented in Table 2 in one column or the other.