A CUBIC FORM

A mathematical vignette

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Let $f(x,y,z) = x^3 + y^3 + z^3 - 3xyz$.

(1) Solve for integers x,y,z, $f(x,y,z) = 0$.

(2) Prove that, for each nonzero integer, $f(x,y,z) = n$ has only finitely many solutions (x,y,z).

(3) Determine necessary and sufficient conditions on the integer n for which there exist integers x,y,z for which $f(x,y,z) = n$.

(4) Observe that

$$f(1,4,4) = 81 = f(5,2,2).$$

Determine a generalization.

(5) Prove that, for any integers u and v for which $u+v$ is a multiple of 3, other than 3 itself, there are at least two triples (x,y,z) of nonzero integers for which $f(x,y,z) = u^3 + v^3$.

Solutions.

We begin with the factorization:

$$\begin{aligned} f(x,y,z) &= (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = p(p^2-3q) \\ &= \tfrac{1}{2}(x+y+z)((x-y)^2+(y-z)^2+(z-x)^2), \end{aligned}$$

where $p = x+y+z$, $q = xy+yz+zx$.

(1) $f(x,y,z) = 0$ if and only $x+y+z = 0$ or $x = y = z$.

(2) If $f(x,y,z) = n \neq 0$, then $p = x+y+z$ must be a divisor of $2n$ and so p can assume at most finitely many values. There are only finitely many ways in which $2n/p$ can be written as the sum of three squares (not all of which work in the equation),

(3) We can solve the equation $f(x,y,z) = n$ if we can find x,y,z for which $(x,y,z) = n$ and $(x-y)^2+(y-z)^2+(z-x)^2 = 2$. The latter equation requires that two of the differences are 1 and the remaining one is 0. If $(x,y,z) = (m,m,m+1)$, then

$$f(m,m,m+1) = 2m^3 + (m+1)^3 - 3m^2(m+1) = 3m+1,$$

and if

$$f(m,m+1,m+1) = 3m^3 + 6m^2 + 6m + 2 - 3m(m^2+2m+1) = 3m+2.$$

Thus, every number n which is not a multiple of 3 leads to a solvable equation.

On the other hand, suppose $f(x,y,z) = p(p^2 - 3q)$ is a multiple of 3. Then p and $p^2 - 3q$ must both be multiples of 3, and so $f(x,y,z)$ is a multiple of 9. Since

$$f(m-1, m, m+1) = 9m,$$

all multiples of 9 lead to a solution. Therefore, $f(x,y,z) = n$ is solvable if and only if n is not a multiple of 3 or is a multiple of 9.

(4) Note that the two solutions for $f(x,y,z) = 81$ are vectors whose coordinate sums are all the same. Observe that

$$\begin{aligned} f(w-x, w-y, w-z) &= (w-x)^3 + (w-y)^3 + (w-z)^3 - 3(w-x)(w-y)(w-z) \\ &= 3w^3 - 3w^2(x+y+z) + 3w(x^2+y^2+z^2) - (x^3+y^3+z^3) \\ &\quad - (3w^3 - 3w^2(x+y+z) + 3w(xy+yz+zx) - 3xyz \\ &= 3w(x^2+y^2+z^2 - xy - yz - zx) \\ &\quad - (x+y+z)(x^2+y^2+z^2 - xy - yz - zx). \end{aligned}$$

If x, y, z are not all equal, $f(w-x, w-y, w-z) = f(x,y,z)$ if an only if $3w = 2(x+y+z)$. Therefore, if we select any x, y, z for which $x+y+z$ is a multiple of 3 and define $w = \frac{2}{3}(x+y+z)$, then $f(x,y,z) = f(w-x, w-y, w-z)$. In the example $(x,y,z) = (1,4,4)$ and $w = 6$.

(5) Suppose that $u + v = 3w$, then from (4), we find that

$$f(2w-u, 2w-v, 2w) = f(u, v, 0) = u^3 + v^3.$$

If $u+v$ is divisible by 3, then $u^3 + v^3$ is divisible by 9, and so there are there is an integer x for which $f(x-1, x, x+1) = 9x = u^3 + v^3$.

Is it possible that $(x-1, x, x+1) = (2w-u, 2w-v, 2w)$ in some order? Taking the sum of the coordinates, we find that $3x = 6w - (u+v) = 3w$, whence $x = w$. This would force $(x-1, x, x+1) = (0, w, 2w) = (0, x, 2x)$, or $x = 1$. Then $f(0,1,2) = 9 = 1^3 + 2^3$. However, if $u + v \neq 3$, then the solutions $(x-1, x, x+1)$ and $(2w-u, 2w-v, 2w)$ must be distinct.